An Optimized Method for Calculating the Linear and Nonlinear Response of SDOF System Subjected to an Arbitrary Base Excitation

Hossein Kabir, Mojtaba Sadeghi

Abstract**—**Finding the linear and nonlinear responses of a typical single-degree-of-freedom system (SDOF) is always being regarded as a time-consuming process. This study attempts to provide modifications in the renowned Newmark method in order to make it more time efficient than it used to be and make it more accurate by modifying the system in its own non-linear state. The efficacy of the presented method is demonstrated by assigning three base excitations such as Tabas 1978, El Centro 1940, and MEXICO CITY/SCT 1985 earthquakes to a SDOF system, that is, SDOF, to compute the strength reduction factor, yield pseudo acceleration, and ductility factor.

Keywords**—**Single-degree-of-freedom system, linear acceleration method, nonlinear excited system, equivalent displacement method.

I. Introduction

FINDING the analytical solution for the motion equation of a typical SDOF system is almost impossible, when a force that is the base excitation, is applied to a non-linear system. Such an evaluation of the system response is often being tackled with the time-stepping methods. The Newmark method [1]-[4] is a method of numerical integration to solve differential equations. This method is widely used in the numerical evaluation of structures' dynamic response and solids such as in finite-element analysis to model dynamic systems. In this specific method, both the linear and nonlinear response of a typical SDOF system is evaluated. Furthermore, Biot [5] implemented a similar work on SDOF system. He just figured out that solving the non-linear system response would essentially depend on implementing basic optimized arithmetic formulation, which is considered in the current study [6]. It is worth mentioning that all the numerical modeling is implemented with the aid of Math Cad programming software [7].

A. Original Newmark Method for Linear Systems

For linear systems, Newmark achieved the following procedure, which is explained with the aid of Table I, to determine the system response.

Hossein Kabir* and Mojtaba Sadeghi are with the Earthquake Center of Excellence, Department of Civil Engineering, Sharif University of Technology, Iran (*Corresponding Author, e-mail: kabir_hussein@mehr.sharif.ir).

TABLE I
ABBREVIATIONS IN THE NEWMARK'S METHOD LINEAR SYSTEM

Symbol	Meaning	Symbol	Meaning
$\mathbf{u_0}$	Initial displacement	p_0	Initial force _Base Excited
Δt	Time step	k	System Lateral Stiffness
$\boldsymbol{\beta}$	Constant Coefficient	γ	Constant Coefficient
c	Damping value	m	System Total Mass
$\mathbf{u_i}$	Displacement at time i		

B. Linear Systems Procedure

1. Initial calculations

$$\frac{\partial^2 u_0}{\partial t^2} = \frac{p_0 - c\times\frac{du_0}{dt} - k\times u_0}{m}$$

Select Δt

$$a_1 = \frac{m}{\beta\times\Delta t^2} + \gamma\frac{c}{\beta\times\Delta t},\ a_2 = \frac{m}{\beta\times\Delta t} + \left(\frac{\gamma}{\beta} - 1\right)\times c,\ a_3 = \left(\frac{1}{2\times\beta} - 1\right)\times m + \Delta t\times\left(\frac{\gamma}{2\times\beta} - 1\right)\times c$$

$$\tilde{k} = k + a_1$$

2. Calculations for each time step, i =0,1, 2…, n.

$$\tilde{p}_{i+1} = p_{i+1} + a_1\times u_i + a_2\times\frac{du_i}{dt} + a_3\times\frac{\partial^2 u_i}{\partial t^2}$$

$$u_{i+1} = \frac{\tilde{p}_{i+1}}{\tilde{k}}$$

$$\frac{du_{i+1}}{dt} = \gamma\times\frac{(u_{i+1}-u_i)}{\beta\times\Delta t^2} + \left(1-\frac{\gamma}{\beta}\right)\times\frac{du_i}{dt} + \Delta t\times\left(1-\frac{\gamma}{2\times\beta}\right)\times\frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial^2 u_{i+1}}{\partial t^2} = \frac{(u_{i+1}-u_i)}{\beta\times\Delta t^2} - \frac{1}{\beta\times\Delta t}\times\frac{du_i}{dt} + \left(1-\frac{1}{2\times\beta}\right)\times\frac{\partial^2 u_i}{\partial t^2}$$

3. Repetition of the next time step by replacing i by i +1.

The Newmark's method would be stable if:

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi\sqrt{2}}\times\frac{1}{\sqrt{\gamma - 2\times\beta}}$$

In the present study, for all of the pre-assigned three earthquake records, the time step is set to be $\Delta t = 0.02$. Hence, for $\beta = \frac{1}{6}, \gamma = \frac{1}{2}$, and Tn > 0.05 , the calculated result would be stable. Therefore, the linear acceleration method is chosen for further calculations.

C. Modified Newmark Method for Nonlinear Systems

For nonlinear systems, it is possible to modify the Newmark's original formulation, achieving the following procedure with the aid of Table II to determine the system response.

TABLE II
ABBREVIATIONS IN THE NEWMARK'S MODIFIED METHOD FOR NONLINEAR SYSTEM

Symbol	Meaning	Symbol	Meaning
$\boldsymbol{u_0}$	Initial displacement	p_0	Initial Force Base_ Excited
Δt	Time step	k	System lateral Stiff.
$\boldsymbol{\beta}$	Constant Coefficient	γ	Constant Coefficient
$\tilde{\boldsymbol{k}}$	Tangent Stiffness	$\tilde{R}_{i+1,j}$	Residual Force
$\boldsymbol{c}$	Damping value	m	System Total Mass
$\boldsymbol{u_i}$	Displacement at time i	$(fs)_{i,j}$	Current Force

D. Modified Nonlinear Systems Procedure

1. Initial calculations

$$\frac{\partial^2 u_0}{\partial t^2} = \frac{p_0 - c \times \frac{du_0}{dt} - (fs)_0}{m}$$

Select Δt

$$a_1 = \frac{m}{\beta \times \Delta t^2} + \gamma \frac{c}{\beta \times \Delta t}, a_2 = \frac{m}{\beta \times \Delta t} + \left(\frac{\gamma}{\beta} - 1\right) \times c, a_3 = \left(\frac{1}{2 \times \beta} - 1\right) \times m + \Delta t \times \left(\frac{\gamma}{2 \times \beta} - 1\right) \times c$$

$$\tilde{k} = k + a_1$$

2. Calculations for each time step, i =0,1, 2..., n.

Initialize j=1, $u_{i+1,j} = u_i$, $(fs)_{i+1,j} = (fs)_{i,j}$, , $(k_T)_{i+1,j} = (k_T)_{i,j}$

$$\tilde{p}_{i+1} = p_{i+1} + a_1 \times u_i + a_2 \times \frac{du_i}{dt} + a_3 \times \frac{\partial^2 u_i}{\partial t^2}$$

3. For each iteration, j =1,2,3 ...

$$\tilde{R}_{i+1,j} = \tilde{p}_{i+1} - (fs)_{i+1,j} - a_1 \times u_{i+1,j}$$

Check the convergence; if the acceptance criteria are not met, implement the steps 3; otherwise, skip these steps and go to the step 4.

$$\widetilde{(k_T)}_{i+1,j} = (k_T)_{i+1,j} + a_1$$

$$\Delta u_j = \frac{\tilde{R}_{i+1,j}}{\widetilde{(k_T)}_{i+1,j}}$$

$$u_{i+1,j+1} = u_{i+1,j} + \Delta u_j$$

State determination $(fs)_{i+1,j+1}$ and $\widetilde{(k_T)}_{i+1,j+1}$

Replace j by j +1 and denote the final value as ui+1.

4. Calculations for velocity and acceleration

$$\frac{du_{i+1}}{dt} = \gamma \times \frac{(u_{i+1} - u_i)}{\beta \times \Delta t^2} + \left(1 - \frac{\gamma}{\beta}\right) \times \frac{du_i}{dt} + \Delta t \times \left(1 - \frac{\gamma}{2 \times \beta}\right) \times \frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial^2 u_{i+1}}{\partial t^2} = \frac{(u_{i+1} - u_i)}{\beta \times \Delta t^2} - \frac{1}{\beta \times \Delta t} \times \frac{du_i}{dt} + \left(1 - \frac{1}{2 \times \beta}\right) \times \frac{\partial^2 u_i}{\partial t^2}$$

5. Repetition of next time step. Replace i by i +1 and implement steps 2 to 4 for the next time step.

It should be noted that the material nonlinearity behavior is modeled as elastic-perfectly plastic, which is shown in Fig. 1 which is denoted in the author's numerical method. This specific feature is incorporated in the Newmark's Method.

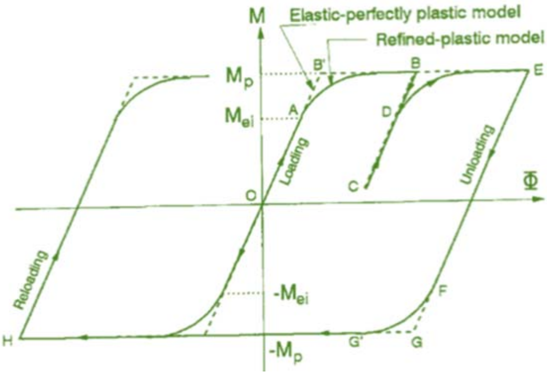

Fig. 1 Elastic-Perfectly Plastic Material Properties

II. PROGRAM ALGORITHM

According to the Newmark Equivalent Energy Theory, for small lateral deformations, the system energy of the both linear and non-linear systems is the same. This theory is the basis especially for finding the response of nonlinear one-degree-of-freedom systems during a base excitation.

First off, the linear displacement is computed using the aforementioned modified Newmark's linear acceleration method, then by choosing the target ductility factor, the correspondent stress reduction factor is calculated.

It is worthwhile to mention that for nonlinear one degree of freedom systems, the overall displacement is dependent to the signature of the multiplication of two consecutive velocities of the elastic perfectly plastic system; hence, if the system becomes mechanism, the response should be corrected regarding Figs. 1 and 2.

$$\text{if } \left|u_{i+1} - y\right| > \frac{fy}{K_{0,0}}$$

$$\text{if } v_{i+1} \cdot v_i < 0$$

$$y \leftarrow u_{i+1} + \text{sign}\left(v_{i+1}\right) \cdot u_i$$

Fig. 2 Corrected displacement for a nonlinear one degree of freedom system

In all the numerical methods, $\beta = \frac{1}{6}$ and $\gamma = \frac{1}{2}$ are assumed while the reason for choosing this option is discussed before.

III. ALGORITHM RESULTS

In Table III, the calculated Strength Reduction Factors (Rμ), using the modified Newmark's Method, are compared together for both Seismo Signal and the Equivalence Energy Methods for El Centro 1940. To represent these comparisons, the Rμ versus the Tn is being sketched in Fig. 3 for El Centro 1940.

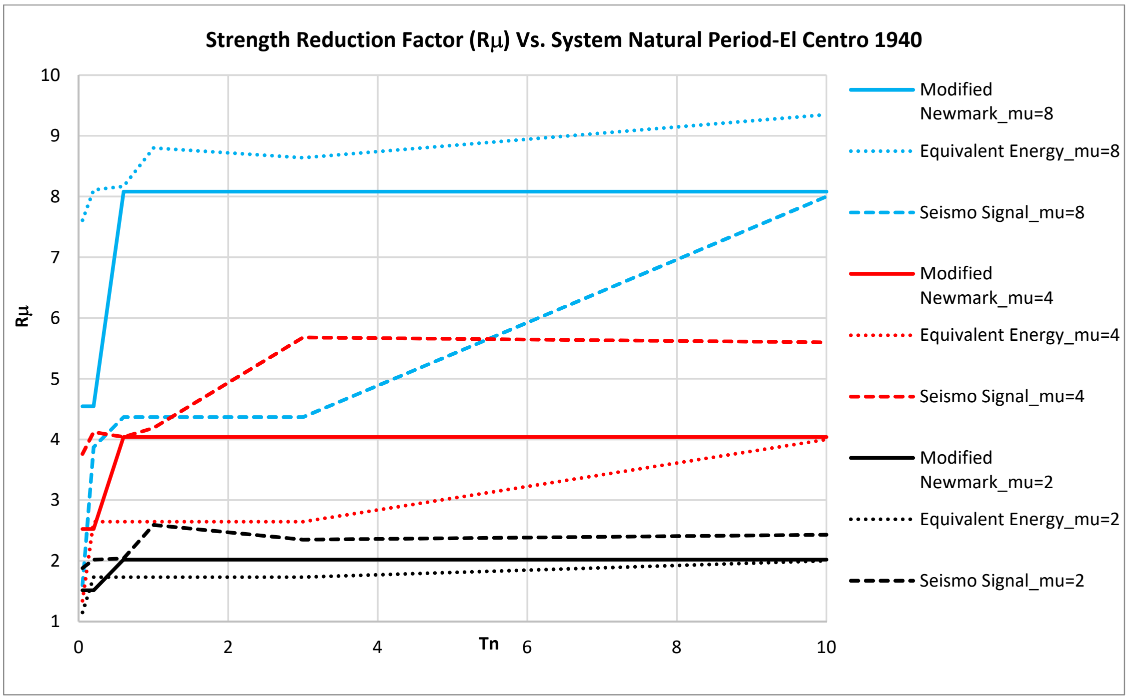

Fig. 3 El Centro 1940 (Rμ)

TABLE III
STRENGTH REDUCTION FACTOR FOR EL CENTRO 1940

Tn (Natural Period)	Rμ_ Modified Newmark's' Method	Rμ_ Seismo Signal	Rμ _Equivalent Energy Method	μ
0.05	1.515	1.88	1.152	
0.2	1.515	2.02	1.732	
0.6	2.02	2.04	1.732	
1	2.02	2.59	1.732	2
3	2.02	2.35	1.732	
10	2.02	2.43	2	
0.05	2.525	3.76	1.342	
0.2	2.525	4.12	2.646	
0.6	4.04	4.04	2.646	
1	4.04	4.19	2.646	4
3	4.04	5.68	2.646	
10	4.04	5.6	4	
0.05	4.545	7.61	1.598	
0.2	4.545	8.11	3.872	
0.6	8.08	8.17	4.368	
1	8.08	8.8	4.368	8
3	8.08	8.64	4.368	
10	8.08	9.35	8	

In Table IV, the calculated Strength Reduction Factor (Rμ), using the modified Newmark's Method, is compared with both Seismo Signal and the Equivalence Energy Method results for Mexico City 1985. To demonstrate these comparisons, the Rμ versus the Tn is sketched in Fig. 4 for Mexico City 1985.

TABLE IV
STRENGTH REDUCTION FACTOR FOR MEXICO CITY 1985

Tn (Natural Period)	Rμ_ Modified Newmark's' Method	Rμ_ Seismo Signal	Rμ _Equivalent Energy Method	μ
0.05	1.515	1.13	1.152	
0.2	1.515	1.19	1.732	
0.6	2.02	1.26	1.732	2
1	2.02	1.44	1.732	
3	2.02	2.06	1.732	
10	2.02	2.04	2	
0.05	2.525	1.48	1.342	
0.2	2.525	1.62	2.646	
0.6	4.04	1.68	2.646	4
1	4.04	1.87	2.646	
3	4.04	4.01	2.646	
10	4.04	3.63	4	
0.05	4.545	2.57	1.598	
0.2	4.545	2.59	3.872	
0.6	8.08	2.71	4.368	8
1	8.08	3.81	4.368	
3	8.08	8.03	4.368	
10	8.08	5.35	8	

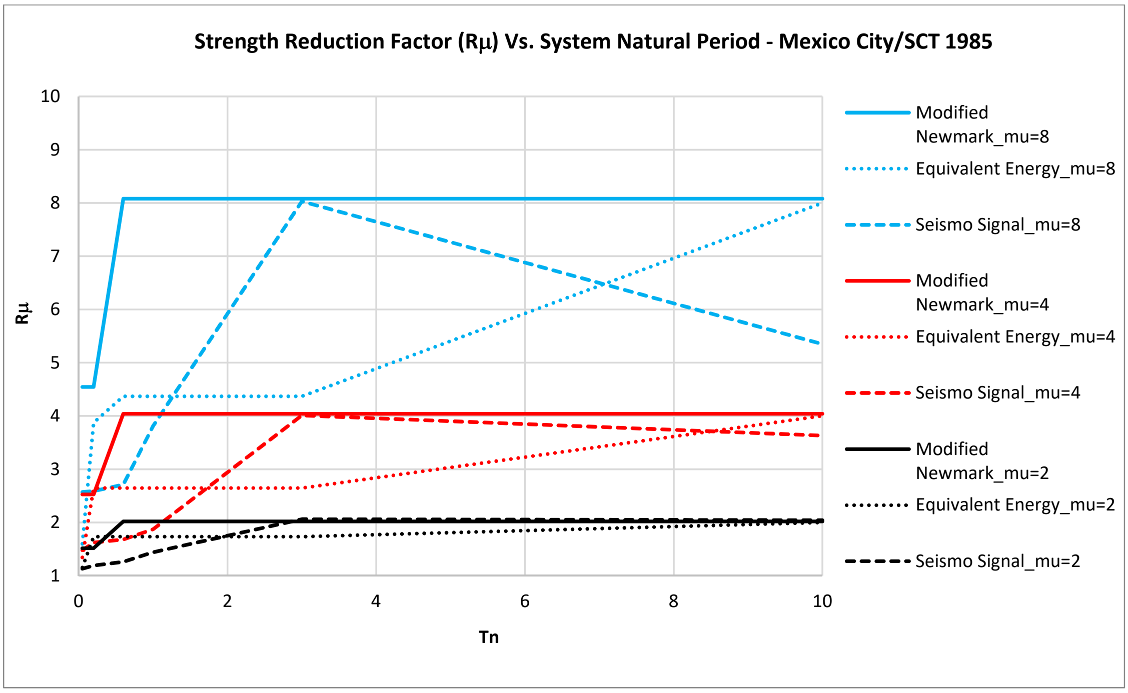

Fig. 4 Mexico City/SCT 1985 (Rμ)

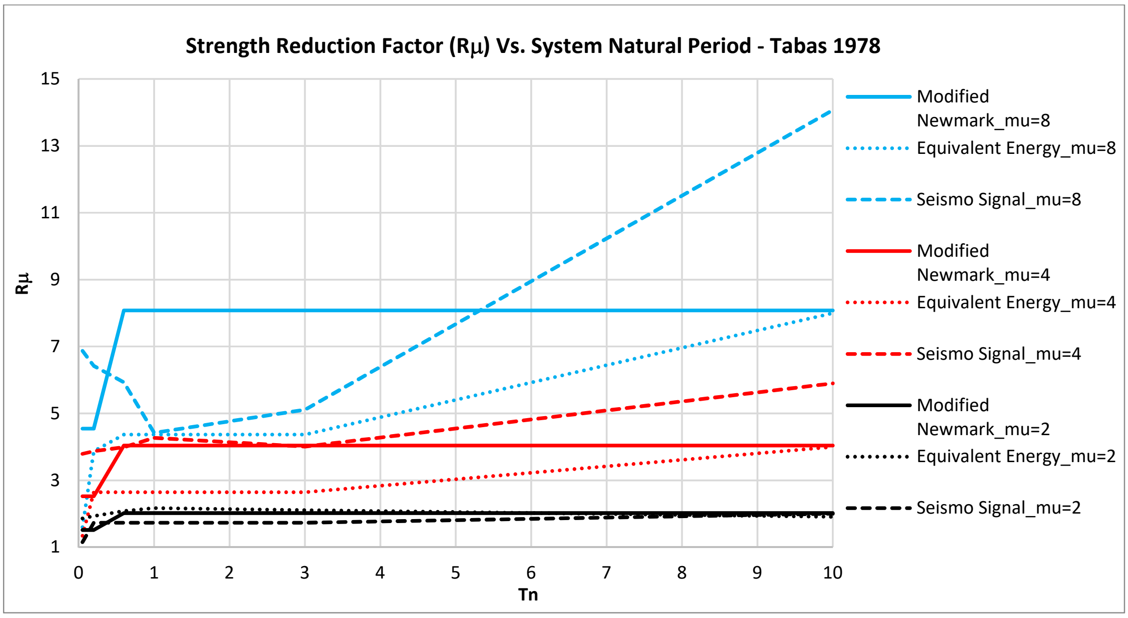

Fig. 5 Tabas 1978 (Rμ)

In Table V, the calculated Modified Newmark's Method Strength Reduction Factor (Rμ) is compared with both Seismo Signal and the Equivalence Energy Method results for Tabas 1978. To represent these comparisons, the Rμ vs Tn is sketched in Fig. 5 for Tabas 1978.

As the results suggest, the calculated Rμ from the modified Newmark's Method is very close to the Rμ computed from the Equivalent Energy Method, but approximately different from the Seismo-Signal results.

First off, the linear displacement is computed using the aforementioned modified Newmark's linear acceleration method, then by choosing the target ductility factor, the correspondent stress reduction factor is calculated. To rationalize this statement, it should be noticed that in Seismo Signal, the maximum absolute displacements for both linear and nonlinear responses are very close to each other; therefore, it is hard to identify the exact Rμ value; thus, it has a considerable error.

TABLE V
STRENGTH REDUCTION FACTOR FOR TABAS 1978

Tn (Natural Period)	Rμ_ Modified Newmark's' Method	Rμ_ Seismo Signal	Rμ _Equivalent Energy Method	μ
0.05	1.515	1.86	1.152	
0.2	1.515	1.93	1.732	
0.6	2.02	2.08	1.732	2
1	2.02	2.17	1.732	
3	2.02	2.11	1.732	
10	2.02	1.91	2	
0.05	2.525	3.79	1.342	
0.2	2.525	3.87	2.646	
0.6	4.04	3.99	2.646	4
1	4.04	4.27	2.646	
3	4.04	4.01	2.646	
10	4.04	5.9	4	
0.05	4.545	6.87	1.598	
0.2	4.545	6.42	3.872	
0.6	8.08	5.93	4.368	8
1	8.08	4.42	4.368	
3	8.08	5.11	4.368	
10	8.08	14.07	8	

IV. VERIFICATION OF THE THEORY OF EQUAL DEFORMATION

In 1960, Newmark [8] only showed that the displacements of an inelastic structure, subjected to earthquake excitations, were similar to the same structure when it behaved elastically. Code writers have merely taken this to develop the equal displacement theory concept that has been the mainstay of the seismic design codes for the past 40 years. Hence, for a long natural period, that is Tn = 10 seconds, the ductility factor must be equal to the correspondent strength reduction factor, that is Rμ=μ, as shown previously in Table V, except for the Seismo Signal results which have a significant error due to the reason that was discussed before.

A. Yield Pseudo Acceleration of the SDOF System with the Proposed Modified Pseudo Acceleration

With the aid of the proposed Modified Newmark Method, the yield pseudo acceleration of the SDOF system is achievable for all of the assumed base excitations, which are presented in Figs. 6-8.

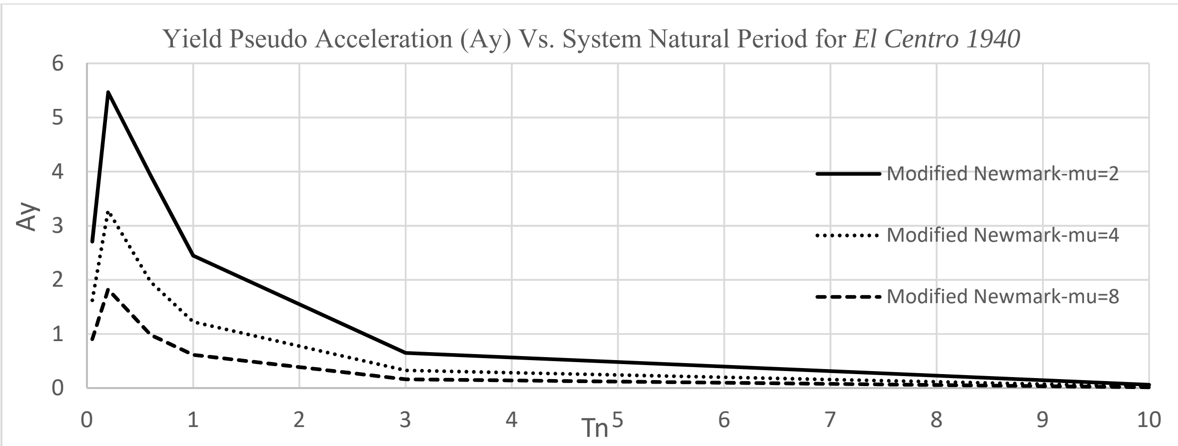

Fig. 6 Ay for El Centro 1940

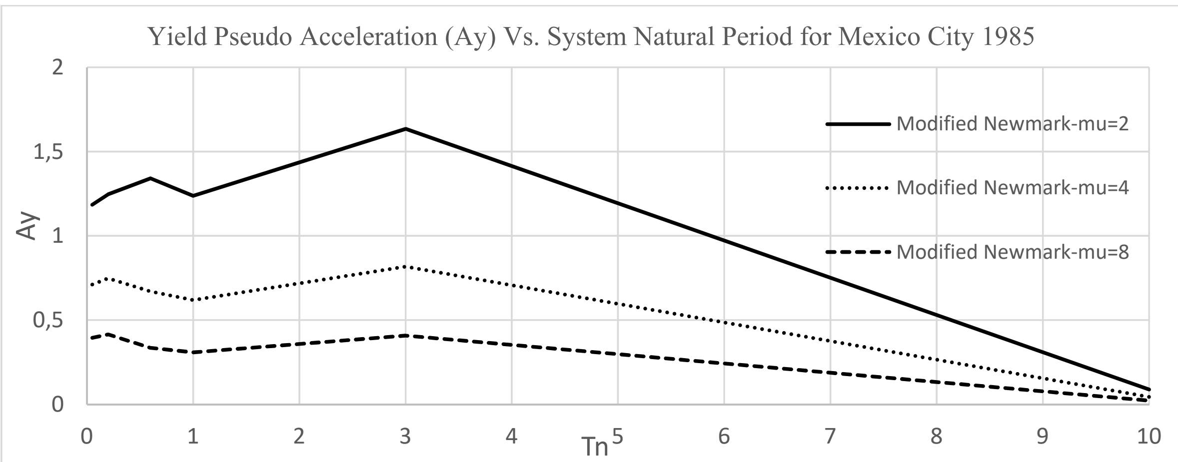

Fig. 7 Ay for Mexico City 1985

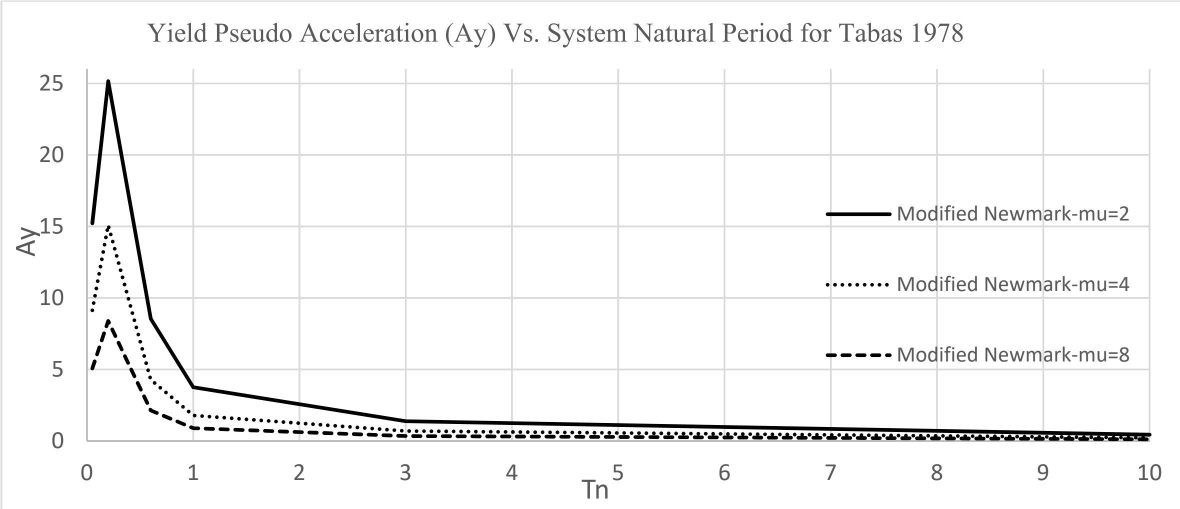

Fig. 8 Ay for Tabas 1978

V. Conclusions

Based on the numerical results, the following conclusions could be made:

- The results convergence to the Equivalent Energy Method proves the Modified Newmark's Method accuracy and efficacy.
- In Modified Newmark's Method for the system with a long natural period "Tn", the stress ductility factor equals to the strength reduction factor with good precision, which proposes the validity of the proposed methodology.
- The yield pseudo acceleration, that is, Ay, of a typical SDOF system under the arbitrary base excitation is also achievable in the proposed method as well.
- It is worth mentioning that so many structures that were built with high performance concrete [9], namely water tank, one story buildings [10], etc. could be assumed as an SDOF system. And with the current methodology, their mechanical response, under the arbitrary applied earthquake, could be evaluated as well.

Acknowledgement

The present study is partially supported by Sharif University of Technology, SUT, during the undergraduate studies of H. K in SUT.

References

[1] K. Chopra, Dynamics of Structures, 4th ed., vol. 1. Prentice Hall, 2012.

[2] Yaghmaei-Sabegh, Saman, and Jorge Ruiz-García. "Nonlinear response analysis of SDOF systems subjected to doublet earthquake ground motions: A case study on 2012 Varzaghan–Ahar events." *Engineering Structures* 110 (2016): 281-292.

[3] Andreaus, Ugo, and Maurizio De Angelis. "Nonlinear dynamic response of a base-excited SDOF oscillator with double-side unilateral constraints." *Nonlinear Dynamics* 84.3 (2016): 1447-1467.

[4] Miranda, Eduardo, and Vitelmo V. Bertero. "Evaluation of strength reduction factors for earthquake-resistant design." *Earthquake spectra* 10.2 (1994): 357-379.

[5] Biot, M. "Theory of elastic systems vibrating under transient impulse with an application to earthquake-proof buildings." *Proceedings of the National Academy of Sciences* 19, no. 2 (1933): 262-268.

[6] Kabir, Hossein. "Numerical Methods of Computing the Nonlinear Response of a Single-Degree-of-Freedom System Subjected to Earthquake Excitation", 2016.

[7] Pritchard, Philip J., and Robert Pritchard. MathCAD: A Tool for Engineering Problem Solving (BEST Series). McGraw-Hill Higher Education, 1998.

[8] Veletsos, A. S., and Nathan M. Newmark. "Effect of inelastic behavior on the response of simple systems to earthquake motions." In Proceedings of the 2nd world conference on earthquake engineering, vol. 2, pp. 895-912. 1960.

[9] Kabir, H., Sadeghi, M. (2017). 'Unconfined Strength of Nano Reactive Silica Sand Powder Concrete'. World Academy of Science, Engineering and Technology, International Science Index 123, International Journal of Civil, Environmental, Structural, Construction and Architectural Engineering, 11(3), 356 - 360.

[10] Kabir, H., Bakhshi, N., Bagheri, A. R., "An Experimental Investigation of Ultra-Fine Aggregate High Strength Concrete (UFAHSC)", International Conference on Architecture, Structure and Civil Engineering (ICASCE'15), (2015): 8-13