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Analysis and Application of in Indirect Minimum Jerk Method for Higher order Differential Equation in Dynamics Optimization Systems

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***Abstract*—**Both the minimum energy consumption and smoothness, which is quantified as a function of jerk, are generally needed in many dynamic systems such as the automobile and the pick-and-place robot manipulator that handles fragile equipments. Nevertheless, many researchers come up with either solely concerning on the minimum energy consumption or minimum jerk trajectory. This research paper considers the indirect minimum Jerk method for higher order differential equation in dynamics optimization proposes a simple yet very interesting indirect jerks approaches in designing the time-dependent system yielding an alternative optimal solution. Extremal solutions for the cost functions of indirect jerks are found using the dynamic optimization methods together with the numerical approximation. This case considers the linear equation of a simple system, for instance, mass, spring and damping. The simple system uses two mass connected together by springs. The boundary initial is defined the fix end time and end point. The higher differential order is solved by Galerkin's methods weight residual. As the result, the 6th higher differential order shows the faster solving time.

***Keywords*—**Optimization, Dynamic, Linear Systems, Jerks.

I. INTRODUCTION

MOST of the robots and advanced mobile machines nowadays are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion, [3]. Consequently, the trajectory planning and designs of these robots are done exclusively through many approaches such as the minimum energy and minimum jerk, [4]. Nevertheless, in some applications, the robot is needed to work very smoothly in order to avoid damaging the specimen that the robot is handling while consuming least amount of energy at the same time. In other words, we may want to minimize the jerk of the movement of the robot as to give it the smoothest motion as well as optimize that robot in the energy consumption issue.

The general format of the dynamic problems is consisting of the equation of motion, the initial conditions, and the boundary conditions. The area of interest in this paper will involve the problems with two-point-boundary-value conditions. Each of the problems may contain many possible solutions depending on the objective of application. Obviously, the robot that aims to run at lowest cost of energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems. Research shows that many of the researchers pay a lot of their attention on the minimization of energy while many tend to seek for the smoothness of the system. According to the second law of Newton's laws, there is a relationship between acceleration and summation of all forces including the control inputs of any linear dynamic system. By taking derivative with respect to time, there is a relationship between derivative of the acceleration called Jerk and derivative of all forces including the derivative of the control inputs of the dynamic system. In this paper, the derivative of the control inputs with respect to time are called indirect jerks.

Therefore, this research paper aims to search for the relationship between the minimum direct jerk and indirect jerk by using the optimization method so that this new alternative can be put into applications.

II. PROBLEM STATEMENT

Dynamic systems can be described as the first order derivative function of state as

$$\dot{x}_i = f_i(x_1,...,x_n,u_1,...u_m,t); \quad i = 1,...,n, \tag{1}$$

where $x \in R^n$, $u \in R^m$ and t are state, control input, and time respectively, [5]. The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum energy or minimum jerk. Note that this paper is focusing on the system with fixed end time and fixed end points. Therefore, states and control inputs that serve the necessary condition must also be able to bring the system from initial conditions $x(t_0)$ at initial time t_0 to the end point $x(t_f)$ at time t_f.

The optimization problem of minimum energy will take the form of

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 dt, \tag{2}$$

where u_i is the control input, which can be force or torque applied to the system, and $i = 1,...,m$. J is the cost function of the energy consumed by the system from initial time t_0 to end time t_f.

The same kind of concept is used to the minimum jerk problem. It is well known that jerk is the change of input force with respect to time. It is, thus, the third derivative with respect to time of x, or first order derivative of control input u. Therefore,

$$Jerk = \dddot{x} \propto \dot{u} \; . \tag{3}$$

Defining

$$\dot{u} = \tilde{u} \; , \tag{4}$$

so that (1) becomes

$$\dot{x}_i = f_i(x_1,...,x_{n+m},\tilde{u}_1,...\tilde{u}_m,t); \quad i=1,...,n+m \; . \tag{5}$$

From now on, $\tilde{u}$ is treated as a variable and as the control input of our dynamic system. Consequently, (2) can be rewritten for the objective function of the minimum indirect jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} \tilde{u}_i^2 dt \; . \tag{6}$$

Similarly, (2) also can be rewritten for the objective function of the minimum direct jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{n} \dddot{x}_i^2 dt \; . \tag{7}$$

This time, *J* is the cost function of the jerks.

III. Necessary Conditions

In this paper, we use the calculus of variations in solving for the extremal solutions of the dynamic system, [1]. Representing the control input with *u*, the principle of calculus of variations helps us solve the optimization problem by finding the time history of the control input that would minimize the cost function of the form

$$J = \phi(t,x_1,...,x_n)_{t_f} + \int_{t_0}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)dt \, , \tag{8}$$

where

$$\phi(t,x_1,...,x_n)_{t_f} \, , \tag{9}$$

is the cost based on the final time and the final states of the system, and

$$\int_{t_i}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)dt \, , \tag{10}$$

is an integral cost dependent on the time history of the state and control variables. Since the cost of the final states would be equal in all feasible time histories of the control input; therefore, the first term of (8) is omitted.
Mathematically,the form of J is

$$J = \int_{t_0}^{t_f} F\left(x_1,..,x_n,\dot{x}_1,..,\dot{x}_n,\ddot{x}_1,..,\ddot{x}_n,\dddot{x}_1,..\dddot{x}_n\right)dt \tag{11}$$

and the variation of such a functional is

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}_i} - \frac{d^3}{dt^3}\frac{\partial F}{\partial \dddot{x}_i} \right) h_i dt + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial F}{\partial \ddot{x}_i} \right) h_i dt \Big|_{t_0}^{t_f} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \ddot{x}_i} \right) \dot{h}_i dt \Big|_{t_0}^{t_f} \tag{12}$$

where $h_i(t)$ are the variations $x_i(t)$. Since $h_i(t)$and $\dot{h}_i(t)$ must vanish at the two end point , the necessary conditions for optimization become

$$\frac{\partial F}{\partial x_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}_i} - \frac{d^3}{dt^3}\frac{\partial F}{\partial \dddot{x}_i} = 0, i = 1,.,n \tag{13}$$

Weighted Residual Methods

Different forms of weighted residual method have been to solve boundary value problems. A summary of these methods is available [1]. These methods can be classified as:(a) those that satisfy the differential equations approximately over the domain but satisfy the boundary conditions exactly such as Galerkin's method, method of moments, collocation method, and method of sub-regions, (b) weak formulations which satisfy the differential equations only partially, and (c) boundary element methods which satisfy the differential equations exactly over the domain but boundary conditions only approximately such as Trefftz method.

Due to the nature of our optimization problem with fixed end point constraints at t_0 and t_f, only the methods classified in category (a) were considered suitable. The underlying fundamental behind this method can be summarized using the following simple example. Consider the problem .

$$\eta(x) - p = 0 \tag{14}$$

where $\eta(\ )$ is a differential operator, x is a function of time, and p is a constant. The solution $x(t)$must also satisfy stated boundary conditions at the initial and final time. In this method, $x(t)$ is approximated as

$$x(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t) \tag{15}$$

where $\alpha_i(t)$ are undeterm\ned parameters and $\phi_i(t)$are linearly independent mode functions selected from a complete set of functions. These functions are usually chosen to satisfy admissibility conditions relating to the boundary conditions.

On substituting (15) in(14), the following error function results

$$\varepsilon = \eta(x) - p \tag{16}$$

This error function $\varepsilon(t)$ is forced to be zero, in the average sense, by setting weighted integrals of the residual equal to zero, i.e.,

$$\int_{t_0}^{t_f} \varepsilon \psi_i dt = 0 \tag{17}$$

where $\psi_i(t)$ are the weighting functions.

The category (a) methods differ primarily in their selection of weighting functions. For example, the method of moments uses weighting function as $t^i, i = 1,..,n.$ Galerkin's method uses weighting functions the same as mode functions. In this paper, Galerkin's method was selected to obtain the approximate solution of the problem because of its generality and ubiquitous use in solving problems of mechanics. The mode functions in this problem are chosen as polynomials due to their simplicity of analytical integration.

$$S_6 \ddot{x}^6 + S_5 \ddot{x}^5 + S_4 \ddot{x}^4 + S_3 \dddot{x} + S_2 \ddot{x} + S_1 \dot{x} + S_0 x = 0 \tag{18}$$

Galerkin's Solution

The approximate solution of the six-order differential must be obtained subjected to the following boundary conditions, $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$, $x(t_f) = x_f$ and $\dot{x}(t_f) = \dot{x}_f$ In order to ensure admissibility of the trial functions, the approximate solution must have the following form [7]:

$$x(t) = \Phi_0(t) + \sum_{i=1}^{m} L_i \phi_i(t) \tag{19}$$

where $\Phi_0(t)$ is an n -dimensional vector of mode functions that satisfies the boundary conditions of the vector x at time t_0 and t_f . $\phi_i(t)$ are mode functions, which vanish at the two end points and also have zero derivatives at the end points. As a result, $x(t)$ always satisfies the boundary conditions of the problem. $L_1,...,L_m$ are n -dimensional constant vectors that are determined by minimizing the residual error.

On substituting (19) in Eq. (18), the following error vector results:

$$\begin{aligned}
\varepsilon(t) = {} & S_6 \ddot{\phi}_0^6 + S_5 \ddot{\phi}_0^5 + S_4 \ddot{\phi}_0^4 + S_3 \dddot{\phi}_0 + S_2 \ddot{\phi}_0 \\
& + S_1 \dot{\phi}_0 + S_0 \phi_0 \\
& + L_1 \big(S_6 \ddot{\phi}_1^6 + S_5 \ddot{\phi}_1^5 + S_4 \ddot{\phi}_1^4 + S_3 \dddot{\phi}_1 \\
& + S_2 \ddot{\phi}_1 + S_1 \dot{\phi}_1 + S_0 \phi_1 \big) \\
& + L_2 \big(S_6 \ddot{\phi}_2^6 + S_5 \ddot{\phi}_2^5 + S_4 \ddot{\phi}_2^4 + S_3 \dddot{\phi}_2 \\
& + S_2 \ddot{\phi}_2 + S_1 \dot{\phi}_2 + S_0 \phi_2 \big) + \ldots.. \\
& + L_m \big(S_6 \ddot{\phi}_m^6 + S_5 \ddot{\phi}_m^5 + S_4 \ddot{\phi}_m^4 + S_3 \dddot{\phi}_m \\
& + S_2 \ddot{\phi}_m + S_1 \dot{\phi}_m + S_0 \phi_m \big)
\end{aligned} \tag{20}$$

In accordance with Galerkin's procedure, the error function must be chosen to be orthogonal to the mode functions

$$\int_{t_0}^{t_f} \varepsilon(t) \phi_i(t) dt = 0, i = 1,..., m \tag{21}$$

This leads to mn scalar equations which can be used to solve for the mn elements of the vector $L_1,..., L_m$ The equation (21) can be written in a matrix form:

$$\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ T_{21} & T_{22} & \cdots & T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mm} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \tag{22}$$

where T_{pl} is a (n x n) matrix sub-block, and R_p is a (nx1) vector defined below:

$$T_{pl} = \int_{t_0}^{t_f} \big(S_6 \ddot{\phi}_l^6 + S_5 \ddot{\phi}_l^5 + S_4 \ddot{\phi}_l^4 + S_3 \dddot{\phi}_l + S_2 \ddot{\phi}_l + S_1 \dot{\phi}_l + S_0 \phi_l \big) \phi_p dt \tag{23}$$

$$R_p = \int_{t_0}^{t_f} \big(S_6 \ddot{\phi}_l^6 + S_5 \ddot{\phi}_l^5 + S_4 \ddot{\phi}_l^4 + S_3 \dddot{\phi}_l + S_2 \ddot{\phi}_l + S_1 \dot{\phi}_l + S_0 \phi_l \big) \phi_p dt \tag{24}$$

The above equation can be inverted to solve for the vectors $L_1,..., L_m$.

Mode Functions: A Particular Choice

It is quite evident that any set of $\Phi_0(t)$ and $\phi_i(t)$ that satisfies the boundary conditions is a valid set of mode functions. In this paper, $\Phi_0(t)$ is chosen as the following cubic function of time

$$\Phi_0(t) = x_0 + \dot{x}_0 t + \left[\frac{3}{t_f^2}(x_f - x_0) - \frac{2}{t_f}\dot{x}_0 - \frac{1}{t_f}\dot{x}_f\right]t^2$$

$$+\left[\frac{2}{t_f^3}(x_f - x_0) + \frac{1}{t_f^2}(\dot{x}_f - \dot{x}_0)\right]t^3 \quad (25)$$

It can be easily verified that $\Phi_0(t_0) = x_0, \Phi_0(t_f) = x_f$ $\dot{\Phi}_0(t_0) = \dot{x}_0, \dot{\Phi}_0(t_f) = \dot{x}_f$ The mode functions $\phi_i(t)$ are selected as

$$\phi_i(t) = t^3(t - t_f)^{i+2}, i = 1,\ldots,m. \quad (26)$$

These mode functions possess the properties

$$\phi_i(t_0) = \phi_i(t_f) = \dot{\phi}_i(t_0) = \dot{\phi}_i(t_f) = 0$$

With these mode functions, the matrix T_{pl} and the right hand side vector R_p can be analytically computed, respectively.

IV. Example Problems

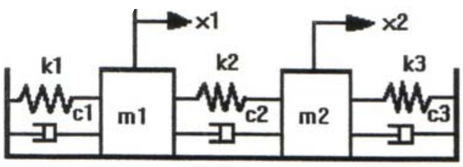

Fig. 1 Two degree of-freedom of spring mass and damper system

The procedure outlined in this paper for dynamic optimization is illustrated with the following example of a two degree-of-freedom spring-mass-damper system sketched in equation as

$$A\dot{x} = Bu \quad (27)$$

The matrices A and B for this system is as follows:

$$A = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I_2 & 0 \end{bmatrix} \quad (28)$$

$$B = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (29)$$

where the matrices M ,C and K are:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad (30)$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad (31)$$

The equation (27) can also be rewritten in the second order differential equation according to the second law of Newton. The parameters used in the model in MKS units are: $m_1 = 2, m_2 = 1, c_1 = c_3 = 1, c_2 = 2$ $k_1 = k_2 = k_3 = 1$,
The boundary conditions are $x(t_0) = (2 \quad 1 \quad 0 \quad 0)^T$ and $x(t_f) = (0\,0\,0\,0)^T$, where $t_0 = 0$ and $t_f = 1.0$.

A. Minimum Direct Jerk Problem

The cost function of minimum direct jerk is defined as

$$J = \int_0^1 \dddot{x}_1^2 + \dddot{x}_2^2 \, dt. \quad (32)$$

In order for the cost function in (32) to be minimized, the Calculus of Variations as stated in previous section has been used.

B. Minimum Indirect Jerk Problem

The cost function of minimum indirect jerk is also defined as:

$$J = \int_0^1 \tilde{u}_1^2 + \tilde{u}_2^2 \, dt. \quad (33)$$

Similarly for (33) to be minimized, the Calculus of Variations must be applied here.

C. Numerical Results

The minimum jerk problem has the exact same format as the minimum energy problem in (2). However, since the time derivative of control inputs are considered, the (27) must be rewritten as to include the consideration of jerk into the system:

$$\begin{aligned} 2\dddot{x}_1 + 3\ddot{x}_1 - 2\ddot{x}_2 + 2\dot{x}_1 - \dot{x}_2 &= \frac{du_1}{dt} = \tilde{u}_1 \\ \dddot{x}_2 - 2\ddot{x}_1 + 3\ddot{x}_2 - \dot{x}_1 + 2\dot{x}_2 &= \frac{du_2}{dt} = \tilde{u}_2. \end{aligned} \quad (34)$$

Therefore, the extra boundary conditions can be applied at both ends that are $u(t_0) = (0\,0)^T$ and $u(t_f) = (0\,0)^T$. These conditions can be applied in the numerical scheme through the original dynamic equations as follow:

$$\begin{aligned} 2\ddot{x}_1 + 3\dot{x}_1 - 2\dot{x}_2 + 2x_1 - x_2 &= u_1 \\ \ddot{x}_2 - 2\dot{x}_1 + 3\dot{x}_2 - x_1 + 2x_2 &= u_2. \end{aligned} \tag{35}$$

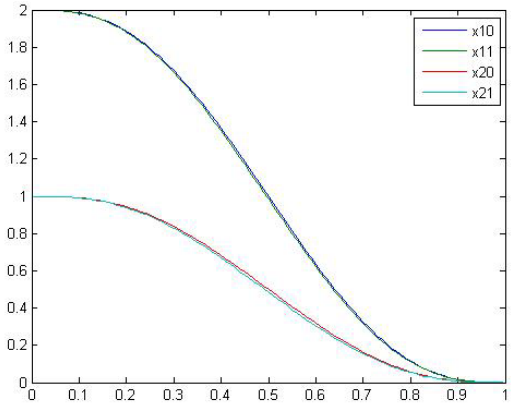

Fig. 2 Compares results of the first state variables of minimum indirect jerk between mode function at 0 and 1

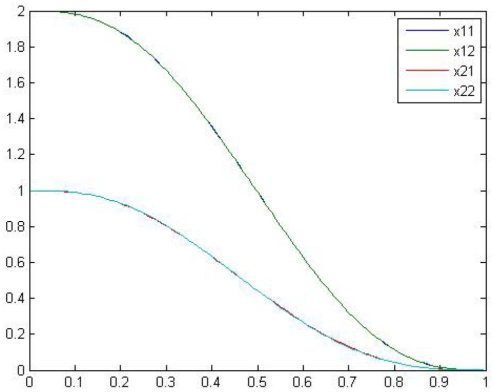

Fig. 3 Compares results of the first state variablesof minimum indirect jerk between mode function at 1 and 2

From the solutions above $x_1(t)$ and $x_2(t)$ from both minimum indirect jerks have exactly the same solutions which can be seen obviously.

In conclusion, the numerical solution of minimum indirect jerk problem becomes much easier and yields to the same results as minimum direct jerk problem since the number of control inputs in dynamic systems must be less than or equal to the number of state variables. Therefore, the variables used in the cost function of the minimum indirect jerk problem will be less than the minimum direct jerk problem when considering the under actuator dynamic or robotic systems.

The results in this paper show that the minimum indirect jerk can be used instead of minimum direct jerk strongly for the linear dynamic systems. However, the nonlinear dynamic problems could be used to compare for the future work which very high expectation that both problems will have the same results.

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