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Analysis on Fractals in Intuitionistic Fuzzy Metric Spaces

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Abstract—This paper investigates the fractals generated by the dynamical system of intuitionistic fuzzy contractions in the intuitionistic fuzzy metric spaces by generalizing the Hutchinson-Barnsley theory. We prove some existence and uniqueness theorems of fractals in the standard intuitionistic fuzzy metric spaces by using the intuitionistic fuzzy Banach contraction theorem. In addition to that, we analyze some results on intuitionistic fuzzy fractals in the standard intuitionistic fuzzy metric spaces with respect to the Hausdorff intuitionistic fuzzy metrics.

Keywords—Fractal Analysis, Fixed Point, Contraction, Iterated Function System, Intuitionistic Fuzzy Metric Space.

I. Introduction

The Fractal Analysis was introduced by Mandelbrot in 1975 [1] and popularized by various mathematicians [2], [3], [4], [5]. Sets with non-integral Hausdorff dimension, which exceeds its topological dimension, are called fractals by Mandelbrot. Hutchinson [2] and Barnsley [3] initiated and developed the Hutchinson-Barnsley theory (HB theory) in order to define and construct the fractal as a compact invariant subset of a complete metric space generated by the IFS of contractions, by using the classical Banach Contraction Theorem.

The theory of fuzzy sets was introduced by Zadeh in 1965 [6]. Many authors have introduced and discussed several notions of fuzzy metric space [7], [8] and fixed point theorems with interesting consequent results in the fuzzy metric spaces [8], [9]. Then the concept of intuitionistic fuzzy metric space was given by Park [10] recently and the subsequent fixed point results in the intuitionistic fuzzy metric spaces were investigated by Alaca and et al. [11] and Mohamad [12]. Recently in [13], [14]; Fractal concepts and HB theory were introduced in fuzzy metric spaces. Here we initiate and discuss fractals in the intuitionistic fuzzy metric spaces through the Hutchinson-Barnsley theory (HB theory).

In this paper, we investigate the intuitionistic fuzzy IFS fractals by proposing a generalization of the HB theory for an iterated function system of intuitionistic fuzzy contractive mappings on a complete intuitionistic fuzzy metric spaces. We also prove some existence and uniqueness theorems of intuitionistic fuzzy fractals in the standard intuitionistic fuzzy metric spaces by using the intuitionistic fuzzy Banach contraction theorem. Finally we prove some results on intuitionistic fuzzy fractals in the standard intuitionistic fuzzy metric spaces with respect to the Hausdorff intuitionistic fuzzy metrics. This paper will lead us to develop more interesting fractal concepts in the intuitionistic fuzzy metric spaces.

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II. Fractals in Metric Space

In this section, we recall the HB theory to define and construct the IFS fractals in the complete metric space.

Definition II.1. *([3], [5]) Let (X, d) be a metric space and $\mathscr{K}_o(X)$ be the collection of all non-empty compact subsets of X.*

Define, $d(x, B) := \inf_{y \in B} d(x, y)$ *and* $d(A, B) := \sup_{x \in A} d(x, B)$ *for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$. The Hausdorff metric or Hausdorff distance (H_d) is a function $H_d : \mathscr{K}_o(X) \times \mathscr{K}_o(X) \longrightarrow \mathbb{R}$ defined by*

$$H_d(A, B) = \max\Big\{d(A, B), d(B, A)\Big\}.$$

Then H_d is a metric on the hyperspace of compact sets $\mathscr{K}_o(X)$ and hence $(\mathscr{K}_o(X), H_d)$ is called a Hausdorff metric space.

We observe that, $\Big(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_d}\Big)$ is also a metric space, where $\mathscr{K}_o(\mathscr{K}_o(X))$ is the hyperspace of all non-empty compact subsets of $(\mathscr{K}_o(X), H_d)$ and $\mathscr{H}_{H_d}$ is the Hausdorff metric on $\mathscr{K}_o(\mathscr{K}_o(X))$ implied by the Hausdorff metric H_d on $\mathscr{K}_o(X)$. That is, for all $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$,

$$\mathscr{H}_{H_d}(\mathscr{A}, \mathscr{B}) = \max\Big\{H_d(\mathscr{A}, \mathscr{B}), H_d(\mathscr{B}, \mathscr{A})\Big\},$$

where $H_d(\mathscr{A}, \mathscr{B}) := \sup_{A \in \mathscr{A}} H_d(A, \mathscr{B})$ and $H_d(A, \mathscr{B}) := \inf_{B \in \mathscr{B}} H_d(A, B)$ for all $A \in \mathscr{K}_o(X)$ and $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$.

Definition II.2. *([2], [3]) Let (X, d) be a metric space and $f_n : X \longrightarrow X$, $n = 1, 2, 3, ..., N_o$ $(N_o \in \mathbb{N})$ be N_o - contraction mappings with the corresponding contractivity ratios k_n, $n = 1, 2, 3, ..., N_o$. The system $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated Function System with the ratio $k = \max_{n=1}^{N_o} k_n$. Then the Hutchinson-Barnsley operator (HB operator) of the IFS is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \text{for} \quad \text{all} \quad B \in \mathscr{K}_o(X).$$

Further, the HB operator (F) is a contraction mapping on $(\mathscr{K}_o(X), H_d)$.

Theorem II.1. *(HB Theorem [2], [3]) Let (X, d) be a complete metric space and $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, there exists only one compact invariant set $A_\infty \in \mathscr{K}_o(X)$ of the HB operator (F) or, equivalently, F has a unique fixed point namely $A_\infty \in \mathscr{K}_o(X)$.*

Definition II.3. *([3]) The fixed point $A_\infty \in \mathscr{K}_o(X)$ of the HB operator F described in the Theorem II.1 is called the Attractor (Fractal) of the IFS. Sometimes $A_\infty \in \mathscr{K}_o(X)$ is called as Fractal generated by the IFS and so called IFS Fractal.*

Theorem II.2. *([4]) Let (X, d) be a metric space. Let $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$ be such that*

$$\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\} \in \mathscr{K}_o(X).$$

Then

$$H_d\Big(\{a \in A : A \in \mathscr{A}\}, \{b \in B : B \in \mathscr{B}\}\Big) \leq \mathscr{H}_{H_d}(\mathscr{A}, \mathscr{B}).$$

Theorem II.3. *(Collage Theorem for IFS in Metric Space [3], [5]) Let (X, d) be a complete metric space. Let $(\mathscr{K}_o(X), H_d)$ be the corresponding Hausdorff metric space and $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IFS with the contractivity ratio k. If $B \in \mathscr{K}_o(X)$, then*

$$H_d(A_\infty, B) \leq \frac{H_d(B, F(B))}{1-k},$$

where F is the HB operator and A_∞ is the attractor of the IFS.

Theorem II.4. *(Falling Leaves Theorem in Metric Space [4]) Let (X, d) be a metric space. Let $(\mathscr{K}_o(X), H_d)$ be the corresponding Hausdorff metric space. If $A, B, C \in \mathscr{K}_o(X)$ are disjoint and $A', B', C' \in \mathscr{K}_o(X)$ are such that $H_d(A, A'), H_d(B, B')$ and $H_d(C, C')$ are all sufficiently small then*

$$\mathscr{H}_{H_d}\Big(\{A, B, C\}, \{A', B', C'\}\Big) = H_d\Big(A \cup B \cup C, A' \cup B' \cup C'\Big).$$

III. Intuitionistic Fuzzy Metric Space

In [6], Zadeh defined a fuzzy set on X as a function $f : X \longrightarrow [0, 1]$. In order to develop some interesting results on Intuitionistic Fuzzy IFS Fractals we have to state the required concepts of intuitionistic fuzzy metric space as follows:

Definition III.1. *([10], [15]) A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm [16], $\diamondsuit$ is a continuous t-conorm [16] and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:*

(a) $M(x, y, t) + N(x, y, t) \leq 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ *if and only if* $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(f) $M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ *is continuous;*
(g) $N(x, y, t) < 1$;
(h) $N(x, y, t) = 0$ *if and only if* $x = y$;
(i) $N(x, y, t) = N(y, x, t)$;
(j) $N(x, y, t) \diamondsuit N(y, z, s) \geq N(x, z, t + s)$;
(k) $N(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1)$ *is continuous;*

for all $x, y, z \in X$ and $t, s > 0$.

Then $(M, N, *, \diamondsuit)$ or simply (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ represents the degree of nearness and the degree of non-nearness between x and y in X with respect to t, respectively.

Definition III.2. *([10]) Let (X, d) be a metric space. Let M_d and N_d be the functions defined on $X^2 \times (0, \infty)$ by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M_d, N_d, *, \diamondsuit)$ is an intuitionistic fuzzy metric space, which is called standard intuitionistic fuzzy metric space, and (M_d, N_d) is called as the standard intuitionistic fuzzy metric induced by the metric d.

Definition III.3. *([10]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. The open ball $B(x, r, t)$ with center $x \in X$ and radius $r, 0 < r < 1$, with respect to $t > 0$, is defined as*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}.$$

Define

$$\tau_{(M,N)} = \Big\{A \subset X : for\ each\ x \in A,\ there\ exists\ t > 0 \\ and\ r \in (0, 1)\ such\ that\ B(x, r, t) \subset A\Big\}.$$

Then $\tau_{(M,N)}$ is a topology on X induced by an intuitionistic fuzzy metric (M, N). Note that the topologies induced by the metric and the corresponding standard intuitionistic fuzzy metric are the same.

Proposition III.1. *([12]) The metric space (X, d) is complete if and only if the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \diamondsuit)$ is complete.*

Definition III.4. *([12]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. We say that the mapping $f : X \longrightarrow X$ is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that*

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

and

$$\frac{1}{N(f(x), f(y), t)} - 1 \geq \frac{1}{k}\left(\frac{1}{N(x, y, t)} - 1\right)$$

for each $x, y \in X$ and $t > 0$.
Here, k is called the intuitionistic fuzzy contractivity ratio of f.

Proposition III.2. *([12]) Let (X,d) be a metric space. The mapping $f : X \longrightarrow X$ is contraction on the metric space (X,d) with contractivity ratio k if and only if f is intuitionistic fuzzy contractive, with intuitionistic fuzzy contractivity ratio k, on the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \diamondsuit)$, induced by d.*

Theorem III.1. *(Intuitionistic Fuzzy Version of the Classical Banach Contraction Theorem [12]) Let $(X, M_d, N_d, *, \diamondsuit)$ be a complete standard intuitionistic fuzzy metric space induced by the metric d on X and let $f : X \longrightarrow X$ an intuitionistic fuzzy contractive mapping. Then f has a unique fixed point.*

IV. Hausdorff Intuitionistic Fuzzy Metric Space

In [15], Gregori et al. defined the Hausdorff intuitionistic fuzzy metric on intuitionistic fuzzy hyperspace $\mathscr{K}_o(X)$ and constructed the Hausdorff intuitionistic fuzzy metric space as follows.

Definition IV.1. *([15]) Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $\tau_{(M,N)}$ be the topology induced by the intuitionistic fuzzy metric (M,N). We shall denote by $\mathscr{K}_o(X)$, the set of all non-empty compact subsets of $(X, \tau_{(M,N)})$.*

Define,

$$M(x,B,t) := \sup_{y\in B} M(x,y,t), \;\; M(A,B,t) := \inf_{x\in A} M(x,B,t)$$

and

$$N(x,B,t) := \inf_{y\in B} N(x,y,t), \;\; N(A,B,t) := \sup_{x\in A} N(x,B,t)$$

for all $x \in X$ and $A, B \in \mathscr{K}_o(X)$.

Then we define the Hausdorff intuitionistic fuzzy metric $(H_M, H_N, *, \diamondsuit)$ as follows:

$$H_M(A,B,t) = \min\Big\{M(A,B,t), M(B,A,t)\Big\}$$

and

$$H_N(A,B,t) = \max\Big\{N(A,B,t), N(B,A,t)\Big\}.$$

Here (H_M, H_N) is an intuitionistic fuzzy metric on the hyperspace of compact sets, $\mathscr{K}_o(X)$, and hence $(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit)$ is called a Hausdorff intuitionistic fuzzy metric space.

We note that, $(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_M}, \mathscr{H}_{H_N}, *, \diamondsuit)$ is also an intuitionistic fuzzy metric space, where $\mathscr{K}_o(\mathscr{K}_o(X))$ is the hyperspace of all non-empty compact subsets of $(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit)$ and $(\mathscr{H}_{H_M}, \mathscr{H}_{H_N})$ is the Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_o(\mathscr{K}_o(X))$ implied by the Hausdorff intuitionistic fuzzy metric (H_M, H_N) on $\mathscr{K}_o(X)$. That is, for all $A \in \mathscr{K}_o(X)$ and $\mathscr{A}, \mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$,

$$\mathscr{H}_{H_M}(\mathscr{A},\mathscr{B}) = \min\Big\{H_M(\mathscr{A},\mathscr{B}), H_M(\mathscr{B},\mathscr{A})\Big\}$$

$$\text{and} \;\; \mathscr{H}_{H_N}(\mathscr{A},\mathscr{B}) = \max\Big\{H_N(\mathscr{A},\mathscr{B}), H_N(\mathscr{B},\mathscr{A})\Big\},$$

where

$$H_M(\mathscr{A},\mathscr{B}) := \inf_{A\in\mathscr{A}} H_M(A,\mathscr{B}),$$

$$H_M(A,\mathscr{B}) := \sup_{B\in\mathscr{B}} H_M(A,B)$$

and

$$H_N(\mathscr{A},\mathscr{B}) := \sup_{A\in\mathscr{A}} H_N(A,\mathscr{B}),$$

$$H_N(A,\mathscr{B}) := \inf_{B\in\mathscr{B}} H_N(A,B).$$

In [17], Rodriguez-Lopez and Romaguera proved the following result for fuzzy metric spaces. Here we generalize the same result for intuitionistic fuzzy metric space. Proof of the following Proposition is very similar to that of fuzzy metric space.

Proposition IV.1. *Let (X,d) be a metric space. Then, the Hausdorff intuitionistic fuzzy metric (H_{M_d}, H_{N_d}) of the standard intuitionistic fuzzy metric (M_d, N_d) coincides with the standard intuitionistic fuzzy metric (M_{H_d}, N_{H_d}) of the Hausdorff metric (H_d) on $\mathscr{K}_o(X)$, i.e., $H_{M_d}(A,B,t) = M_{H_d}(A,B,t)$ and $H_{N_d}(A,B,t) = N_{H_d}(A,B,t)$ for all $A, B \in \mathscr{K}_o(X)$ and $t > 0$.*

Proposition IV.2. *Let (X,d) be a metric space and let $(\mathscr{K}_o(X), H_d)$ and $(\mathscr{K}_o(\mathscr{K}_o(X)), \mathscr{H}_{H_d})$ be the corresponding Hausdorff metric spaces. Then, the Hausdorff intuitionistic fuzzy metric $(\mathscr{H}_{\mathscr{M}_{H_d}}, \mathscr{H}_{\mathscr{N}_{H_d}})$ of the standard intuitionistic fuzzy metric $(\mathscr{M}_{H_d}, \mathscr{N}_{H_d})$ coincides with the standard intuitionistic fuzzy metric $(\mathscr{M}_{\mathscr{H}_{H_d}}, \mathscr{N}_{\mathscr{H}_{H_d}})$ of the Hausdorff metric $(\mathscr{H}_{H_d})$ on $\mathscr{K}_o(\mathscr{K}_o(X))$, i.e., $\mathscr{H}_{\mathscr{M}_{H_d}}(\mathscr{A},\mathscr{B},t) = \mathscr{M}_{\mathscr{H}_{H_d}}(\mathscr{A},\mathscr{B},t)$ and $\mathscr{H}_{\mathscr{N}_{H_d}}(\mathscr{A},\mathscr{B},t) = \mathscr{N}_{\mathscr{H}_{H_d}}(\mathscr{A},\mathscr{B},t)$ for all $\mathscr{A},\mathscr{B} \in \mathscr{K}_o(\mathscr{K}_o(X))$ and $t > 0$.*

Proof.

Proposition IV.1 completes the proof. □

Now we define the intuitionistic fuzzy furthest-distance function and prove the following theorem in the case of intuitionistic fuzzy metric space to that of metric space [4].

Definition IV.2. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $x \in X$ and $A \in \mathscr{K}_o(X)$. We say that the function $(\mathcal{F}_\mathcal{M}, \mathcal{F}_\mathcal{N})$ is intuitionistic fuzzy furthest-distance function for A from x, if for all $t > 0$,*

$$\begin{aligned}\mathcal{F}_\mathcal{M}(x,A,t): &= M(A,\{x\},t)\\ &= \inf_{a\in A}\sup_{x\in\{x\}} M(a,x,t)\\ &= \inf_{a\in A} M(a,x,t)\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_\mathcal{N}(x,A,t): &= N(A,\{x\},t)\\ &= \sup_{a\in A}\inf_{x\in\{x\}} N(a,x,t)\\ &= \sup_{a\in A} N(a,x,t)\end{aligned}$$

The following theorem relates that the Hausdorff intuitionistic fuzzy metric and the intuitionistic fuzzy furthest-distance function.

Theorem IV.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and let $(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit)$ denote the space of all non-empty compact subsets of X together with the Hausdorff intuitionistic fuzzy metric. Then for all $A, B \in \mathscr{K}_o(X)$,*

$$H_M(A,B,t) \geq \min\left\{\sup_{x\in B} \mathcal{F}_{\mathcal{M}}(x,A,t), \inf_{x\in B} M(x,A,t)\right\}$$

and

$$H_N(A,B,t) \leq \max\left\{\inf_{x\in B} \mathcal{F}_{\mathcal{N}}(x,A,t), \sup_{x\in B} N(x,A,t)\right\}.$$

Proof.

Fix $t > 0$. Let $A, B \in \mathscr{K}_o(X)$.
We note that

$$\begin{aligned} M(A,B,t) &= \inf_{a\in A}\sup_{b\in B} M(a,b,t) \\ &\geq \sup_{b\in B}\inf_{a\in A} M(a,b,t) \\ &= \sup_{b\in B} \mathcal{F}_{\mathcal{M}}(b,A,t) \end{aligned}$$

and

$$\begin{aligned} N(A,B,t) &= \sup_{a\in A}\inf_{b\in B} N(a,b,t) \\ &\leq \inf_{b\in B}\sup_{a\in A} N(a,b,t) \\ &= \inf_{b\in B} \mathcal{F}_{\mathcal{N}}(b,A,t) \end{aligned}$$

It follows that

$$H_M(A,B,t) \geq \min\left\{\sup_{x\in B} \mathcal{F}_{\mathcal{M}}(x,A,t), \inf_{x\in B} M(x,A,t)\right\}$$

and

$$H_N(A,B,t) \leq \max\left\{\inf_{x\in B} \mathcal{F}_{\mathcal{N}}(x,A,t), \sup_{x\in B} N(x,A,t)\right\}. \quad \square$$

The above Theorem IV.1 shows that the nearness Hausdorff intuitionistic fuzzy metric from A to B is always greater than the smaller of the maximum value (on B) of the function $\mathcal{F}_{\mathcal{M}}(x, A, t)$ and the minimum value (on B) of the function $M(x, A, t)$ and the non-nearness Hausdorff intuitionistic fuzzy metric from A to B is always lesser than the higher of the minimum value (on B) of the function $\mathcal{F}_{\mathcal{N}}(x, A, t)$ and the maximum value (on B) of the function $N(x, A, t)$.

V. IF-IFS Fractals in Intuitionistic Fuzzy Metric Space

Our primer interests are intuitionistic fuzzy fractals. More precisely, we are interested in the existence and the topological structure of fractal sets for systems of intuitionistic fuzzy contractive functions. So, we define the Intuitionistic Fuzzy IFS and Intuitionistic Fuzzy IFS Fractals in the intuitionistic fuzzy metric space. Also we discuss some existence and uniqueness of the Intuitionistic Fuzzy IFS Fractals in the standard intuitionistic fuzzy metric space.

Definition V.1. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and $f_n : X \longrightarrow X,\ n = 1, 2, 3, ..., N_o\ (N_o \in \mathbb{N})$ be N_o - intuitionistic fuzzy contractive mappings with the corresponding contractivity ratios $k_n,\ n = 1, 2, 3, ..., N_o$. The system $\{X; f_n,\ n = 1, 2, 3, ..., N_o\}$ is called an Intuitionistic Fuzzy Iterated Function System (IF-IFS) in the intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$. Then the Intuitionistic Fuzzy Hutchinson-Barnsley operator (IF-HB operator) of the IF-IFS is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = \bigcup_{n=1}^{N_o} f_n(B), \quad \forall\ B \in \mathscr{K}_o(X).$$

Definition V.2. *Let $(X, M, N, *, \diamondsuit)$ be a complete intuitionistic fuzzy metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IF-IFS and F be the IF-HB operator of the IF-IFS. We say that the set $A_\infty \in \mathscr{K}_o(X)$ is Intuitionistic Fuzzy Attractor (Intuitionistic Fuzzy Fractal) of the given IF-IFS, if A_∞ is a unique fixed point of the IF-HB operator F. Such $A_\infty \in \mathscr{K}_o(X)$ is also called as Fractal generated by the IF-IFS and so called as IF-IFS Fractal.*

Definition V.3. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space and let $C \in \mathscr{K}_o(X)$. Define a function f_o : $\mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ by $f_o(B) = C$ for all $B \in \mathscr{K}_o(X)$. Then f_o is called an intuitionistic fuzzy condensation function and C is called the associated intuitionistic fuzzy condensation set.*

Observe that an intuitionistic fuzzy condensation function is an intuitionistic fuzzy contractive mapping on the Hausdorff intuitionistic fuzzy metric space $(\mathscr{K}_o(X), H_M, H_N, *, \diamondsuit)$ and that is possesses a unique fixed point, namely the intuitionistic fuzzy condensation set.

Definition V.4. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Let $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IF-IFS and f_o : $\mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ be an intuitionistic fuzzy condensation function. The system $\{X; f_n,\ n = 0, 1, 2, 3, ..., N_o\}$ is called an Intuitionistic Fuzzy Iterated Function System (IF-IFS) with condensation. Then the Intuitionistic Fuzzy Hutchinson-Barnsley operator (IF-HB operator) with condensation is a function $F : \mathscr{K}_o(X) \longrightarrow \mathscr{K}_o(X)$ defined by*

$$F(B) = f_o(B) \cup \bigcup_{n=1}^{N_o} f_n(B) = \bigcup_{n=0}^{N_o} f_n(B), \quad \forall\ B \in \mathscr{K}_o(X).$$

Theorem V.1. *(Generalization of HB Theorem for IF-IFS with Condensation in Standard Intuitionistic Fuzzy Metric Space) Let $(X, M_d, N_d, *, \diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d, where (X, d) is a complete metric space. Let $\{X; f_n,\ n = 0, 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IF-IFS with condensation on $(X, M_d, N_d, *, \diamondsuit)$. Then, there exists only one compact invariant set $A_\infty \in \mathscr{K}_o(X)$ of the IF-HB operator (F) or, equivalently, F has a unique fixed point namely $A_\infty \in \mathscr{K}_o(X)$.*

Proof.

Since (X,d) is a complete metric space, we have $(\mathscr{K}_o(X),H_d)$ is a complete metric space. It follows from the Proposition III.1 that, $(\mathscr{K}_o(X),M_{H_d},N_{H_d},*,\diamondsuit)$ is a complete standard intuitionistic fuzzy metric space induced by the Hausdorff metric H_d. By the Proposition IV.1, we conclude that $(\mathscr{K}_o(X),H_{M_d},H_{N_d},*,\diamondsuit)$ is a complete standard intuitionistic fuzzy metric space.

Since $\{X;f_n,\ n=0,1,2,3,...,N_o;N_o\in\mathbb{N}\}$ is an IF-IFS with condensation on $(X,M_d,N_d,*,\diamondsuit)$ and f_o is an intuitionistic fuzzy contractive mapping on $(X,M_d,N_d,*,\diamondsuit)$, for each $n\in\{0,1,2,3,...,N_o\}$, we have f_n is an intuitionistic fuzzy contractive mapping on $(X,M_d,N_d,*,\diamondsuit)$. By Proposition III.2, we obtain that f_n is a contraction mapping on (X,d), for each $n\in\{0,1,2,3,...,N_o\}$. Then $\{X;f_n,\ n=0,1,2,3,...,N_o;N_o\in\mathbb{N}\}$ is an IFS on (X,d). Since the HB operator F of the IFS is a contraction function on $(\mathscr{K}_o(X),H_d)$, again by Proposition III.2, F is an intuitionistic fuzzy contractive mapping on $(\mathscr{K}_o(X),M_{H_d},N_{H_d},*,\diamondsuit)$. Proposition IV.1 concludes that, F is an intuitionistic fuzzy contractive mapping on $(\mathscr{K}_o(X),H_{M_d},H_{N_d},*,\diamondsuit)$.

Finally we obtain, F is an intuitionistic fuzzy contractive mapping on a complete standard intuitionistic fuzzy metric space $(\mathscr{K}_o(X),H_{M_d},H_{N_d},*,\diamondsuit)$. By using the Theorem III.1 (Intuitionistic Fuzzy Version of the Classical Banach Contraction Theorem), F has a unique fixed point in the standard intuitionistic fuzzy metric space. This completes the proof. □

Theorem V.2. *(Existence and Uniqueness Theorem of IF-IFS Fractals in Standard Intuitionistic Fuzzy Metric Space) Let $(X,M_d,N_d,*,\diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d, where (X,d) is a complete metric space. Let $\{X;f_n,\ n=1,2,3,...,N_o;N_o\in\mathbb{N}\}$ be an IF-IFS on $(X,M_d,N_d,*,\diamondsuit)$. Then, there exists only one compact invariant set $A_\infty\in\mathscr{K}_o(X)$ of the IF-HB operator (F) or, equivalently, F has a unique fixed point namely $A_\infty\in\mathscr{K}_o(X)$.*

The above Theorem V.2 is a particular case of the Theorem V.1 for IF-IFS without condensation function. Also the Theorem V.2 shows that the existence of Fractals in the Intuitionistic Fuzzy Space. Hence we will discuss the properties of Fractals in the Intuitionistic Fuzzy Metric Space.

Remarks

1) If B is a non-empty compact set in (X,τ_d), where τ_d is a topology induced by a metric d on X, then B is also a non-empty compact set in $(X,\tau_{(M_d,N_d)})$, where $\tau_{(M_d,N_d)}$ is a topology induced by a standard intuitionistic fuzzy metric (M_d,H_d) on X; and conversely.
2) Let $(X,M_d,H_d.*,\diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d on X. Then, A_∞ is an intuitionistic fuzzy attractor (intuitionistic fuzzy fractal) of the IF-IFS on $(X,M_d,H_d.*,\diamondsuit)$ if and only if A_∞ is an attractor (fractal) of the IFS on (X,d).

The above theorems and remarks exhibited that all deterministic fractals generated by IFS in a metric space such as Cantor set, Sierpinski gasket, Koch curve, etc. are the examples of intuitionistic fuzzy fractals in a standard intuitionistic fuzzy metric space.

VI. Some Results in Standard Intuitionistic Fuzzy Metric Space

In this section, we prove some results such as Falling Leaves Theorem and Collage Theorem for IF-IFS in the Standard Intuitionistic Fuzzy Metric Space.

Now, we discuss about the relationships between the hyperspaces $\mathscr{K}_o(X)$ and $\mathscr{K}_o(\mathscr{K}_o(X))$ together with their corresponding standard Hausdorff intuitionistic fuzzy metrics.

Theorem VI.1. *Let $(X,M_d,N_d,*,\diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d, where (X,d) is a metric space. Let $(\mathscr{K}_o(X),M_{H_d},N_{H_d},*,\diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the Hausdorff metric H_d on $\mathscr{K}_o(X)$. Let $\mathscr{A},\mathscr{B}\in\mathscr{K}_o(\mathscr{K}_o(X))$ be such that*

$$\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\}\in\mathscr{K}_o(X).$$

Then for all $t>0$,

$$H_{M_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\geq\mathscr{H}_{\mathscr{M}_{H_d}}(\mathscr{A},\mathscr{B},t)$$

and

$$H_{N_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\leq\mathscr{H}_{\mathscr{N}_{H_d}}(\mathscr{A},\mathscr{B},t).$$

Proof.

Fix $t>0$. Now by Theorem II.2 and Propositions IV.1 & IV.2, we have

$$\begin{aligned}&H_{M_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\\ &=M_{H_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\\ &=\frac{t}{t+H_d\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\}\Big)}\\ &\geq\frac{t}{t+\mathscr{H}_{H_d}(\mathscr{A},\mathscr{B})}\\ &=\mathscr{M}_{\mathscr{H}_{H_d}}(\mathscr{A},\mathscr{B},t)\\ &=\mathscr{H}_{\mathscr{M}_{H_d}}(\mathscr{A},\mathscr{B},t).\end{aligned}$$

Similarly,

$$\begin{aligned}&H_{N_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\\ &=N_{H_d}\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\},t\Big)\\ &=\frac{H_d\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\}\Big)}{t+H_d\Big(\{a\in A:A\in\mathscr{A}\},\{b\in B:B\in\mathscr{B}\}\Big)}\\ &\leq\frac{\mathscr{H}_{H_d}(\mathscr{A},\mathscr{B})}{t+\mathscr{H}_{H_d}(\mathscr{A},\mathscr{B})}\\ &=\mathscr{N}_{\mathscr{H}_{H_d}}(\mathscr{A},\mathscr{B},t)\\ &=\mathscr{H}_{\mathscr{N}_{H_d}}(\mathscr{A},\mathscr{B},t).\quad\square\end{aligned}$$

The above Theorem VI.1 shows that, in standard intuitionistic fuzzy metric space, the H_{M_d} is a 'stronger' degree of nearness Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_0(X)$ than $\mathscr{H}_{\mathscr{M}_{H_d}}$ on $\mathscr{K}_0(\mathscr{K}_0(X))$ and $\mathscr{H}_{\mathscr{N}_{H_d}}$ is a 'stronger' degree of non-nearness Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_0(\mathscr{K}_0(X))$ than H_{M_d} on $\mathscr{K}_0(X)$.

Now we consider the situation that Leaves fall from the sky, the sun is setting, and the shadows of three leaves float down a white wall. At one instant the set of leaf shadows is represented by $\mathscr{A} = \{A, B, C\} \in \mathscr{K}_0(\mathscr{K}_0(X))$ while at a later instant it is represented by $\mathscr{A}' = \{A', B', C'\} \in \mathscr{K}_0(\mathscr{K}_0(X))$. Here A and A' represent the shadows of a given leaf, B and B' represent the shadows of the second leaf and C and C' represent the shadows of a third leaf.

Suppose that the leaf shadows A, B, C are disjoint, then the following theorem tells us that when the two time instants taken are sufficiently close, the Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_0(X)$ between the union of the shadows at a first instant and the union of the shadows at a next instant is the same as the Hausdorff intuitionistic fuzzy metric on $\mathscr{K}_0(\mathscr{K}_0(X))$ between $\mathscr{A}$ and $\mathscr{B}$ in the standard intuitionistic fuzzy metric space.

Theorem VI.2. *(Falling Leaves Theorem in Standard Intuitionistic Fuzzy Metric Space) Let $(X, M_d, N_d, *, \diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d, where (X, d) is a metric space. Let $(\mathscr{K}_o(X), M_{H_d}, N_{H_d}, *, \diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the Hausdorff metric H_d on $\mathscr{K}_o(X)$. If $A, B, C \in \mathscr{K}_o(X)$ are disjoint and $A', B', C' \in \mathscr{K}_o(X)$ are such that $H_{M_d}(A, A')$, $H_{M_d}(B, B')$ & $H_{M_d}(C, C')$ are all sufficiently large and $H_{N_d}(A, A')$, $H_{N_d}(B, B')$ & $H_{N_d}(C, C')$ are all sufficiently small then for all $t > 0$,*

$$\mathscr{H}_{\mathscr{M}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big) = H_{M_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big)$$

and

$$\mathscr{H}_{\mathscr{N}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big) = H_{N_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big).$$

Proof.

Fix $t > 0$. By the Theorem II.4, we have

$$\mathscr{H}_{H_d}\Big(\{A, B, C\}, \{A', B', C'\}\Big) = H_d\Big(A \cup B \cup C, A' \cup B' \cup C'\Big).$$

Now by using the Propositions IV.1 & IV.2,

$$\begin{aligned} H_{M_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big) &= M_{H_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big) \\ &= \frac{t}{t + H_d\Big(A \cup B \cup C, A' \cup B' \cup C'\Big)} \\ &= \frac{t}{t + \mathscr{H}_{H_d}\Big(\{A, B, C\}, \{A', B', C'\}\Big)} \\ &= \mathscr{M}_{\mathscr{H}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big) \\ &= \mathscr{H}_{\mathscr{M}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} H_{N_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big) &= N_{H_d}\Big(A \cup B \cup C, A' \cup B' \cup C', t\Big) \\ &= \frac{H_d\Big(A \cup B \cup C, A' \cup B' \cup C'\Big)}{t + H_d\Big(A \cup B \cup C, A' \cup B' \cup C'\Big)} \\ &= \frac{\mathscr{H}_{H_d}\Big(\{A, B, C\}, \{A', B', C'\}\Big)}{t + \mathscr{H}_{H_d}\Big(\{A, B, C\}, \{A', B', C'\}\Big)} \\ &= \mathscr{N}_{\mathscr{H}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big) \\ &= \mathscr{H}_{\mathscr{N}_{H_d}}\Big(\{A, B, C\}, \{A', B', C'\}, t\Big). \quad \Box \end{aligned}$$

We have framed the above result for three sets, but we can easily see that it is true for any finite number of sets.

The following theorem gives an idea of how good an approximation a set is to the Fractal set of an IF-IFS.

Theorem VI.3. *(Collage Theorem for IF-IFS in Standard Intuitionistic Fuzzy Metric Space) Let $(X, M_d, H_d, *, \diamondsuit)$ be a standard intuitionistic fuzzy metric space induced by the metric d, where (X, d) is a complete metric space. Let $(\mathscr{K}_o(X), H_{M_d}, H_{N_d}, *, \diamondsuit)$ be the corresponding Hausdorff fuzzy metric space and $\{X; f_n,\ n = 1, 2, 3, ..., N_o; N_o \in \mathbb{N}\}$ be an IF-IFS with the contractivity ratio k. Then for all $B \in \mathscr{K}_o(X)$ and $t > 0$,*

$$H_{M_d}(A_\infty, B, t) \geq H_{M_d}\Big(B, F(B), t(1-k)\Big)$$

and

$$H_{N_d}(A_\infty, B, t) \leq H_{N_d}\Big(B, F(B), t(1-k)\Big)$$

where F is the IF-HB operator and A_∞ is the intuitionistic fuzzy attractor (intuitionistic fuzzy fractal) of the IF-IFS.

Proof.

Fix $t > 0$. By the Theorem II.3, for $t > 0$ and $k \in (0, 1)$ we have

$$\frac{t}{t + H_d(A_\infty, B)} \geq \frac{t(1-k)}{t(1-k) + H_d\Big(B, F(B)\Big)}$$

and

$$\frac{H_d(A_\infty, B)}{t + H_d(A_\infty, B)} \leq \frac{H_d\Big(B, F(B)\Big)}{t(1-k) + H_d\Big(B, F(B)\Big)}.$$

Now by using the Proposition IV.1,

$$\begin{aligned} H_{M_d}(A_\infty, B, t) &= M_{H_d}(A_\infty, B, t) \\ &= \frac{t}{t + H_d(A_\infty, B)} \\ &\geq \frac{t(1-k)}{t(1-k) + H_d\Big(B, F(B)\Big)} \\ &= M_{H_d}\Big(B, F(B), t(1-k)\Big) \\ &= H_{M_d}\Big(B, F(B), t(1-k)\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} H_{N_d}(A_\infty, B, t) &= N_{H_d}(A_\infty, B, t) \\ &= \frac{H_d(A_\infty, B)}{t + H_d(A_\infty, B)} \\ &\leq \frac{H_d\Big(B, F(B)\Big)}{t(1-k) + H_d\Big(B, F(B)\Big)} \\ &= N_{H_d}\Big(B, F(B), t(1-k)\Big) \\ &= H_{N_d}\Big(B, F(B), t(1-k)\Big). \quad \square \end{aligned}$$

A consequence of Theorem VI.3 is that any non-empty compact set of X can be approximated arbitrarily by an invariant set in the sense of standard Hausdorff intuitionistic fuzzy metric. Also it gives an upper bound and lower bound to the nearness and non-nearness Hausdorff intuitionistic fuzzy distance, respectively, between the intuitionistic fuzzy fractal of the given IF-IFS and a non-empty compact subset of a standard intuitionistic fuzzy metric space.

VII. Conclusion

In this study, we investigated the intuitionistic fuzzy IFS fractals in the intuitionistic fuzzy metric spaces by generalizing the Hutchinson-Barnsley theory for an iterated function system of intuitionistic fuzzy contractions. We proved some existence and uniqueness theorems of fractals in the standard intuitionistic fuzzy metric spaces by using the intuitionistic fuzzy Banach contraction Theorem. Further, we discussed some results on intuitionistic fuzzy fractals in the standard intuitionistic fuzzy metric spaces with respect to the Hausdorff intuitionistic fuzzy metrics. This will help us to develop more interesting results on fractals in general intuitionistic fuzzy metric spaces.

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