

Analysis of a Spatiotemporal Phytoplankton Dynamics: Higher Order Stability and Pattern Formation

Randhir Singh Baghel, Joydip Dhar and Renu Jain

Abstract—In this paper, for the understanding of the phytoplankton dynamics in marine ecosystem, a susceptible and an infected class of phytoplankton population is considered in spatiotemporal domain. Here, the susceptible phytoplankton is growing logistically and the growth of infected phytoplankton is due to the instantaneous Holling type-II infection response function. The dynamics are studied in terms of the local and global stabilities for the system and further explore the possibility of Hopf -bifurcation, taking the half saturation period as (i.e., α) the bifurcation parameter in temporal domain. It is also observe that the reaction diffusion system exhibits spatiotemporal chaos and pattern formation in phytoplankton dynamics, which is particularly important role play for the spatially extended phytoplankton system. Also the effect of the diffusion coefficient on the spatial system for both one and two dimensional case is obtained. Furthermore, we explore the higher-order stability analysis of the spatial phytoplankton system for both linear and no-linear system. Finally, few numerical simulations are carried out for pattern formation.

Keywords—Phytoplankton dynamics, Reaction-diffusion system, Local stability, Hopf-bifurcation, Global stability, Chaos, Pattern Formation, Higher-order stability analysis.

I. INTRODUCTION

PHYTOPLANKTONS are the staple items for the food web and they are the recycler of most of the energy that flows through the ocean ecosystem. It has a major role in stabilizing the environment as it consumes half of the universal carbon dioxide and releases oxygen. So far, there is a number of studies which show the presence of pathogenic viruses in the plankton community [1], [18], [20]. A good review of the nature of marine viruses and their ecological as well as their biological effects is given in [6]. Marine viruses infect not only plankton but cultivated stocks of Crabs, Oysters, Mussels, Clams shrimp, Salmon and Catfish, etc. are all susceptible to various kinds of viruses. We observed that the viruses are nonliving organisms, in the sense, they have no metabolism when out side the host and they can reproduce only by infecting the living organisms. Viral infection of the phytoplankton cell is of two types, namely, Lysogenic and Lytic. In lytic viral infection, when a virus injects its DNA into

Randhir Singh Baghel is with the School of Mathematics and Allied Sciences, Jiwaji University, Gwalior, (M.P.)-474011, India. (E-mail: randhirsng@gmail.com).

Joydip Dhar is with the Department of Applied Sciences, ABV-Indian Institute of Information Technology and Management, Gwalior, (M.P.)-474010, India. (E-mail: jdhar@iiitm.ac.in).

Renu Jain is with the School of Mathematics and Allied Sciences, Jiwaji University, Gwalior, (M.P.)-474011, India. (E-mail: renujain3@rediffmail.com).

a cell, it hijacks the cell's replication machinery and produces large a number of viruses. As a result, they rupture the host and are released into the environment. On the other hand, in lysogenic viral infection, the DNA of the viruses do not use the machinery of the host themselves, but their genes are duplicated each time as the host cell divides. Many papers have already been developed which have used this kind of lysogenic viral infection [1], [3], [10], [19].

Plankton pattern formation is dependent on the interplay of various physical (temperature, light) and biological (nutrient supply, fish predation) factors [22]. The pattern formation focuses on environment, social and technological sciences where the nonlinearities conspire to from spatial patterns observe The pattern formation in living systems is probably one of the most exciting subjects in modern biology and ecology. We observed that the pattern formations in the population dynamics of both aquatic system and natural environment. Our mainly study the pattern formation in marine ecosystem taking the phytoplankton dynamics. Plankton system is study an important area for research in marine ecology. Sometimes are stationary, spiral, traveling or disordered in space and time often referred as spatiotemporal chaos. The diffusion of population is capturing the spatial distribution (i.e. pattern) of both susceptible and infected class of population. The reaction-diffusion equations modeling predator-prey interactions show a wide spectrum of ecologically relevant behavior resulting from intrinsic factors alone[14].

This study is partially motivated by few works, namely, (i) a SIAM review paper [12] that considers the reaction - diffusion system as a model for marine plankton dynamics, (ii) a study on diffusion induced chaos [15] and (iii) a phytoplankton dynamics with susceptible and infective classes [5]. Our mathematical model is an extension of temporal model presented by [5], in spatiotemporal domain.

In this paper, we have introduced the spatially extended of the system and analyze four different cases of the system. In the first case, explore the Hopf bifurcation and global stability of the temporal system, in second case, determined the chaos and pattern formation in 1-D and 2-D and also studied the effect of diffusion coefficient on the spatiotemporal system and third case, we introduce higher order stability analysis for the linear and non-linear system. Fourth case, conclusion and discuss are given.

II. MATHEMATICAL MODEL

A mathematical model of phytoplankton dynamics is proposed by considering the population densities of susceptible and infected phytoplankton as P_s and P_i respectively, at any instant of time t . The population of susceptible phytoplankton is assumed to be growing logistically with intrinsic growth rate r and carrying capacity K . Let a_1 be the disease contact rate and d_1 be the removal rate of the diseased phytoplankton population, out of which c_1 fraction of infected phytoplankton rejoin the susceptible phytoplankton population, because, dead infected phytoplankton become nutrients for the growth of susceptible phytoplankton after bacterial decomposition and partially through natural recovery process in the ecosystem. The proposed mathematical model is given as follows:

$$\frac{dP_s}{dT} = rP_s \left(1 - \frac{P_s}{K} \right) - \frac{a_1 P_s P_i}{\eta + P_s} + c_1 P_i, \quad (1)$$

$$\frac{dP_i}{dT} = \frac{a_1 P_s P_i}{\eta + P_s} - d_1 P_i, \quad (2)$$

$$P_s(0) > 0, \quad P_i(0) > 0,$$

where a_1, c_1, d_1, η are all positive constant. The Holling type-II the functional response $\frac{a_1 P_s P_i}{\eta + P_s}$ is used [8] and many other researchers.

A. Analysis of Dynamical Behaviour

We study the dynamic behaviour of the system (1)-(2) and non-dimensionalizing the system (1)-(2) using $u = \frac{P_s}{K}, v = \frac{a_1 P_i}{rK}, t = rT$ and $\xi = K, \alpha = \frac{\eta}{K}, \gamma = \frac{c_1}{a_1}, \omega = \frac{c_1 K}{r}, \beta = \frac{d_1}{r}$, corresponding temporal model is given by

$$\frac{du}{dt} = u(1-u) - \frac{\xi uv}{(u+\alpha)} + \gamma v, \quad (3)$$

$$\frac{dv}{dt} = \frac{\omega uv}{(u+\alpha)} - \beta v. \quad (4)$$

The parameters $\alpha, \beta, \gamma, \xi, \omega$ are strictly positive constants.

There are three biologically feasible steady states for the system (3)-(4), namely (i) $E_0 = (0, 0)$, (ii) $E_1 = (1, 0)$, (iii) $E_2 = (u^*, v^*)$ where $u^* = (\frac{\alpha\beta}{\omega-\beta})$, $v^* = \{\frac{u^*(1-u^*)}{\frac{\xi u^*}{u^*+\alpha} - \gamma}\}$. The non-trivial equilibrium $E_2 = (u^*, v^*)$ exists if $\omega > \beta, \alpha < \frac{\omega-\beta}{\beta}$ and $(\beta\xi - \omega\gamma) > 0$.

The general variation matrix corresponding to the system (3)-(4) is given by

$$J = \begin{bmatrix} 1 - 2u - \frac{\alpha\xi v}{(u+\alpha)^2} & -\frac{\xi\beta}{\omega} + \gamma \\ \frac{\omega\alpha v}{(u+\alpha)^2} & 0 \end{bmatrix}.$$

The characteristic equation for the equilibrium $E_0 = (0, 0)$, is $\lambda(1-\lambda) = 0$ and corresponding eigenvalues are $\lambda = 0, 1$. Therefore the equilibrium $E_0 = (0, 0)$ is unstable. The characteristic equation for the equilibrium $E_1 = (1, 0)$, is $\lambda(1+\lambda) = 0$ and corresponding eigenvalues are $\lambda = 0, -1$. Therefore the equilibrium $E_1 = (1, 0)$, is unstable in v-direction and stable in u-direction. The characteristic equation for the equilibrium $E^* = (u^*, v^*)$, is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0 \quad (5)$$

where

$$a_1 = \frac{\alpha\xi v^*}{(u^* + \alpha)^2} + 2u^* - 1, \quad a_2 = \frac{\alpha v^*}{(u^* + \alpha)^2} (\xi\beta - \omega\gamma).$$

Since $a_2 > 0$ from the existence of equilibrium E^* , therefore using Routh-Hurwitz criteria the equilibrium E^* is locally asymptotically stable, if $a_1 > 0$.

Now, we explore the possibility of Hopf-bifurcation in the system (3)-(4), taking α (the half saturation period) as the bifurcation parameter. In a two dimensional dynamical system, the necessary and sufficient conditions for the existence of the Hopf-bifurcation are:

- (1) if there exists $\alpha = \alpha_0$, such that $TrDf(u(\alpha_0)) = 0$ and $detDf(u(\alpha_0)) > 0$,
- (2) $\frac{d}{d\alpha}\{Re(\lambda(\alpha))\}_{\alpha=\alpha_0} \neq 0$.

The existence of a pair of purely imaginary eigenvalues of the Jacobian matrix is ensured from the condition (1). Moreover, if the transversality condition (2) is satisfied, then the Hopf-bifurcation occurs in the system.

We claim that system (3)-(4) exhibits the Hopf-bifurcation. Now, we will verify the conditions (1) and (2) for the Hopf-bifurcation in the system.

(a) $1 - 2u - \frac{\alpha\xi v}{(u+\alpha)^2} = 0$, (b) $\frac{\alpha v}{(u+\alpha)^2} (\xi\beta - \omega\gamma) > 0$. These condition always hold because equilibrium v^* is positivity condition. Now we will check the bifurcation point from condition (1). After substituting of the values of u^* and v^* and solving it for α , it reduces to

$$\alpha = \left\{ \frac{(\omega - \beta)(\xi\beta^2 - \omega^2\gamma)}{\beta^2\xi(\omega + \beta) - 2\omega\gamma} \right\} \quad (6)$$

For the understanding of the above result, taking the parameter values: $\xi = 1, \gamma = 0.0001, \omega = 2.0$, and $\beta = 0.6$, we get a positive root $\alpha = 0.5381$ of the equation (6). Now, we check the condition (2), then

$$\lambda^2 - trDf\lambda + detDf = 0. \quad (7)$$

Put $\lambda = x + iy$ in (7), we get $(x + iy)^2 - trDf(x + iy) + detDf = 0$, on separating real parts, then we get $\frac{d\lambda}{d\alpha} = \frac{(2x-T)(x\frac{dT}{d\alpha}) - \frac{dD}{d\alpha}}{(2x-T)^2 + (2y)^2} + \frac{2y^2\frac{dT}{d\alpha}}{(2x-T)^2 + (2y)^2}$, where $T = (1 - 2u - \frac{\alpha\xi v}{(u+\alpha)^2})$ and $D = \frac{\alpha v}{(u+\alpha)^2} (\xi\beta - \omega\gamma)$ at $\alpha = \alpha_0, x(\alpha_0) = 0, y(\alpha_0) = \sigma$ where $\lambda = \pm i\sigma$ then $\frac{d\lambda}{d\alpha} = (\frac{T\frac{dD}{d\alpha} + 2\sigma^2\frac{dT}{d\alpha}}{(T^2 + 4\sigma^2)}) \neq 0$. This ensures that the above system has a hopf-bifurcation. It is shown graphically in figure 1.

B. Global Stability

In this subsection, we describe the global stability behavior of the system (3)-(4) without help of lyapunov function. Our global stability analysis based on a purely algebraic criterion provided by [24], [25], which is an application of floquet theory and the poincare-benedixon theorem. For this global stability analysis our system (3)-(4) in the following form:

$$\frac{du}{dt} = uR(u) - v(S(u) - \gamma), \quad \frac{dv}{dt} = v\left(\frac{\omega}{\xi}S(u) - \beta\right), \quad (8)$$

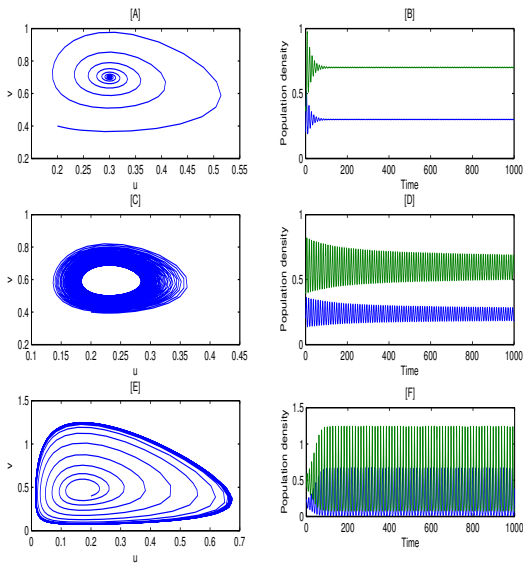


Fig. 1. Phase plane and Time series for parameter values: $\xi = 1, \gamma = 0.0001, \omega = 2.0, \beta = 0.6$, with (A) – (B) : $\alpha = 0.7$, (C) – (D) : $\alpha = 0.5381$ and (E) – (F) : $\alpha = 0.4$.

where

$$R(u) = (1 - u), \quad S(u) = \frac{\xi u}{(u + \alpha)}. \quad (9)$$

The system (8) will be globally stable around the locally asymptotically stable positive interior equilibrium point E^* if and only if the following condition holds,

$$\frac{d}{du} \left(\frac{Q(u) - Q(u^*)}{S(u) - S(u^*)} \right) \leq 0, \quad (10)$$

for $0 \leq x \leq 1, Q(u)$ is define as follows:

$$Q(u) = \frac{d}{du}(uR(u)) - u \frac{R(u)}{S(u)} \frac{d}{du} S(u). \quad (11)$$

Now next is to find the inequality for $Q(u) - Q(u^*)$ in the following form. Substituting for $R(u)$ and $S(u)$ from (9) in (11) then we get

$$Q(u) = \left(1 - 2u - \frac{\alpha(1-u)}{(u+\alpha)} \right) \quad (12)$$

Inequality for $Q(u) - Q(u^*)$ can be simplified as follows:

$$\begin{aligned} Q(u) - Q(u^*) &= \left(1 - 2u - \frac{\alpha(1-u)}{(u+\alpha)} \right) \\ &\quad - \left(1 - 2u^* - \frac{\alpha(1-u^*)}{(u^*+\alpha)} \right) \\ &= -\frac{(u-u^*)}{(u+\alpha)(u^*+\alpha)} (2(u+\alpha)(u^*+\alpha) \\ &\quad + (\alpha^* - \alpha)). \end{aligned}$$

From (9) we get

$$S(u) - S(u^*) = \frac{\xi(u - u^*)}{(u + \alpha)(u^* + \alpha)}. \quad (13)$$

Then substituting values for $S(u) - S(u^*)$ and $Q(u) - Q(u^*)$ in the (10) we get

$$\begin{aligned} \frac{d}{du} \left(\frac{Q(u) - Q(u^*)}{S(u) - S(u^*)} \right) &= -\frac{1}{\xi} \frac{d}{du} (2(u + \alpha)(u^* + \alpha) + (\alpha^2 - \alpha)) \\ &= -\frac{2}{\xi} (u^* + \alpha) \frac{d}{du} (u + \alpha) \end{aligned}$$

The condition for the global stability (10) is holds as follows:

$$\frac{d}{du} \left(\frac{Q(u) - Q(u^*)}{S(u) - S(u^*)} \right) = -\frac{2}{\xi} (u^* + \alpha) \leq 0. \quad (14)$$

Hence, it is clear that the positive interior equilibrium E^* is global stable.

III. MATHEMATICAL MODEL WITH DIFFUSION

Now, we will study the phytoplankton dynamics (1)-(2) with movement (i.e., diffusion) and since the phytoplankton population are not uniform in throughout the habitat. Therefore the population densities, i.e., P_s and P_i are become space and time dependent. Keeping in view of the above, our mathematical model can state by the following reaction diffusion equations:

$$\frac{\partial P_s}{\partial T} = rP_s \left(1 - \frac{P_s}{K} \right) - \frac{a_1 P_s P_i}{\eta + P_s} + cP_i + \delta_1 \nabla^2 P_s, \quad (15)$$

$$\frac{\partial P_i}{\partial T} = \frac{a_1 P_s P_i}{\eta + P_s} - dP_i + \delta_2 \nabla^2 P_i, \quad (16)$$

$$P_s(0) > 0, \quad P_i(0) > 0,$$

where ∇^2 is the usual laplacian operator for two dimensional, $\delta_i, (i = 1, 2)$ are diffusion coefficients and a, b, c, d are positive constants same as above section.

It is much easier to work with equation that have been scaled to non-dimensional form, in above system, we take $u = \frac{P_s}{K}, v = \frac{a_1 P_i}{rK}, t = rT, x_i = X_i \left(\frac{r}{\delta_1} \right)^{\frac{1}{2}}$ and re-scaling the parameters via, $\xi = K, \alpha = \frac{\eta}{K}, \gamma = \frac{c}{a}, \delta = \frac{\delta_1}{\delta_2}, \omega = \frac{cK}{r}, \beta = \frac{d}{r}$. Hence our spatial model reduces to

$$\frac{\partial u}{\partial t} = u(1-u) - \frac{\xi uv}{(u+\alpha)} + \gamma v + \Delta u, \quad (17)$$

$$\frac{\partial v}{\partial t} = \frac{\omega uv}{(u+\alpha)} - \beta v + \delta \Delta v, \quad (18)$$

$$u(x, y, 0) > 0, \quad v(x, y, 0) > 0, \quad (x, y) \in \Omega \quad \text{and} \quad (19)$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad (x, y) \in \partial\Omega, \quad t > 0, \quad (20)$$

where, n is the outward normal to $\partial\Omega$. and the parameters $\alpha, \beta, \gamma, \xi, \omega$ are positive constant. We assume that the system is defined on two dimensional bounded domain, denoted by Ω and consider the zero-flux boundary conditions.

Now, we will study the effect of diffusion in the spatially homogenous equilibrium $E^* = (u^*, v^*)$ of the reaction diffusion system. Obviously, the interior equilibrium point E^* for the non-spatial system (3)-(4) is a spatially homogeneous steady-state for the reaction-diffusion system (17)-(18). We assume that E^* is stable in the non-spatial system, which means that

the spatially homogeneous equilibrium is stable with respect to spatially homogeneous perturbations.

The conditions for the diffusion instability to occur in system (17) and (18), we take small heterogeneous perturbation following form:

$$u(x, y, t) = u^* + \epsilon \exp((kx + ky)i + \lambda_k t), \quad (21)$$

$$v(x, y, t) = v^* + \eta \exp((kx + ky)i + \lambda_k t), \quad (22)$$

where ϵ and η are chosen to be small and $k = (k_x, k_y)$ is the wave number. Substituting (21)-(22) into (17) and (18), linearizing the system around the interior equilibrium E^* , we get the characteristic equation as follows:

$$|J_k - \lambda_k I_2| = 0, \quad (23)$$

with

$$J_k = \begin{bmatrix} a_{11} - k^2 & a_{12} \\ a_{21} & a_{22} - \delta k^2 \end{bmatrix},$$

where, I_2 and k are second order identity matrix and wave number respectively and

$$a_{11} = 1 - 2u^* - \frac{\alpha \xi v^*}{(u^* + \alpha)^2}, \quad a_{12} = -\frac{\xi \beta}{\omega} + \gamma, \quad a_{21} = \frac{\omega \alpha v^*}{(u^* + \alpha)^2}, \quad a_{22} = 0.$$

The diffusion instability conditions when at least one of (23) the eigenvalues of the systems matrix crosses the imaginary axis. The characteristic equation following form:

$$\lambda_k^2 - (a_{11} + a_{22} - k^2(1 + \delta))\lambda_k + (a_{11} - k^2)(a_{22} - \delta k^2) - a_{12}a_{21} = 0. \quad (24)$$

We obtain that a change in stability will occur when at least one of the following two inequalities does not hold:

$$a_{11} + a_{22} - (1 + \delta)k^2 < 0, \quad (25)$$

$$h(k^2) \equiv (a_{11} - k^2)(a_{22} - \delta k^2) - a_{12}a_{21} > 0, \quad (26)$$

where a_{ij} is the elements of the matrix J^* . Since δ and k^2 are positive, both the inequalities always holds as $a_{11} = \text{tr}(J^*) < 0$ by the stability condition of the non-spatial steady state. Hence in this system, the diffusion-driven instability never occurs.

Numerical simulation is carried out for the linear stability of the system (17)-(18) taking same parameter values as in the previous subsection. Moreover, it is observed that the increases of diffusivity ratio coefficient stabilizes the system (see Fig. 2).

Now, in the next two subsection, we perform numerical simulation of the spatiotemporal system (17)-(18) evolution of pattern with respect to the time T and also obtain the effect of diffusion on the system for one and two dimensional cases.

A. One Dimensional Case

In this subsection, we will study of the following system for one dimensional case.

$$\frac{\partial u}{\partial t} = u(1 - u) - \frac{\xi uv}{(u + \alpha)} + \gamma v + \frac{\partial^2 u}{\partial x^2}, \quad (27)$$

$$\frac{\partial v}{\partial t} = \frac{\omega uv}{(u + \alpha)} - \beta v + \delta \frac{\partial^2 u}{\partial x^2}, \quad (28)$$

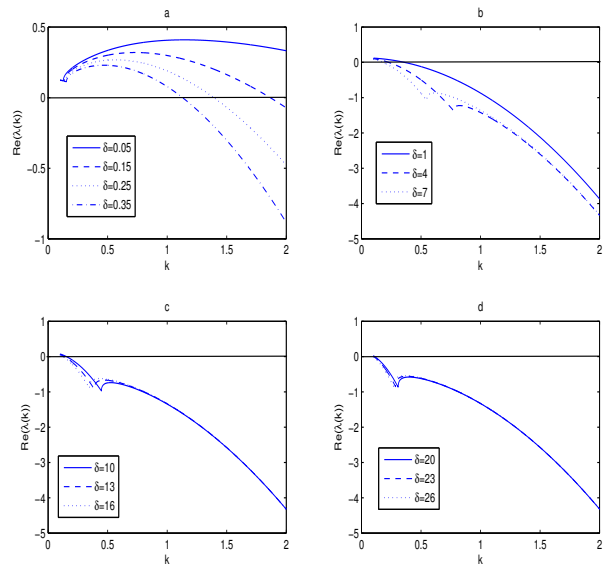


Fig. 2. Plot of max $\text{Re}(\lambda(k))$ against k , parameter values are given in text.

$$u(x, 0) > 0, \quad v(x, 0) > 0, \quad \text{for } x \in [0, R] \quad \text{and} \quad (29)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0, \quad (30)$$

The numerical solutions of the phytoplankton dynamics (i.e., u, v) are plotted with one space coordinate and time. Computer experiments are done in one dimension with domain size 6000 and we checked the sensitivity of the results to the choice of the time and space steps and their values are chosen sufficiently small. The parameter values and initial data: $\omega = 2, \beta = 0.8, \xi = 1, \gamma = 0.0001, \delta = 1, h = 4, \Delta t = 10^{-2}, \alpha = 0.3$. Varying the time to the four basic one-dimensional dynamics, namely stationary, intermittent chaos and chaos covering (almost all) of the domain (see Fig. 3).

We shown that the numerically effect of diffusion constant on the system (27)-(28) for one dimensional case . We observed that the system is stabilized if the increases the diffusivity coefficient. It is universal truth that diffusion process is stabilized the system(see Fig.4).

B. Two-Dimensional Case

The numerical solutions of the phytoplankton dynamics (17)-(18) are plotted for two dimensional (i.e.,) space x and y coordinate with time t . We use the square domain (400×400) for figure 5. The reaction diffusion equation is solved using finite difference technique semi implicit in time along with zero flux boundary condition and non-zero asymmetrical initial condition. The parameter values are $\xi = 1, \alpha = 0.4, \beta = 0.6, \omega = 2.0, \gamma = 0.0001, \delta = 0.5, h = 4, \Delta t = 1/6$ and

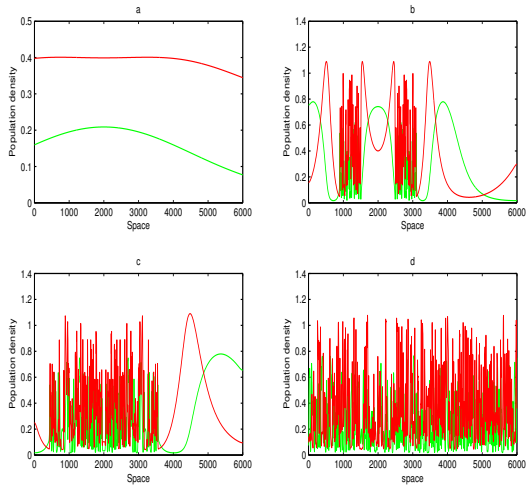


Fig. 3. The red lines for susceptible phytoplankton, and green lines for infected phytoplankton population density. Simulations are obtained for different time scales and other parametric values are given in text. In figure (a)T=5, in (b) T=800, in (c) T=2000, in (d) T=10000.

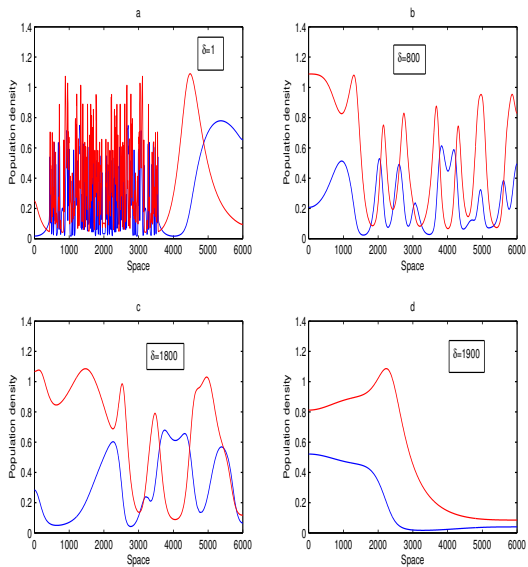


Fig. 4. The red lines for susceptible phytoplankton, and blue lines for infected phytoplankton population. Simulations are obtained for fixed time T and different diffusivity constant. The other parametric values are given in the text.

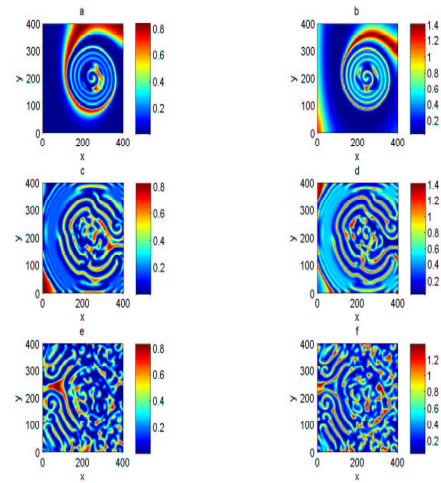


Fig. 5. Susceptible phytoplankton densities [first column] and infected phytoplankton densities [second column] population density of the system. Spatial patterns are obtained different time scales and other parametric values are given in text. Plots show the population density of (a)-(b)T = 300, (c)-(d) T = 600, (e)-(f)T = 800.

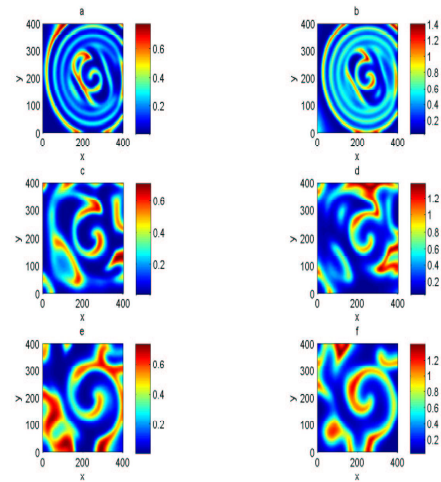


Fig. 6. Susceptible phytoplankton densities [first column] and infected phytoplankton densities [second column] population density of the system. Spatial patterns are obtained for fixed time T=300 and different values of diffusivity constant. Plots show the population density of (a)-(b) $\delta = 5$, (c)-(d) $\delta = 15$, (e)-(f) $\delta = 25$.

initial condition (19)-(20). The time evolution of the system led to the formation of spiral patterns, followed by irregular patches covering the whole domain (see Fig. 5). The size of these patches has been related to the characteristic length of observed plankton patterns in the ocean.

We show the numerically effect of the diffusion coefficient on the system (17)-(18). We observed that if the increases the diffusion coefficient then the system is stabilized(see Fig.6).

IV. HIGHER ORDER STABILITY ANALYSIS

In this subsection, we will determine the instability condition by the higher-order spatiotemporal perturbation terms [21]. We choose a general two non-dimensional reaction-diffusion system. Taking the system (3)-(4) is recalled with specific choice of parameter values. The reaction diffusion system with two dimensional is described as follows:

$$u_t = f(u, v) + u_{xx} + u_{yy}, \tag{31}$$

$$v_t = g(u, v) + \delta(v_{xx} + v_{yy}), \tag{32}$$

with no-flux boundary conditions and initial distribution of population within 2D bounded domain. The interior equilibrium point E^* for the non-spatial system corresponding to the model (31)-(32) is a spatially homogeneous equilibrium for the system (31)-(32). we consider E^* is locally asymptotically stable equilibrium for the temporal model. The stability of temporal system for requires following two conditions:

$$f_u + g_v < 0, \quad f_u g_v - f_v g_u > 0, \tag{33}$$

here, f_u denotes partial derivative of $f(u, v)$ with respect to u evaluated at (u^*, v^*) , f_v stands for partial derivative of $f(u, v)$ with respect to v evaluated at (u^*, v^*) and so on. Taking the spatiotemporal perturbations $u(t, x, y)$ and $v(t, x, y)$ on the steady states u^*, v^* defined by $u = u^* + n(t, x, y)$, $v = v^* + p(t, x, y)$ and then expanding the temporal part in Taylor series up to second order around the steady state, we find following two expressions:

$$n_t = f_u n + f_v p + \frac{f_{uu}}{2} n^2 + \frac{f_{vv}}{2} p^2 + f_{uv} np + n_{xx} + n_{yy}, \tag{34}$$

$$p_t = g_u n + g_v p + \frac{g_{uu}}{2} n^2 + \frac{g_{vv}}{2} p^2 + g_{uv} np + \delta(p_{xx} + p_{yy}). \tag{35}$$

Now, we taking spatiotemporal perturbation in the form $n(t, x, y) = n(t) \cos k_x x \cos k_y y$, $p(t, x, y) = p(t) \cos k_x x \cos k_y y$ with no-flux boundary condition leads to the following two system of equations:

$$n_t = f_u n + f_v p + \frac{f_{uu}}{2} n^2 + \frac{f_{vv}}{2} p^2 + f_{uv} np - k^2 n, \tag{36}$$

$$p_t = g_u n + g_v p + \frac{g_{uu}}{2} n^2 + \frac{g_{vv}}{2} p^2 + g_{uv} np - \delta k^2 p. \tag{37}$$

It is clear from above two equations that the growth or decay of first-order perturbation terms depends upon the second-order perturbation terms. Further, we need the dynamical equations for second-order perturbation terms involved in (36)-(37). Multiplying (36) by $2u$ and neglecting the contribution of third-order perturbation terms, we find the dynamical equation for u^2 as

$$(n^2)_t = 2f_u n^2 + 2f_v np - 2k^2 n^2, \tag{38}$$

and proceeding in a similar fashion, the dynamical equations for remaining second-order perturbations are given by

$$(p^2)_t = 2g_u np + 2g_v p^2 - 2k^2 p^2, \tag{39}$$

$$(np)_t = g_u n^2 + f_v p^2 + (f_u + g_v) np - k^2(1 + \delta) np. \tag{40}$$

The truncation of third- and higher-order terms in Taylor series expansion and neglecting of third and higher-order

perturbation terms during derivation of dynamical equations (36)-(40) leads us to a closed system of equations for n, p, n^2, p^2, np . Otherwise, one cannot avoid infinite hierarchy of dynamical equations for perturbation terms. Truncation of higher-order terms does not affect the understanding of the role of leading-order non-linearity. Applicability and significance of the analysis can be justified with the perturbation terms up to order three for the system (3)-(4) with suitable choice of parameter values. Consideration of third- and higher-order perturbation terms may be required for this type of analysis use in other system. It also depends upon the non-linearity involved. The dynamical equations (36)-(40) can be written into a compact matrix form as follows:

$$\frac{dX}{dt} = AX, \tag{41}$$

where, $X = [n, p, n^2, p^2, np]^T$ and

$$A = \begin{bmatrix} a_{11} & f_v & \frac{f_{uu}}{2} & \frac{f_{vv}}{2} & f_{uv} \\ g_u & a_{22} & \frac{g_{uu}}{2} & \frac{g_{vv}}{2} & g_{uv} \\ 0 & 0 & a_{33} & 0 & 2f_v \\ 0 & 0 & 0 & a_{44} & 2g_u \\ 0 & 0 & g_u & f_v & a_{55} \end{bmatrix},$$

with $a_{11} = f_u - k^2$, $a_{22} = g_v - \delta k^2$, $a_{33} = 2(f_u - k^2)$, $a_{44} = 2(g_v - \delta k^2)$, $a_{55} = (f_u + g_v) - k^2(1 + \delta)$. Taking solution of the system of (41) in the form $X(t) \sim e^{\lambda t}$ one can obtain the characteristic equation for the matrix A

$$|A - \lambda I_5| = 0, \tag{42}$$

where, $\lambda \equiv \lambda(k)$ are the eigenvalues of A. Thus, required instability condition demands at least one of the eigenvalues of matrix A must have positive real part, i.e. $Re(\lambda(k)) > 0$ for at least one $k \in (1, 2, \dots, 5)$. Existence of at least one eigenvalue having positive real part implies that spatiotemporal perturbation diverge with the advancement of time. The complicated structure of the matrix A prevent us to find the eigenvalues analytically. Therefore, using numerical simulations, we find an interval for k where at least one eigenvalues of A have positive real part.

Now, we consider system (17)-(18) and choosing some parameter values $\xi = 1$, $\gamma = 0.0001$, $\omega = 2.0$, $\beta = 0.6$, $\alpha = 0.4$ for different values of δ and its interior equilibrium point $E^* = (0.5714, 0.4164)$ is locally asymptotically stable for the temporal model corresponding to the model (17)-(18). We now calculate the eigenvalues of the matrix A for the model system (17)-(18) around the steady state $u^* = 0.5714$, $v^* = 0.4164$. We found that one eigenvalue having positive real part for a range of values of k in figure 7(a, b, c), we have plotted largest $Re\{\lambda(k)\} \equiv$ linear obtained by solving (23) along with largest $Re\{\lambda(k)\} \equiv$ higher order computed numerically for the characteristic equation (42) for a range of wavelengths. It is clear that linear and higher order are positive for $k \in (0.1, 1)$ over the entire range. These results ensure the existence of realistic parameter values, where the eigenvalues obtained from linear and non linear analysis possesses negative real part but one eigenvalue of the matrix A have positive real part within the same range of values for k (see figure 7).

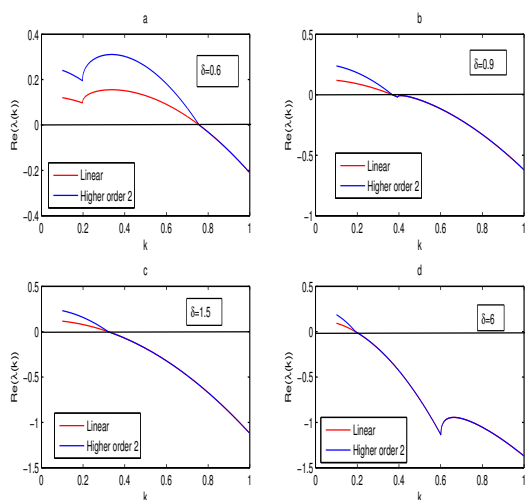


Fig. 7. For holling type-II functional response, Maximum of $\text{Re}(\lambda(k))$ plots for (red line) linear and (blue line) non-linear system versus k . The parametric values are given in the text.

V. DISCUSSION AND CONCLUSION

In this paper, a phytoplankton dynamics namely, susceptible and infected population with spatial movement have been studied. Dynamic behavior of the steady state were studied and also explored the existence of Hopf bifurcation. It is established that when the half saturation period (α) of Holling-II functional response crossed its threshold value α_0 , the susceptible and infected phytoplankton population start oscillating around the endemic equilibrium (see Fig. 1 for different values of α). In particular, we observed that the endemic equilibrium is stable, when $\alpha < 0.5381$ (Fig. 1(A)-(B)), but when α crosses the threshold value $\alpha_0 = 0.5381$, the above system exhibits Hopf-bifurcation (see Fig. 1(C, D, E, F)).

Also, we have studied the reaction diffusion model in both one and two dimension space coordinates. For one dimensional case, we shown that how modest changes in a single parameter of the system time T , can lead to dramatic changes in the qualitative dynamics of solutions(see Fig.3) and observed that the effect of diffusion on the spatial system, if the use of sufficient large values of diffusion constant then it can be made stable of the spatial system(see Fig. 4).

Furthermore, the dynamics of the spatially extended system are complicated and will depend on the system parameters, the initial data, and also the specifics of habitat geometry. There are situations where the local dynamics of solutions gives us important clues to the behavior in the spatially extended situation. For two dimensional case, numerical experiments for different values time T and different types of initial conditions for obtained different types of pattern (see Fig.5). Also shown that effect of the diffusion on the spatiotemporal system for two dimensional case, if the sufficient large diffusion constant increases then the stabilized the system(see Fig.2, 6).

We observed that the unstable spatial system can be made stable by increasing diffusivity coefficient to a sufficiently

large value(see Fig. 7). Also obtained the stability of the linear and non linear system to predict the stable or not with the help of higher-order stability analysis(see Fig.7).

Hence, the the rate of growth of susceptible phytoplankton due to the dead infected phytoplankton (which become nutrients of susceptible phytoplankton after bacterial decomposition) is a major factor for the stability of the system.

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