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An Iterated Function System for Reich Contraction in Complete b Metric Space

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Abstract—In this paper, we introduce R Iterated Function System and employ the Hutchinson Barnsley theory (HB) to construct a fractal set as its unique fixed point by using Reich contractions in a complete b metric space. We discuss about well posedness of fixed point problem for b metric space.

Keywords—Fractals, Iterated Function System, Compact set, Reich Contraction, Well posedness.

I. Introduction

THE term fractal was introduced in 1975 by Benoit Mandelbrot, a pioneer in the field of fractal geometry [1]. At one time, most people believed that the geometry of nature centered on simple figures such as lines, circles, conic sections, polygons, spheres, quadratic surfaces, and so on. But many objects in nature are so complicated and irregular that it is hopeless to use just the familiar objects from classical geometry to model them. For instance, how do we model mountains and trees in geometrical terms? Imagine the complexity of the network of paths that supply blood to and from every cell in the human body. Imagine also the intricate treelike structures of lungs and kidneys in the body. Likewise, dynamical behavior in nature can be complicated and irregular. Fractals and mathematical chaos are the appropriate tools for analyzing many of these questions [2].

The study of fractals is an exciting science that offers research possibilities in any number of application areas and in pure mathematics itself. One of the more exciting and profound developments in the construction of fractals sets, is the use of iterated function systems. The mathematics was developed by John Hutchinson [3] and the method was popularized by Michael Barnsley [4] and others. The iterated function system approach provides a good theoretical framework from which to pursue the mathematics of many classical fractals as well more general types. It is to be kept in mind from the outset that the output of an iterated function system called the attractor. Nevertheless, iterated function systems produce an amazing variety of fractals and are important to fractal theory. The mathematics of iterated functions is particularly appealing and is embedded in the general theory of dynamical systems, an area of great importance in mathematics. Many important fractals, including the classical Cantor set, Koch snowflake and Sierpinski gasket can be generated as attractors of iterated function systems. Hutchinson [3] and Barnsley [4] initiated an ingenius way to define and construct fractals as compact invariant subsets of an abstract complete metric space with respect to the union of contractions. That is, Hutchinson introduced an operator on hyperspace of nonempty compact sets called as Hutchinson Barnsley operator (HB operator) to define a fractal set as a unique fixed point by using the Banach Contraction Theorem in the metric spaces. Adrian Petrusel has discussed about finite family of single valued and multivalued operators satisfying MeirKeeler mapping by using Hutchinson Barnsley to prove the fixed point theorem in fractals [5]. S.L. Singh et al. have presented a development of the Hutchinson Barnsley (HB) theory for a system of single valued and multivalued contractions on metric spaces [6]. Sahu et al.[7] have introduced K Iterated Function System using Kannan mapping to prove fixed point theorem in complete metric space and theorem. Easwaramoorthy et al. has investigated the fractals generated by the iterated function system of fuzzy contractions in the fuzzy metric spaces by generalizing the Hutchinson Barnsley theory [8]. In this paper, we introduce R Iterated Function System and employ the Hutchinson Barnsley theory (HB) to construct a fractal set as its unique fixed point by using Reich contractions in a complete b metric space. We discuss about well posedness of fixed point problem for b metric space.

II. Preliminaries

A. Metric Fractals

Definition 1 [9] Let (X, d) be a metric space with distance function d and T be a mapping from X into itself. Then T is called a contraction mapping if there is a constant $0 \leq k < 1$ such that

$$d\,(Tx, Ty) \leq kd\,(x, y)$$

$\forall x, y \in X$ the constant k is called contractivity factor for T.

Theorem 1 [9] Let $T : X \to X$ be a contraction mapping, with contractivity factor 'k', on a complete metric space (X, d). Then T possesses exactly unique fixed point $u \in X$.

Definition 2 [4] Let (X, d) be a metric space and $\mathbb{P}(X)$ be the set of all nonempty compact subsets of X. $d\,(a, B) = inf\,\{d\,(a, b) : b \in B\}$, $h\,(A, B) = sup\,\{d\,(a, B) : a \in A\}$, The Hausdorff metric or Hausdorff distance (H_d) is a function $H_d : \mathbb{P}(X) \times \mathbb{P}(X) \longrightarrow \mathbb{R}$ defined by

$$H_d(A, B) = \max\Big\{h(A, B), h(B, A)\Big\}.$$

Then H_d is a metric on the hyperspace of compact sets $\mathbb{P}(X)$ and hence $(\mathbb{P}(X), H_d)$ is called a Hausdorff metric space.

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Note: Throughout this paper the notation $u \vee v$ means the maximum and $u \wedge v$ denotes the minimum of the pair of real numbers u and v.

Theorem 2 ([3], [4]) If (X, d) is a complete metric space then $(\mathbb{P}(X), H_d)$ is also a complete metric space.

Definition 3 [3],[4] Let (X, d) be a metric space and $T_n : X \to X$, $n = 1, 2, 3......N_o$ $(N_o \in \mathbb{N})$ be N_o contraction mappings with the contractivity ratios k_n, $n = 1, 2, 3......N_o$. The system $\{X; T_n, n = 1, 2, 3......N_o\}$ is called an Iterated Function System (IFS) or Hyperbolic Iterated function system with the ratio $k = max_{n=1}^{N_o} k_n$.

Then the Hutchinson Barnsley operator (HB operator) of the IFS is a function $\widehat{T} : \mathbb{P}(X) \to \mathbb{P}(X)$ defined by $\widehat{T}(B)=\bigcup_{n=1}^{N_o} T_n(B)$, for all $B \in \mathbb{P}(X)$.

Theorem 3 [3],[4] Let (X, d) be a metric space. Let $\{X; T_n, n = 1, 2, 3......N_o; N_o \in \mathbb{N}\}$ be an IFS. Then, the HB operator $\widehat{T}$ is a contraction mapping on $(\mathbb{P}(X), H_d)$.

Theorem 4 [3],[4] Let (X, d) be a metric space and $\{X; T_n, n = 1, 2, 3......N_o; N_o \in \mathbb{N}\}$ be an IFS. Then there exists only one compact invariant set $A_\infty \in \mathbb{P}(X)$ of the HB operator $\widehat{T}$ (or) equivalently $\widehat{T}$ has a unique fixed point $A_\infty \in \mathbb{P}(X)$.

The fixed point $A_\infty \in \mathbb{P}(X)$ of the HB operator $\widehat{T}$ described in theorem 4 is called an attractor (Fractal) of the IFS. So $A_\infty \in \mathbb{P}(X)$ is called a fractal generated by IFS of classical Banach contraction [3],[4].

b Metric space

The concept of b metric space appeared in I.A.Bakhtin, S.Czerwik [10], [11].

Definition 4 [11] Let X be a non empty set and $t \geq 1$ be a real number. A function $d : X \times X \to R^+$ is said to be a b metric if $\forall x, y, z \in X$,

$$\begin{aligned} d(x, y) &= 0 \text{ iff } x = y; \\ d(x, y) &= d(y, x) \\ d(x, z) &\leq t[d(x, y) + d(y, z)] \end{aligned}$$

The pair (X, d) is called b metric space. In fact, the class of b metric spaces is effectively larger than that of metric spaces, since a b metric is a metric when $t = 1$. The following example of Singh and Prasad [12] shows that a b metric on X need not be a metric on X.

Example 1 Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$, $d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$.

$$d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$$

for all $x, y, z \in X$. If $m > 2$ the ordinary triangle inequality does not hold. We recall the notion of some basic definition of b metric space.

Definition 5 Let (X, d) be a b metric space. Then a sequence $(x_n)_{n\in N}$ is called:

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$

Definition 6 The b metric space (X, d) is complete if every Cauchy sequence in X converges.

Lemma 1 [11] Let (X, d) be a b metric space. Then for all $A, B, C \in \mathbb{P}(X)$ we have
$H(A, C) \leq t[H(A, B) + H(B, C)]$.

Theorem 5 [11] If (X, d) is a complete b metric space then $(\mathbb{P}(X), H_d)$ is also a complete b metric space.

III. R Iterated Function System

In 1971, [13] S.Reich introduced a mapping, which was an enhancement over contraction mapping. We shall try to investigate the possibility of development in IFS by replacing contraction condition by a more general condition known as Reich mapping defined as follows.

There exists nonnegative numbers a, b, c satisfying $a+b+c < 1$ such that for each $x, y \in X$

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y)$$

Theorem 6 Let X be a complete b metric space with metric d, let $T : X \to X$ be a Reich contraction $d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y)$ for each $x, y \in X$, where a, b, c are nonnegative real numbers and satisfying $a+b+c < 1$. Then T has a unique fixed point.

Proof:

$$\begin{aligned} d\left(T^n x, T^{n+1} x\right) &= d\left(T\left(T^{n-1} x\right), T\left(T^n x\right)\right) \\ &\leq ad\left(T^{n-1} x, T^n x\right) \\ &\quad +bd\left(T^n x, T^{n+1} x\right) \\ &\quad +cd\left(T^{n-1} x, T^n x\right) \\ d\left(T^n x, T^{n+1} x\right) &\leq \left(\frac{a+c}{1-b}\right) d\left(T^{n-1} x, T^n x\right) \\ d\left(T^n x, T^{n+1} x\right) &\leq kd\left(T^{n-1} x, T^n x\right) \end{aligned}$$

$$where\ k = \frac{a+c}{1-b},\ 0 < k < 1$$

$$\begin{aligned} d\left(T^n x, T^{n+1} x\right) &\leq kd\left(T^{n-1} x, T^n x\right) \\ &\leq k\left[kd\left(T^{n-2} x, T^{n-1} x\right)\right] \\ &= k^2 d\left(T^{n-2} x, T^{n-1} x\right) \\ &\leq k^2\left[kd\left(T^{n-3} x, T^{n-2} x\right)\right] \\ &= k^3 d\left(T^{n-3} x, T^{n-2} x\right) \\ &\quad . \\ &\quad . \\ &\leq k^n d(x, Tx) \end{aligned}$$

$$\begin{aligned} d(T^n x, T^m x) &\leq td\left(T^n x, T^{n+1} x\right) + t^2 d\left(T^{n+1} x, T^{n+2} x\right) \\ &\quad +....... + t^{m-n-1} d\left(T^{n+m-1} x, T^m x\right) \\ &\leq tk^n d(x, Tx) + t^2 k^{n+1} d(x, Tx) \\ &\quad +..... + t^{m-n-1} k^{m-1} d(x, Tx) \\ d(T^n x, T^m x) &\leq \frac{k^n t}{1-tk} d(x, Tx)\ \forall m, n \geq N \end{aligned}$$

So $T^n x$ is a cauchy sequence, since X is complete b metric space, $T^n x$ will converges to a fixed point $u \in X$.

$$\begin{aligned} d(u,Tu) &\leq t\left[d\left(u,T^{n+1}x\right)\right. \\ &\quad \left.+d\left(T\left(T^n x\right),Tu\right)\right] \\ d(u,Tu) &\leq t\left[d\left(u,T^{n+1}x\right)\right. \\ &\quad +ad\left(T^n x,T^{n+1}x\right)+bd(u,Tu) \\ &\quad \left.+cd\left(T^n x,u\right)\right] \\ d(u,Tu)-bd(u,Tu) &\leq t\left[d\left(u,T^{n+1}x\right)\right. \\ &\quad +ad\left(T^n x,T^{n+1}x\right) \\ &\quad \left.+cd\left(T^n x,u\right)\right] \\ d(u,Tu) &\leq \frac{1}{1-bt}\left[td\left(u,T^{n+1}x\right)\right. \\ &\quad \left.+tak^n d(x,Tx)+tcd\left(T^n x,u\right)\right] \end{aligned}$$

Letting $n \to \infty$ which implies that $d(u,Tu)=0$ i.e. u is the fixed point of T.
Uniqueness: On the contrary let u and v be two fixed points of T. Then $u=Tu$ and $v=Tv$

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq ad(u,Tu)+bd(v,Tv)+cd(u,v) \\ &\leq ad(u,u)+bd(v,v)+cd(u,v) \\ d(u,v) &\leq cd(u,v) \\ d(u,v)-cd(u,v) &\leq 0 \\ (1-c)\,d(u,v) &\leq 0 \\ d(u,v) &= 0 \end{aligned}$$

since $(1-c)>0$, therefore $u=v$.
On the origin of the definition of hyperbolic iterated function system given by Barnsley, we now introduce R Iterated function system.
Definition 7 A Reich Iterated function system (R IFS) consists of a complete b metric space (X,d) together with a finite set of Reich contraction $T_n : X \to X$ with R contractive factors $a_n, b_n, c_n,\ n=1,2,3.....N_o$.
Definition 8 Let (X,d) be a b metric space. Let $\{X; T_n, n=1,2,3......N_o, N_o \in \mathbb{N}\}$ be an R IFS consists of finite number of Reich contraction. Then HB operator of the R IFS is a function $\widehat{T}:\mathbb{P}(X)\to\mathbb{P}(X)$ defined by

$$\widehat{T}(B)=\bigcup_{n=1}^{N_o} T_n(B) \text{ for all B} \in \mathbb{P}\text{(X)}.$$

Lemma 2 Let $T : X \to X$ be a Reich contraction on the b metric space (X,d) with R contractivity factors a,b,c. Then $\widehat{T}:\mathbb{P}(X)\to\mathbb{P}(X)$ defined by $\widehat{T}(B)=\{T(x): x\in B\}$ for every $B\in\mathbb{P}(X)$ is a Reich contraction on $(\mathbb{P}(X),H_d)$ with contractivity factors a,b,c.
Proof:
Let $B,C \in H(X)$ then

$$\begin{aligned} H_d(T(B),T(C)) &= d(T(B),T(C)) \vee d(T(C),T(B)) \\ &\leq [ad(B,T(B))+bd(C,T(C)) \\ &\quad +cd(B,C)] \\ &\quad \vee [ad(C,T(C))+bd(B,T(B)) \\ &\quad +cd(C,B)] \\ &\leq [ad(B,T(B))+ \\ &\quad bd(C,T(C))] \vee [ad(C,T(C)) \\ &\quad +bd(B,T(B))] \\ &\quad +c[d(B,C)\vee d(C,B)] \\ &= [ad(B,T(B))+bd(C,T(C))] \\ &\quad +cH_d(B,C) \\ &\leq aH_d(B,T(B))+bH_d(C,T(C)) \\ &\quad +cH_d(B,C). \\ H_d(T(B),T(C)) &\leq aH_d(B,T(B))+bH_d(C,T(C)) \\ &\quad +cH_d(B,C) \end{aligned}$$

Lemma 3 Let (X,d) be a b metric space. Let $\{T_n; n=1,2,3,........N_o, N_o\in\mathbb{N}\}$ be Reich contraction on $(\mathbb{P}(X),H_d)$. Let a_n,b_n,c_n contractive factors for T_n for each n. Define $\widehat{T}:\mathbb{P}(X)\to\mathbb{P}(X)$ by $\widehat{T}(B) = T_1(B)\bigcup T_2(B)\bigcup T_3(B)\bigcup..........\bigcup T_n(B) = \bigcup_{n=1}^{N_0} T_n(B)$ for each $B\in\mathbb{P}(X)$. Then $\widehat{T}$ is a Reich contraction with R contractivity factors
$a,b,c = max\{a_n,b_n,c_n;\ n=1,2,3,.....N_o; N_o\in\mathbb{N}\}$.
Proof:
We shall prove the theorem by mathematical induction and using the properties of metric h. For $N=1$ the statement is obviously true now for $N=2$ we see that

$$\begin{aligned} H_d\left(\widehat{T}(B),\widehat{T}(C)\right) &= H_d(T_1(B)\cup T_2(B),T_1(C)\cup T_2(C)) \\ &\leq H_d(T_1(B),T_1(C)) \vee H_d(T_2(B),T_2(C)) \\ &\leq [a_1H_d(B,T_1(B))+b_1H_d(C,T_1(C)) \\ &\quad +c_1H_d(B,C)] \\ &\quad \vee [a_2H_d(B,T_2(B))+b_2H_d(C,T_2(C)) \\ &\quad +c_2H_d(B,C)] \\ &\leq [a_1H_d(B,T_1(B))\vee a_2H_d(B,T_2(B))] \\ &\quad +[b_1H_d(C,T_1(C))\vee b_2H_d(C,T_2(C))] \\ &\quad +[c_1H_d(B,C)\vee c_2H_d(B,C)] \\ &\leq (a_1\vee a_2)[H_d(B,T_1(B)) \\ &\quad \vee H_d(B,T_2(B))] \\ &\quad +(b_1\vee b_2)[H_d(C,T_1(C)) \\ &\quad veeH_d(C,T_2(C))] \\ &\quad +(c_1\vee c_2)[H_d(B,C)\vee H_d(B,C)] \\ &= [aH_d(B,T_1(B)\cup T_2(B)) \\ &\quad +bH_d(C,T_1(C)\cup T_2(C)) \\ &\quad +cH_d(B,C)] \\ &= aH_d(B,T(B))+bH_d(C,T(C)) \\ &\quad +cH_d(B,C). \end{aligned}$$

Thus, from all the above results and the definitions of R Iterated Function System (R IFS), we are in the position to present the following theorem for R IFS.

Theorem 7 Let $\{X : T_1, T_2, T_3................T_{N_o}; N_o \in \mathbb{N}\}$, be a R iterated function system with R Contractivity factors a, b, c. Then the transformation $\widehat{T} : \mathbb{P}(X) \to \mathbb{P}(X)$ defined by $\widehat{T}(B) = \bigcup_{n=1}^{N_o} T_n(B)$ for all $B \in \mathbb{P}(X)$ be a Reich contraction on the complete b metric space $(\mathbb{P}(X), H_d)$ with contractivity factors has a unique fixed point, which is also called an attractor, $A_\infty \in \mathbb{P}(X)$ obeys

$$A_\infty = \widehat{T}(A) = \bigcup_{n=1}^{N_o} T_n(A)$$

and is given by $A_\infty = n \stackrel{lim}{\to} \infty \widehat{T}^{on}(B)$ for any $B \in \mathbb{P}(X)$.

Proof: (X, d) is complete b metric space, then by theorem 5 $(\mathbb{P}(X), H_d)$ is a complete Hausdorff b metric space. Also by lemma 3 the HB operator $\widehat{T}$ is Reich contraction mapping. Then Reich fixed point theorem 6, we conclude that $\widehat{T}$ has a unique fixed point.

Definition 9 The fixed point $A_\infty \in \mathbb{P}(X)$ of the HB operator $\widehat{T}$ described in theorem 7 is called the attractor $(Fractal)$ of the R IFS.

IV. Well posedness

Let (X, d) be a b metric space. $T : X \to X$ be an operator.

Definition 10 The fixed point problem for an operator T is well posed iff

(i) $F_T = \{u\}$

(ii) If $x_n \in X,\ n \in N$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $d(x_n, u) \to 0$ as $n \to \infty$.

The notion of well posed fixed point problem for single valued operators was defined by F.S. De Blasi, J.Myjak [14] and S.Reich, A.J.Zaslavski [15].

Example 2 Let (X, d) be a b metric space. $T : X \to X$ be an Reich contraction. Then the fixed point problem is well posed for the operator T. Indeed $F_T = \{u\}$ and let $x_n \in X,\ n \in N$ be such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$

we have

$$\begin{aligned}
d(x_n, u) &\leq td(x_n, Tx_n) + d(Tx_n, Tu) \\
&\leq t[d(x_n, Tx_n) + ad(x_n, Tx_n) \\
&\quad +bd(u, Tu) + cd(x_n, u)] \\
d(x_n, u) - ctd(x_n, u) &\leq td(x_n, Tx_n) + atd(x_n, Tx_n) \\
(1 - ct)d(x_n, u) &\leq (t + at)\, d(x_n, Tx_n) \\
d(x_n, u) &\leq \frac{t + at}{1 - ct} d(x_n, Tx_n) \to 0 \\
as\ n \to \infty &
\end{aligned}$$

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