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An Interval Type-2 Dual Fuzzy Polynomial Equations and Ranking Method of Fuzzy Numbers

Nurhakimah Ab. Rahman, Lazim Abdullah

Abstract**—**According to fuzzy arithmetic, dual fuzzy polynomials cannot be replaced by fuzzy polynomials. Hence, the concept of ranking method is used to find real roots of dual fuzzy polynomial equations. Therefore, in this study we want to propose an interval type-2 dual fuzzy polynomial equation (IT2 DFPE). Then, the concept of ranking method also is used to find real roots of IT2 DFPE (if exists). We transform IT2 DFPE to system of crisp IT2 DFPE. This transformation performed with ranking method of fuzzy numbers based on three parameters namely value, ambiguity and fuzziness. At the end, we illustrate our approach by two numerical examples.

Keywords**—**Dual fuzzy polynomial equations, Interval type-2, Ranking method, Value.

I. Introduction

POLYNOMIALS play a major role in various areas such as mathematics, engineering and social sciences [1]. Previous studies by [2], [3], [6], [14], [15] showed that there are some methods to find solution for fuzzy polynomials. They are Newton-Raphson method, ranking method, modified Adomian decomposition method and fuzzy neural network.

The solution of fuzzy polynomials by ranking method is investigated by [3], by finding real roots of polynomial equation $A_1x + A_2x^2 + ... + A_nx^n = A_0$ for $x \in R$ where $A_0, A_1, ..., A_n$ are fuzzy numbers. The numerical solution of fuzzy polynomial equations by fuzzy neural network is solved in [6]. Also, the numerical solutions of dual fuzzy polynomial equations by a fuzzy neural network are shown in [7] and linear and nonlinear fuzzy equations are shown by [8]-[10].

Hence, we are interested to use the ranking method of fuzzy numbers to achieve the aim of this study. The ranking method was firstly introduced by [11], [12]. They introduced three real indices called value, ambiguity, and fuzziness to obtain simple fuzzy numbers that could be used to represent more arbitrary fuzzy numbers. This ranking method provides the canonical representation of the solution of fuzzy linear systems [5]. In 2011, [4] have found real roots of dual polynomial equation such as

$$A_1'x + A_2'x^2 + ... + A_n'x^n = A_1''x + A_2''x^2 + ... + A_n''x^n + A_0$$

for $x \in R$ where $A_0, A_1', ..., A_n', A_1'', ...A_n''$ are fuzzy numbers, also by using a ranking method of fuzzy numbers.

Therefore, in this study, we propose an interval type-2 dual fuzzy polynomial equation (IT2 DFPE) and using ranking method to find real roots of IT2 DFPE (if exists). An IT2 DFPE is performed because there are no research attempts about type-2 fuzzy polynomials so far. Hence, these value, ambiguity, and fuzziness were used and IT2 DFPE is transformed to system of crisp interval type-2 dual fuzzy polynomial equation. Then by solving this system, real roots of IT2 DFPE is found.

The structure of this paper is organized as follows. Section II gives an overview of the basic notation and definitions used in this paper, including three of basic parameters which are value, ambiguity, and fuzziness. Section II presents an IT2 DFPE, solving using ranking method of fuzzy numbers. Then, two numerical applications including trapezoidal and triangular of IT2 DFPE are considered in Section IV and Conclusion comes in Section V.

II. Preliminaries

This section introduces the basic definitions relating to triangular and trapezoidal fuzzy numbers, interval type-2 fuzzy sets, and three parameters which is value, ambiguity and fuzziness.

A. Fuzzy Numbers

In this subsection, we briefly review definition of fuzzy numbers, trapezoidal fuzzy numbers and triangular fuzzy numbers.

Definition 1: Fuzzy number [3]

A fuzzy subset A of the real line R with membership function $A(x): R \to [0,1]$ is called a fuzzy number if:

i. A is normal, there exist an element x_0 such that $A(x_0) = 1$.

ii. A is fuzzy convex, $A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\} \forall x_1, x_2 \in R, \forall \lambda \in [0,1]$

iii. $A(x)$ is upper semi continuous

iv. Support A is bounded, where $\sup port\ A = cl\{x \in R : A(x) > 0\}$ and cl is the closure operator.

Definition 2: Fuzzy number [3]

Nurhakimah Ab. Rahman is with the Department of Mathematics, Faculty of Sciences and Technology, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Malaysia (corresponding author to provide phone: +6013-9476797; e-mail: hamikahrun@yahoo.com).

Lazim Abdullah is with the Department of Mathematics, Faculty of Sciences and Technology, Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Malaysia (e-mail: lazim_m@umt.edu.my).

A fuzzy set A on R is a fuzzy number, provided:

i. Its membership function is upper semi-continuous.

ii. There exists three intervals $[a,b],[b,c],[c,d]$ such that μ is increasing on $[a,b]$, equal to 1 on $[b,c]$, decreasing on $[c,d]$, and equal to 0 elsewhere.

Definition 3: Trapezoidal fuzzy numbers [14]

A popular fuzzy number is the trapezoidal fuzzy number $\mu = (x_0, y_0, \alpha, \beta)$ with interval defuzzifier $[x_0, y_0]$ and left fuzziness α and right fuzziness β where the membership function is

$$\mu(x) = \begin{cases} \dfrac{x - x_0 + \alpha}{\alpha} & x_0 - \alpha \le x \le x_0 \\ 1 & x \in [x_0, y_0] \\ \dfrac{y_0 - x + \beta}{\beta} & y_0 \le x \le y_0 + \beta \\ 0 & otherwise \end{cases}$$

Its parametric form is:

$\underline{u}(r) = x_0 - \alpha + \alpha r$, $\overline{u}(r) = y_0 + \beta - \beta r$

Let $v = (x_0, y_0, \alpha, \beta)$, be a trapezoidal fuzzy number and $x_0 = y_0$, then v is called a triangular fuzzy number and is denoted by $v = (x_0, \partial, \beta)$

Definition 4: Triangular fuzzy numbers [15]

The triangular fuzzy number $U = (m, \alpha, \beta)$ for $\alpha > 0, \beta > 0$ with membership function

$$\mu_u(x) = \begin{cases} \dfrac{x - m}{\alpha} + 1 & m - \alpha \le x \le m \\ \dfrac{m - x}{\beta} + 1 & m \le x \le m + \beta \\ 0 & otherwise \end{cases}$$

Its parametric form is

$\underline{u}(h) = m + \alpha(h - 1) \qquad \overline{u}(h) = m + \beta(h - 1)$

B. Interval Type-2 Fuzzy Set

Let $\tilde{A}$ be a type-1 trapezoidal fuzzy set, $\tilde{A} = (a_1, a_2, a_3, a_4; H_1(A), H_2(A))$ where $H_1\left(\tilde{A}\right)$ denotes the membership value of the element a_2, $H_2\left(\tilde{A}\right)$ denotes the membership value of the element a_3, $0 \le H_2(A) \le 1$. If $a_2 = a_3$, then the type-1 fuzzy set $\tilde{A}$ becomes a triangular type-1 fuzzy set.

Definition 5**:** Type-2 fuzzy sets [13]

A type-2 fuzzy set, denoted $\tilde{A}$ is characterized by a type-2 membership function $\mu_{\tilde{A}}(x,u)$ where $x \in X$ and $\mu \in J_x \subseteq [0,1]$

$$\tilde{A} = \left\{ \left((x,u), \mu_{\tilde{A}}(x,u) \right) \middle| \forall x \in X, \forall u \in J_x \subseteq [0,1] \right\}$$

In which $0 \le \mu_{\tilde{A}}(x,u) \le 1$. $\tilde{A}$ can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u) \big/ (x,u) \qquad J_x \subseteq [0,1]$$

where $\int\int$ denotes union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

In Definition 5, the first restriction that $\forall u \in J_x \subseteq [0,1]$ is consistent with the type-1 constraint that $0 \le \mu_{\tilde{A}}(x,u) \le 1$. When uncertainties disappear a type-2 membership function must reduce to a type-1 membership function, in which case the variable u equals $\mu_A(x)$ and $0 \le \mu_A(x) \le 1$. The second restriction that $0 \le \mu_{\tilde{A}}(x,u) \le 1$ is consistent with the fact that the amplitudes of a membership function should lie between or be equal to 0 and 1.

Definition 6: Interval type-2 fuzzy sets [13]

When all $\mu_{\tilde{A}}(x,u) = 1$, then $\tilde{A}$ is an interval type-2 fuzzy set (IT2 FS).

C. Ranking Method and Its Parameters

References [11], [12] introduced three parameters, value, ambiguity and fuzziness for a fuzzy number in order to capture the relevant information about the fuzzy number's ill-defined magnitude. These parameters are defined as follows:

A function $s:[0,1] \to [0,1]$ is said to be a reducing function, if s is increasing, $s(0) = 0$ and $s(1) = 1$. The simplest and most natural reducing function is the uniform $s(r) = r$.

1) Value and Ambiguity

The value notion is given to associate a real value with any fuzzy number. This value essentially assesses a fuzzy number by assigning a number to the ill-defined magnitude represented by the fuzzy number. On the other hand, ambiguity measures how much vagueness is present in the ill-defined magnitude of the fuzzy number.

Definition 7: Value of fuzzy number [11]

Let μ be a fuzzy number with r-cut representation $L_\mu(\cdot), R_\mu(\cdot)$ and let $s:[0,1] \to [0,1]$ be a reducing function. Then the value of μ (with respect to s) is

$$V_s(\mu) = \int_0^1 s(r)\left[L_\mu(r) + R_\mu(r)\right] dr$$

Let $s(r) = r$, then

$$V(\mu) = \int_0^1 r[L(r) + R(r)]d \qquad (1)$$

Definition 8: Ambiguity of fuzzy numbers [11], [12]

Let μ be a fuzzy number with r-cut representation $L_\mu(\cdot), R_\mu(\cdot)$ and let s be a reducing function. Then the ambiguity of μ with respect to s is:

$$A_s(\mu) = \int_0^1 r[R_\mu(r) - L_\mu(r)]\, dr$$

which, under our assumption $s(r) = r$ becomes

$$A(\mu) = \int_0^1 r[R(r) - L(r)]\, dr \qquad (2)$$

Again, for simplicity, the reference to the reducing function is removed.

Hence, it follows from the above definitions that

$$V_s(\mu) - A_s(\mu) = 2\int_0^1 s(r)L(r)dr$$

$$V_s(\mu) + A_s(\mu) = 2\int_0^1 s(r)R(r)dr$$

and therefore

$$V_s(\mu) = 2\int_0^1 s(r)L(r)dr + A_s(\mu)\,,\; V_s(\mu) = 2\int_0^1 s(r)R(r)dr - A_s(\mu)$$

$V_s(\mu)$ may be seen as a central value that from a global point of view is the value of the (ill-defined) magnitude that the fuzzy number represents. Observe also that $R(r) - L(r)$ is the length of the r-cut interval $[L(r), R(r)]$ and thus $A_s(\mu)$ may be seen as the global spread of the membership function of μ (weighted through s(r)). This is obviously a measure of the vagueness of μ.

2) A Fuzziness Measure

The fuzziness has been described and assessed in several ways. It mostly involves a comparison between the fuzzy set and its complement [16]. Notice that the 'most fuzzy' subset is the one with a constant membership function equal to half and coincides with its complement. A natural way of expressing fuzziness is in terms of the lack of distinction between the fuzzy set and its complement.

Definition 9: Complement of a fuzzy number [12]

Suppose μ is a fuzzy number with $Supp(\mu) = [p, q]$. Let $L_\mu(\cdot), R_\mu(\cdot)$ be the r-cut representation of μ and let μ_c be the complement of μ with r-cut representation $L_\mu(\cdot), R_\mu(\cdot)$. We define the fuzziness of μ (with respect to a reducing function s), $F_s(\mu)$ by the global difference between μ and μ_c, can be assessed by:

$$F_s(\mu) = \int_0^1 s(r)(q-p)\, dr -$$
$$\left\{\int_{1/2}^1 s(r)[L_c(r) - p]\, dr + \int_{1/2}^1 s(r)[R(r) - L(r)]\, dr + \right.$$
$$\int_{1/2}^1 s(r)[q - R_c(r)]\, dr + \int_0^{1/2} s(r)[L(r) - p]\, dr +$$
$$\left.\int_0^{1/2} s(r)[R_c(r) - L_c(r)]\, dr + \int_0^{1/2} s(r)[q - R(r)]\, dr\right\}$$

A routine calculation shows that for $s(r) = r$ this expression reduces to

$$F(\mu) = \int_0^{1/2} [R(r) - L(r)]dr + \int_{1/2}^1 [L(r) - R(r)]dr \qquad (3)$$

where, as usual the suffix s is removed for $s(r) = r$

Note that for any two trapezoidal fuzzy numbers, A_1 and A_2 and any real number k we have

$$V(kA_1 + A_2) = kV(A_1) + V(A_2) \qquad (4)$$

$$A(kA_1 + A_2) = |k|A(A_1) + A(A_2) \qquad (5)$$

$$F(kA_1 + A_2) = |k|F(A_1) + F(A_2) \qquad (6)$$

Algorithm 1: [3], [4], [17]

As a ranking method, we compare two fuzzy numbers, A_1 and A_2 to the following steps:

1. Compare $V(A_1)$ and $V(A_2)$. If they are equal, then go to the next step. Otherwise ranking A_1 and A_2 is according to the relative position of $V(A_1)$ and $V(A_2)$.
2. Compare $A(A_1)$ and $A(A_2)$. If they are equal, then go to the next step. Otherwise ranking A_1 and A_2 is according to the relative position of $A(A_1)$ and $A(A_2)$.
3. Compare $F(A_1)$ and $F(A_2)$. If they are equal, then go to the next step. Otherwise ranking A_1 and A_2 is according to the relative position of $F(A_1)$ and $F(A_2)$.

Note that, $A_1 = A_2$ if and only if:

$$\begin{aligned} V(A_1) &= V(A_2) \\ A(A_1) &= A(A_2) \\ F(A_1) &= F(A_2) \end{aligned} \qquad (7)$$

III. The Proposed Ranking Method for Interval Type-2 Dual Fuzzy Polynomial Equations

In this section, we propose an IT2 DFPE and perform solution for the proposed equations using the ranking method of fuzzy numbers. In 2011, [4] had described the dual fuzzy polynomial equations clearly. However, in this study we present interval type-2 dual fuzzy numbers in dual fuzzy polynomial equations. Hence, an IT2 DFPE is defined as below.

Definition 10: Interval type-2 dual fuzzy polynomial equations [4], [7]

$$\tilde{\tilde{A}}'_1 x + \tilde{\tilde{A}}'_2 x^2 + \ldots + \tilde{\tilde{A}}'_n x^n = \tilde{\tilde{A}}''_1 x + \tilde{\tilde{A}}''_2 x^2 + \ldots + \tilde{\tilde{A}}''_n x^n + \tilde{\tilde{A}}_0 \quad (8)$$

where $x \in R$, the coefficient $\tilde{\tilde{A}}'_1, \ldots, \tilde{\tilde{A}}'_n, \tilde{\tilde{A}}''_1, \ldots, \tilde{\tilde{A}}''_n$ and $\tilde{\tilde{A}}_0$ are interval type-2 fuzzy numbers.

Let

$$\tilde{\tilde{A}}'_i = \left(\tilde{A}'^U_i, \tilde{A}'^L_i\right) \text{ and } \tilde{\tilde{A}}''_i = \left(\tilde{A}''^U_i, \tilde{A}''^L_i\right)$$

where

$$\tilde{\tilde{A}}_i = \left\{ \begin{array}{l} \left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right) \\ \left(\left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^L\right), H_2\left(\tilde{A}_i^L\right)\right)\right) \end{array} \right\}$$

Therefore, (8) can also be written as follows:

$$\left\{ \begin{array}{l} \left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) x + \left(\tilde{A}'^U_2, \tilde{A}'^L_2\right) x^2 + \ldots. + \left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) x^n = \\ \left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) x + \left(\tilde{A}''^U_2, \tilde{A}''^L_2\right) x^2 + \ldots. + \left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) x^n \\ + \left(\tilde{A}_0^U, \tilde{A}_0^L\right) \end{array} \right\}$$

Let $\left(\overline{x}, \underline{x}\right)$ be a real solution for (8). Then we have:

$$\begin{cases} \left(\overline{V}, \underline{V}\right)\left(\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) x + \left(\tilde{A}'^U_2, \tilde{A}'^L_2\right) x^2 + \ldots. + \left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) x^n\right) = \\ \left(\overline{V}, \underline{V}\right)\left(\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) x + \left(\tilde{A}''^U_2, \tilde{A}''^L_2\right) x^2 + \ldots. + \left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) x^n + \left(\tilde{A}_0^U, \tilde{A}_0^L\right)\right) \\ \left(\overline{A}, \underline{A}\right)\left(\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) x + \left(\tilde{A}'^U_2, \tilde{A}'^L_2\right) x^2 + \ldots. + \left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) x^n\right) = \\ \left(\overline{A}, \underline{A}\right)\left(\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) x + \left(\tilde{A}''^U_2, \tilde{A}''^L_2\right) x^2 + \ldots. + \left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) x^n + \left(\tilde{A}_0^U, \tilde{A}_0^L\right)\right) \\ \left(\overline{F}, \underline{F}\right)\left(\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) x + \left(\tilde{A}'^U_2, \tilde{A}'^L_2\right) x^2 + \ldots. + \left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) x^n\right) = \\ \left(\overline{F}, \underline{F}\right)\left(\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) x + \left(\tilde{A}''^U_2, \tilde{A}''^L_2\right) x^2 + \ldots. + \left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) x^n + \left(\tilde{A}_0^U, \tilde{A}_0^L\right)\right) \end{cases}$$

which implies for

$$\begin{cases} \left(\overline{V}, \underline{V}\right)\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) x + \ldots + \left(\overline{V}, \underline{V}\right)\left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) x^n = \\ \left(\overline{V}, \underline{V}\right)\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) x + \ldots + \left(\overline{V}, \underline{V}\right)\left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) x^n + \left(\overline{V}, \underline{V}\right)\left(\tilde{A}_0^U, \tilde{A}_0^L\right) \\ \left(\overline{A}, \underline{A}\right)\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) |x| + \ldots + \left(\overline{A}, \underline{A}\right)\left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) \left|x^n\right| = \\ \left(\overline{A}, \underline{A}\right)\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) |x| + \ldots + \left(\overline{A}, \underline{A}\right)\left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) \left|x^n\right| + \left(\overline{A}, \underline{A}\right)\left(\tilde{A}_0^U, \tilde{A}_0^L\right) \\ \left(\overline{F}, \underline{F}\right)\left(\tilde{A}'^U_1, \tilde{A}'^L_1\right) |x| + \ldots + \left(\overline{F}, \underline{F}\right)\left(\tilde{A}'^U_n, \tilde{A}'^L_n\right) \left|x^n\right| = \\ \left(\overline{F}, \underline{F}\right)\left(\tilde{A}''^U_1, \tilde{A}''^L_1\right) |x| + \ldots + \left(\overline{F}, \underline{F}\right)\left(\tilde{A}''^U_n, \tilde{A}''^L_n\right) \left|x^n\right| + \left(\overline{F}, \underline{F}\right)\left(\tilde{A}_0^U, \tilde{A}_0^L\right) \end{cases} \quad (9)$$

Hence, for finding the real solution for (8), it is enough to solve the above system of crisp interval type-2 dual polynomial equations with crisp methods. Of course, though we have a system of crisp interval type-2 dual polynomial equations, it is only enough to solve one equation for (8) and finding its real roots. Then the real roots of that equation that satisfies both other equations are real solutions of system (9). Thus, for solving (9) we can consider two states:

The first, suppose that $x \geq 0$, then we obtain positive real roots. The second, suppose that $x < 0$ then we obtain negative real roots.

As fuzzy polynomials have trapezoidal and triangular fuzzy numbers, hence according to [4], we are giving the following formula to obtain value, ambiguity and fuzziness for interval type-2 dual trapezoidal fuzzy numbers.

Let

$$\tilde{\tilde{A}}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right) \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^L\right), H_2\left(\tilde{A}_i^L\right)\right)$$

be an interval type-2 trapezoidal fuzzy number. Hence, the value, ambiguity and fuzziness for IT2 DFPE can be obtain as follows.

$$\underline{V}\left(\tilde{A}_i'^{L}\right) = \frac{a_{i1}'^{L} + a_{i2}'^{L}}{2} + \frac{a_{i4}'^{L} - \alpha_{i3}'^{L}}{6} \qquad \overline{V}\left(\tilde{A}_i'^{U}\right) = \frac{a_{i1}'^{U} + a_{i2}'^{U}}{2} + \frac{a_{i4}'^{U} - \alpha_{i3}'^{U}}{6}$$

$$\underline{A}\left(\tilde{A}_i'^{L}\right) = \frac{a_{i2}'^{L} - a_{i1}'^{L}}{2} + \frac{a_{i4}'^{L} + a_{i3}'^{L}}{6}, \qquad \overline{A}\left(\tilde{A}_i'^{U}\right) = \frac{a_{i2}'^{U} - a_{i1}'^{U}}{2} + \frac{a_{i4}'^{U} + a_{i3}'^{U}}{6}$$

$$\underline{F}\left(\tilde{A}_i'^{L}\right) = \frac{a_{i4}'^{L} + a_{i3}'^{L}}{4} \qquad \overline{F}\left(\tilde{A}_i'^{U}\right) = \frac{a_{i4}'^{U} + a_{i3}'^{U}}{4} \tag{10}$$

$$\overline{V}\left(\tilde{A}_i''^{U}\right) = \frac{a_{i1}''^{U} + a_{i2}''^{U}}{2} + \frac{a_{i4}''^{U} - \alpha_{i3}''^{U}}{6} \qquad \underline{V}\left(\tilde{A}_i''^{L}\right) = \frac{a_{i1}''^{L} + a_{i2}''^{L}}{2} + \frac{a_{i4}''^{L} - \alpha_{i3}''^{L}}{6}$$

$$\overline{A}\left(\tilde{A}_i''^{U}\right) = \frac{a_{i2}''^{U} - a_{i1}''^{U}}{2} + \frac{a_{i4}''^{U} + a_{i3}''^{U}}{6}, \qquad \underline{A}\left(\tilde{A}_i''^{L}\right) = \frac{a_{i2}''^{L} - a_{i1}''^{L}}{2} + \frac{a_{i4}''^{L} + a_{i3}''^{L}}{6}$$

$$\overline{F}\left(\tilde{A}_i''^{U}\right) = \frac{a_{i4}''^{U} + a_{i3}''^{U}}{4} \qquad \underline{F}\left(\tilde{A}_i''^{L}\right) = \frac{a_{i4}''^{L} + a_{i3}''^{L}}{4} \tag{11}$$

$$\overline{V}\left(\tilde{A}_0^{U}\right) = \frac{a_{01}^{U} + a_{02}^{U}}{2} + \frac{a_{04}^{U} - a_{03}^{U}}{6} \qquad \underline{V}\left(\tilde{A}_0^{L}\right) = \frac{a_{01}^{L} + a_{02}^{L}}{2} + \frac{a_{04}^{L} - a_{03}^{L}}{6}$$

$$\overline{A}\left(\tilde{A}_0^{U}\right) = \frac{a_{02}^{U} - a_{01}^{U}}{2} + \frac{a_{04}^{U} + a_{03}^{U}}{6}, \qquad \underline{A}\left(\tilde{A}_0^{L}\right) = \frac{a_{02}^{L} - a_{01}^{L}}{2} + \frac{a_{04}^{L} + a_{03}^{L}}{6}$$

$$\overline{F}\left(\tilde{A}_0^{U}\right) = \frac{a_{04}^{U} + a_{03}^{U}}{4} \qquad \underline{F}\left(\tilde{A}_0^{L}\right) = \frac{a_{04}^{L} + a_{03}^{L}}{4} \tag{12}$$

Now, let considering triangular fuzzy number of IT2 DFPE. Let

$$\tilde{\tilde{A}}_i = \left(\tilde{A}_i^{U}, \tilde{A}_i^{L}\right) = \left\{\left(a_{i1}^{U}, a_{i2}^{U}, a_{i3}^{U}; H_1\left(\tilde{A}_i^{U}\right)\right)\left(a_{i1}^{L}, a_{i2}^{L}, a_{i3}^{L}; H_1\left(\tilde{A}_i^{L}\right)\right)\right\}$$

Hence, the value, ambiguity and fuzziness for triangular fuzzy number of IT2 DFPE can be obtained as follows.

$$\overline{V}\left(\tilde{A}_i'^{U}\right) = a_{i1}'^{U} + \frac{a_{i3}'^{U} - a_{i2}'^{U}}{6} \qquad \underline{V}\left(\tilde{A}_i'^{L}\right) = a_{i1}'^{L} + \frac{a_{i3}'^{L} - a_{i2}'^{L}}{6}$$

$$\overline{A}\left(\tilde{A}_i'^{U}\right) = \frac{a_{i3}'^{U} + a_{i2}'^{U}}{6}, \qquad \underline{A}\left(\tilde{A}_i'^{L}\right) = \frac{a_{i3}'^{L} + a_{i2}'^{L}}{6}$$

$$\overline{F}\left(\tilde{A}_i'^{U}\right) = \frac{a_{i3}'^{U} + a_{i2}'^{U}}{4} \qquad \underline{F}\left(\tilde{A}_i'^{L}\right) = \frac{a_{i3}'^{L} + a_{i2}'^{L}}{4} \tag{13}$$

and

$$\overline{V}\left(\tilde{A}_i''^{U}\right) = a_{i1}''^{U} + \frac{a_{i3}''^{U} - a_{i2}''^{U}}{6} \qquad \underline{V}\left(\tilde{A}_i''^{L}\right) = a_{i1}''^{L} + \frac{a_{i3}''^{L} - a_{i2}''^{L}}{6}$$

$$\overline{A}\left(\tilde{A}_i''^{U}\right) = \frac{a_{i3}''^{U} + a_{i2}''^{U}}{6}, \qquad \underline{A}\left(\tilde{A}_i''^{L}\right) = \frac{a_{i3}''^{L} + a_{i2}''^{L}}{6}$$

$$\overline{F}\left(\tilde{A}_i''^{U}\right) = \frac{a_{i3}''^{U} + a_{i2}''^{U}}{4} \qquad \underline{F}\left(\tilde{A}_i''^{L}\right) = \frac{a_{i3}''^{L} + a_{i2}''^{L}}{4} \tag{14}$$

and

$$\overline{V}\left(\tilde{A}_0^{U}\right) = a_{01}^{U} + \frac{a_{03}^{U} - a_{02}^{U}}{6} \qquad \underline{V}\left(\tilde{A}_0^{L}\right) = a_{01}^{L} + \frac{a_{03}^{L} - a_{02}^{L}}{6}$$

$$\overline{A}\left(\tilde{A}_0^{U}\right) = \frac{a_{03}^{U} + a_{02}^{U}}{6}, \qquad \underline{A}\left(\tilde{A}_0^{L}\right) = \frac{a_{03}^{L} + a_{02}^{L}}{6}$$

$$\overline{F}\left(\tilde{A}_0^{U}\right) = \frac{a_{03}^{U} + a_{02}^{U}}{4} \qquad \underline{F}\left(\tilde{A}_0^{L}\right) = \frac{a_{03}^{L} + a_{02}^{L}}{4} \tag{15}$$

A. Illustrative Example

In order to illustrate our approach, the following interval type-2 fuzzy polynomial equation is considered.

Example 1

This example is taken from [4] and we present the original equation into the form of interval type-2 fuzzy numbers where we have upper and lower equations.

$$\begin{aligned}&\{(2,4,6;1)(1,3,5;1)\}x + \{(0,2,4;1)(0,1,3;1)\}x^2 + \{(3,6,8;1)(4,5,7;1)\}x^3 \\ &+ \{(-3,0,2;1)(-1,0,1;1)\}x^4 = \{(1,2,3;1)(0,1,2;1)\}x + \{(0,1,2;1)(0,1,1;1)\}x^2 \\ &+ \{(1,2,3;1)(0,1,2;1)\}x^3 + \{(-2,0,1;1)(-1,0,1;1)\}x^4 + \{(2,7,11;1)(4,5,6;1)\}\end{aligned}$$

According to (13)-(15), the following value, ambiguity and fuzziness are obtained. For upper side, we have:

$\overline{V}\left(\tilde{A}_1'\right)=2.333$ $\quad \overline{V}\left(\tilde{A}_2'\right)=0.333$ $\quad \overline{V}\left(\tilde{A}_3'\right)=3.333$

$\overline{A}\left(\tilde{A}_1'\right)=1.667$, $\quad \overline{A}\left(\tilde{A}_2'\right)=1.000$, $\quad \overline{A}\left(\tilde{A}_3'\right)=2.333$,

$\overline{F}\left(\tilde{A}_1'\right)=2.500$ $\quad \overline{F}\left(\tilde{A}_2'\right)=1.500$ $\quad \overline{F}\left(\tilde{A}_3'\right)=3.500$

$\overline{V}\left(\tilde{A}_4'\right)=-2.667$ $\quad \overline{V}\left(\tilde{A}_1''\right)=1.167$ $\quad \overline{V}\left(\tilde{A}_2''\right)=0.167$

$\overline{A}\left(\tilde{A}_4'\right)=0.333$ $\quad \overline{A}\left(\tilde{A}_1''\right)=0.833$ $\quad \overline{A}\left(\tilde{A}_2''\right)=0.500$

$\overline{F}\left(\tilde{A}_4'\right)=0.500$ $\quad \overline{F}\left(\tilde{A}_1''\right)=1.250$ $\quad \overline{F}\left(\tilde{A}_2''\right)=0.750$

$\overline{V}\left(\tilde{A}_3''\right)=1.167$ $\quad \overline{V}\left(\tilde{A}_4''\right)=-1.833$

$\overline{A}\left(\tilde{A}_3''\right)=0.833$ $\quad \overline{A}\left(\tilde{A}_4''\right)=0.167$

$\overline{F}\left(\tilde{A}_3''\right)=1.250$ $\quad \overline{F}\left(\tilde{A}_4''\right)=0.250$

$\overline{V}\left(\tilde{A}_0\right)=2.667$, $\overline{A}\left(\tilde{A}_0\right)=3.000$, $\overline{F}\left(\tilde{A}_0\right)=4.500$

For lower side, we have:

$\underline{V}\left(\tilde{A}_1'\right)=1.333$ $\quad \underline{V}\left(\tilde{A}_2'\right)=0.333$ $\quad \underline{V}\left(\tilde{A}_3'\right)=4.333$

$\underline{A}\left(\tilde{A}_1'\right)=1.333$ $\quad \underline{A}\left(\tilde{A}_2'\right)=0.667$ $\quad \underline{A}\left(\tilde{A}_3'\right)=2.000$

$\underline{F}\left(\tilde{A}_1'\right)=2.000$ $\quad \underline{F}\left(\tilde{A}_2'\right)=1.000$ $\quad \underline{F}\left(\tilde{A}_3'\right)=3.000$

$\underline{V}\left(\tilde{A}_4'\right)=-0.833$ $\quad \underline{V}\left(\tilde{A}_1''\right)=0.167$ $\quad \underline{V}\left(\tilde{A}_2''\right)=0$

$\underline{A}\left(\tilde{A}_4'\right)=0.167$ $\quad \underline{A}\left(\tilde{A}_1''\right)=0.500$ $\quad \underline{A}\left(\tilde{A}_2''\right)=0.333$

$\underline{F}\left(\tilde{A}_4'\right)=0.250$ $\quad \underline{F}\left(\tilde{A}_1''\right)=0.750$ $\quad \underline{F}\left(\tilde{A}_2''\right)=0.500$

$\underline{V}\left(\tilde{A}_3''\right)=0.167$ $\quad \underline{V}\left(\tilde{A}_4''\right)=-0.833$

$\underline{A}\left(\tilde{A}_3''\right)=0.500$ $\quad \underline{A}\left(\tilde{A}_4''\right)=0.167$

$\underline{F}\left(\tilde{A}_3''\right)=0.750$ $\quad \underline{F}\left(\tilde{A}_4''\right)=0.250$

$\underline{V}\left(\tilde{A}_0\right)=4.167$, $\underline{A}\left(\tilde{A}_0\right)=1.833$, $\underline{F}\left(\tilde{A}_0\right)=2.750$

For the upper membership systems, we have the system of crisp IT2 DFPE, as follows.

$$\begin{cases} 2.333x+0.333x^2+3.333x^3-2.667x^4=1.167x+0.167x^2 \\ +1.167x^3-1.833x^4+2.667 \\ 1.667|x|+|x^2|+2.333|x^3|+0.333|x^4|=0.833|x|+0.5|x^2|+ \\ 0.833|x^3|+0.167|x^4|+3.000 \\ 2.500|x|+1.500|x^2|+3.500|x^3|+0.5|x^4|=1.250|x|+0.750|x^2|+ \\ 1.250|x^3|+0.250|x^4|+4.500 \end{cases}$$

and for the lower membership systems, the system of crisp IT2 DFPE, obtained is as follows.

$$\begin{cases} 1.333x+0.333x^2+4.333x^3-0.833x^4=0.167x+ \\ 0.167x^3-0.833x^4+4.167 \\ 1.333|x|+0.667|x^2|+2.000|x^3|+0.167|x^4|=0.500|x|+ \\ 0.333|x^2|+0.500|x^3|+0.167|x^4|+1.833 \\ 2.000|x|+|x^2|+3.000|x^3|+0.250|x^4|=0.750|x|+0.5|x^2|+ \\ 0.750|x^3|+0.250|x^4|+2.750 \end{cases}$$

After solving the upper and lower systems, the exact solution obtained is

$$\overline{x}=[-1,1]$$
$$\underline{x}=[-1,1]$$

This result shows that the exact solution obtained for triangular IT2 DFPE is similar to the exact solution for the triangular dual fuzzy polynomial equations, which were produced by [4].

Example 2

This example is taken from [7] and we present the original equation into the form of interval type-2 fuzzy numbers where we have upper and lower equations.

$$\{(-2,0,3;1)(-1,1,2;1)\}x+\{(-2,0,3;1)(-1,1,2;1)\}x^2=$$
$$\{(-1,0,2;1)(0,1,2;1)\}x+\{(-1,0,2;1)(0,1,1;1)\}x^2+$$
$$\{(-2,0,2;1)(0,0,1;1)\}$$

According to (13)-(15), the following value, ambiguity and fuzziness are obtained.

For upper side, we have:

$\overline{V}\left(\tilde{A}_1'\right) = -1.50$ $\overline{V}\left(\tilde{A}_2'\right) = -1.50$ $\overline{V}\left(\tilde{A}_1''\right) = -0.667$

$\overline{A}\left(\tilde{A}_1'\right) = 0.50$ $\overline{A}\left(\tilde{A}_2'\right) = 0.50$ $\overline{A}\left(\tilde{A}_1''\right) = 0.333$

$\overline{F}\left(\tilde{A}_1'\right) = 0.75$ $\overline{F}\left(\tilde{A}_2'\right) = 0.75$ $\overline{F}\left(\tilde{A}_1''\right) = 0.50$

$\overline{V}\left(\tilde{A}_2''\right) = -0.667$ $\overline{V}\left(\tilde{A}_0\right) = -1.667$

$\overline{A}\left(\tilde{A}_2''\right) = 0.333$ $\overline{A}\left(\tilde{A}_0\right) = 0.333$

$\overline{F}\left(\tilde{A}_2''\right) = 0.50$ $\overline{F}\left(\tilde{A}_0\right) = 0.50$

For lower side, we have:

$\underline{V}\left(\tilde{A}_1'\right) = -0.833$ $\underline{V}\left(\tilde{A}_2'\right) = -0.833$ $\underline{V}\left(\tilde{A}_1''\right) = 0.167$

$\underline{A}\left(\tilde{A}_1'\right) = 0.50$ $\underline{A}\left(\tilde{A}_2'\right) = 0.5$ $\underline{A}\left(\tilde{A}_1''\right) = 0.500$

$\underline{F}\left(\tilde{A}_1'\right) = 0.75$ $\underline{F}\left(\tilde{A}_2'\right) = 0.75$ $\underline{F}\left(\tilde{A}_1''\right) = 0.750$

$\underline{V}\left(\tilde{A}_2''\right) = 0$ $\underline{V}\left(\tilde{A}_0\right) = 0.167$,

$\underline{A}\left(\tilde{A}_2''\right) = 0.333$ $\underline{A}\left(\tilde{A}_0\right) = 0.167$

$\underline{F}\left(\tilde{A}_2''\right) = 0.500$ $\underline{F}\left(\tilde{A}_0\right) = 0.25$

For the upper membership systems, we have the system of crisp IT2 DFPE, as follows:

$$\begin{cases} -1.5x - 1.5x^2 = -0.667 - 0.667x^2 - 1.667 \\ 0.5|x| + 0.5\left|x^2\right| = 0.333|x| + 0.333\left|x^2\right| + 0.333 \\ 0.75|x| + 0.75\left|x^2\right| = 0.5|x| + 0.5\left|x^2\right| + 0.5 \end{cases}$$

and for the lower membership systems, the system of crisp IT2 DFPE, obtained is as follows:

$$\begin{cases} -0.833x - 0.833x^2 = 0.167x + 0.167 \\ 0.5|x| + 0.5\left|x^2\right| = 0.5|x| + 0.333\left|x^2\right| + 0.167 \\ 0.75|x| + 0.75\left|x^2\right| = 0.75|x| + 0.5\left|x^2\right| + 0.25 \end{cases}$$

After solving the upper and lower systems, the exact solution obtained is

$$\overline{x} = [-1, 1]$$
$$\underline{x} = [-1, 1]$$

IV. The Proposed IT2 DFPE and Its Solutions

In this section, we illustrate our approach of IT2 DFPE by giving two numerical examples. We propose two numerical examples to illustrate the solutions for each type of fuzzy polynomials. Trapezoidal fuzzy numbers and triangular fuzzy numbers are used to define each type of fuzzy polynomials.

Example 1

In this example, we consider the trapezoidal fuzzy numbers of IT2 DFPE:

$$\{(2,4,6,8;1,1)(1,3,5,7;1,1)\}x = \{(1,2,3,4;1,1)(0,2,1,4;1,1)\}x + \{(1,2,3,4;1,1)(1,1,4,3;1,1)\}$$

According to (10)-(12), the following value, ambiguity and fuzziness are obtained. For upper side, we have:

$\overline{V}\left(\tilde{A}_1'\right) = 3.333$ $\overline{V}\left(\tilde{A}_2''\right) = 1.667$ $\overline{V}\left(\tilde{A}_0\right) = 1.667$

$\overline{A}\left(\tilde{A}_1'\right) = 3.333$, $\overline{A}\left(\tilde{A}_2''\right) = 1.667$, $\overline{A}\left(\tilde{A}'_0\right) = 1.667$

$\overline{F}\left(\tilde{A}_1'\right) = 3.500$ $\overline{F}\left(\tilde{A}_2''\right) = 1.75$ $\overline{F}\left(\tilde{A}'_0\right) = 1.75$

For lower side, we have:

$\underline{V}\left(\tilde{A}_1'\right) = 2.333$ $\underline{V}\left(\tilde{A}_1''\right) = 1.500$ $\underline{V}\left(\tilde{A}_0\right) = 0.833$

$\underline{A}\left(\tilde{A}_1'\right) = 3.000$, $\underline{A}\left(\tilde{A}_1''\right) = 1.833$, $\underline{A}\left(\tilde{A}_0\right) = 1.167$

$\underline{F}\left(\tilde{A}_1'\right) = 3.000$ $\underline{F}\left(\tilde{A}_1''\right) = 1.250$ $\underline{F}\left(\tilde{A}_0\right) = 1.750$

For the upper membership systems, we have the system of crisp IT2 DFPE, as follows:

$$\begin{cases} 3.333x = 1.667x + 1.667 \\ 3.333|x| = 1.667|x| + 1.667 \\ 3.5|x| = 1.75|x| + 1.75 \end{cases}$$

For the lower membership systems, we have the system of crisp IT2 DFPE, as follows:

$$\begin{cases} 2.333x = 1.5x + 0.833 \\ 3.0|x| = 1.833|x| + 1.167 \\ 3.0|x| = 1.25|x| + 1.75 \end{cases}$$

After solving the upper and lower systems, the exact solution obtained is:

$$\overline{x} = [-1,1]$$
$$\underline{x} = [-1,1]$$

Example 2

In this example, we consider the triangular fuzzy numbers of IT2 DFPE:

$$\{(3,5,7;1)(1,4,6;1)\}x = \{(1,3,4;1)(-1,2,1;1)\}x + \{(2,2,3;1)(2,2,5;1)\}$$

According to (13)- (15), the following value, ambiguity and fuzziness are obtained. For upper side, we have:

$$\overline{V}\left(\tilde{A}'_1\right) = 3.333 \quad \overline{V}\left(\tilde{A}''_2\right) = 1.167 \quad \overline{V}\left(\tilde{A}_0\right) = 2.167$$
$$\overline{A}\left(\tilde{A}'_1\right) = 2.000\,, \quad \overline{A}\left(\tilde{A}''_2\right) = 1.167\,, \quad \overline{A}\left(\tilde{A}'_0\right) = 0.833$$
$$\overline{F}\left(\tilde{A}'_1\right) = 3.000 \quad \overline{F}\left(\tilde{A}''_2\right) = 1.75 \quad \overline{F}\left(\tilde{A}'_0\right) = 1.25$$

For lower side, we have:

$$\underline{V}\left(\tilde{A}'_1\right) = 1.333 \quad \underline{V}\left(\tilde{A}''_1\right) = -1.167 \quad \underline{V}\left(\tilde{A}_0\right) = 2.500$$
$$\underline{A}\left(\tilde{A}'_1\right) = 1.667\,, \quad \underline{A}\left(\tilde{A}''_1\right) = 0.500\,, \quad \underline{A}\left(\tilde{A}_0\right) = 1.167$$
$$\underline{F}\left(\tilde{A}'_1\right) = 2.500 \quad \underline{F}\left(\tilde{A}''_1\right) = 0.750 \quad \underline{F}\left(\tilde{A}_0\right) = 1.750$$

For the upper membership systems, we have the system of crisp IT2 DFPE, as follows:

$$\begin{cases} 3.333x = 1.167x + 2.167 \\ 2.000|x| = 1.167|x| + 0.833 \\ 3.0|x| = 1.75|x| + 1.25 \end{cases}$$

For the lower membership systems, we have the system of crisp IT2 DFPE, as follows:

$$\begin{cases} 1.333x = -1.167x + 2.500 \\ 1.667|x| = 0.5|x| + 1.167 \\ 2.5|x| = 0.75|x| + 1.75 \end{cases}$$

After solving the upper and lower systems, the exact solution obtained is:

$$\overline{x} = [-1,1]$$
$$\underline{x} = [-1,1]$$

V. Conclusion

As mentioned in the beginning of this study, to the best of our knowledge there has been no research carried out to date on the IT2 DFPE. Hence, we have introduced an IT2 DFPE and it has been solved using the ranking method of fuzzy numbers that is transformed to a system of crisp IT2 DFPE. Based on two numerical examples given, it shows that the solutions for the IT2 DFPE exist and an exact solution has been produced.

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