

# An Efficient Collocation Method for Solving the Variable-Order Time-Fractional Partial Differential Equations Arising from the Physical Phenomenon

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**Abstract**—In this work, we present an efficient approach for solving variable-order time-fractional partial differential equations, which are based on Legendre and Laguerre polynomials. First, we introduced the pseudo-operational matrices of integer and variable fractional order of integration by use of some properties of Riemann-Liouville fractional integral. Then, applied together with collocation method and Legendre-Laguerre functions for solving variable-order time-fractional partial differential equations. Also, an estimation of the error is presented. At last, we investigate numerical examples which arise in physics to demonstrate the accuracy of the present method. In comparison results obtained by the present method with the exact solution and the other methods reveals that the method is very effective.

**Keywords**—Collocation method, fractional partial differential equations, Legendre-Laguerre functions, pseudo-operational matrix of integration.

## I. INTRODUCTION

In recent decades, fractional calculus has emerged that many phenomena in various branches of science such as bioengineering, biology, economics, medicine, earthquake, colored noise, signal processing, electromagnetism, electrochemistry, dynamic of viscoelastic materials, continuum and statistical mechanics, solid mechanics, fluid-dynamic traffic model and seepage flow in porous media [1]-[15]. Many researchers have been attention to solve fractional differential equations, fractional integro-differential equations and fractional partial differential equations such as, Kumar and Agrawal [16], have been used an approximate method for numerical solution of fractional differential equations, Liu et al. [17], have been solved space fractional Fokker-Planck equation, Keshavarz et al. [18], presented Bernoulli wavelet operational matrix to solve the fractional order differential equations, Kazem et al. [19], introduced fractional-order Legendre functions for solving fractional-order differential equations, Chen et al. [20] applied generalized fractional-order Legendre functions to solving fractional partial differential equations with variable coefficients, Wang et al. [21] have been solved fractional partial differential equations numerically by Haar wavelet method, Authors in [22] introduced new approximations for solving the Caputo-type fractional partial differential equations, Zhou and Xu [23] have been used the third kind Chebyshev wavelets collocation method for

solving the time-fractional convection diffusion equations with variable coefficients, readers who are interested in learning more about this topic can refer to [24]-[28].

Recently, variable-order fractional derivative and integration field have received considerable attention, which is created from constant-order fractional derivative and integration. Variable-order fractional derivative introduced in several physical branches [29]-[32]. In this context, the proposed equations are variable-order fractional differential equations, variable-order fractional partial differential equations and variable-order fractional functional boundary value problems, which are dealing with by different methods such as spline finite difference [33], cubic spline approximation [34], Legendre wavelets [35] and other numerical methods introduced in [36]-[41].

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### A. Applications

Fractional order partial differential equations appear in many physical phenomena such as:

- fractional order mobile-immobile advection-dispersion model, which is appeared to simulate solute transport in watershed catchments and rivers. Schumer et al. [42] considered the following fractional-order mobile-immobile model for the total concentration:

$$\frac{\partial C}{\partial t} + \beta \frac{\partial^\gamma C}{\partial t^\gamma} = -V \frac{\partial C}{\partial x} + \frac{\partial^2 C}{\partial x^2}, \quad 0 < \gamma < 1,$$

where  $C$  denotes the solute concentration in the total (mobile + immobile) phase, and  $\beta > 0$  is the fractional capacity coefficient. Here  $V > 0$  and  $D > 0$  are the velocity and dispersion coefficient for the mobile phase. The time drift term  $\frac{\partial C}{\partial t}$  describes the motion time and thus helps to distinguish the status of particles conveniently. When  $\gamma \rightarrow 1$ , the fractional-order advection-dispersion equation reduces to the advection-dispersion equation with a retardation factor  $\beta + 1$ . For more information can refer to [43], which explained the properties of four fractional-order advection-dispersion equation (FADE) models. Recently, numerical and analytical solution of variable order fractional mobile-immobile advection-dispersion model (vofMIAD) considered in various article for instance, Jiang et al. in [40] presented a new numerical method to obtain the approximation solution for the time vofMIAD

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model based on reproducing kernel theory and collocation method, Zhang et al. in [41], [44] applied numerical analysis for solving vofMIAD equation, Ma et al. [45] have been used Jacobi spectral collocation method for the time vofMIAD and for more information can see in [46], [47].

- The first and still most significant soliton (solitary waves) systems arose prior to the 1970s in the context of outstanding problems in applied science. Foremost among these are the Korteweg-de Vries (KdV) equation, the sine-Gordon (SG) equation and the nonlinear Schrodinger (NLS) equation. SG equation appeared in augmenting the linear wave equation with following elementary form [48]

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin(u),$$

where normalizing units have been used to measure  $x, t$ , and  $u$ . This equation has many physical applications including the propagation of crystal defects, domain walls in ferromagnetic and ferroelectric materials, a one-dimensional model for elementary particles, the propagation of splay waves on a biological (lipid) membrane, self-induced transparency of short optical pulses and the propagation of quantum units of magnetic flux (called fluxons) on long Josephson (super-conducting) transmission lines [48], [49]. In the continuum limit the problem is reduced to dynamical equations with fractional derivatives resulting from the fractional power of the long-range interaction. Fractional SG and wave-Hilbert nonlinear equations have been found for classical lattice dynamics. In the other words, the dynamics on the 1D lattice can be equivalent to the corresponding fractional nonlinear equation in the long-wave limit [50]. We consider the time fractional SG with the following form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^\gamma u}{\partial t^\gamma} = \sin(u), \quad 1 < \gamma < 2.$$

Numerical and analytical methods have been existed for solving this equation such as diagonally implicit Runge-Kutta-Nystrom [51], reproducing kernel Hilbert space method [52], variational homotopy perturbation method [53] and implicit RBF meshless approach [54].

### B. The Main Goal of This Paper

This paper is to develop a collocation method, Legendre pseudo-operational matrix and Laguerre pseudo-operational matrix of the variable-order fractional integration for solving variable-order time fractional partial differential equations. Also, by using proposed method, we investigate the approximate solution of the variable fractional order of mobile-immobile advection-dispersion model and SG equation. By using the pseudo-operational matrix and collocation points, we have a system of nonlinear algebraic equations with unknown Legendre-Laguerre coefficients.

The plan of this paper is listed as follows. In Section II, we present some necessary definitions of the variable-order fractional calculus. In Section III, introduce some properties of

Legendre-Laguerre functions. In Section IV, we derive integral pseudo-operational matrixes of the integer and variable fractional order for Legendre-Laguerre functions. In Section V, consider a brief description of the collocation method. In Section VI, the error analysis is given. In Section VII, we apply the proposed method to some problems and report our numerical finding and conclusions are drawn in Section VIII. The advantages of the proposed approach are:

- 1) According to physics models the time of the occurrence of an event doesn't have fix domain. So for approximate the time functions in the problem, we applying the Laguerre polynomials, which defined in  $[0, \infty)$ .
- 2) We introduce a new technique to obtain the operational matrixes. In the calculation of these operational matrixes, less approximation is used.
- 3) By using a few terms of Legendre-Laguerre functions approximate solution converges to the exact solution.

## II. PRELIMINARIES

We give some basic definitions and properties of the variable-order fractional calculus theory.

*Definition 1:* The Riemann-Liouville variable-order fractional integral operator with order  $\gamma(x, t) > 0$  of  $u(x, t)$  is defined as [29], [30]

$$I_t^{\gamma(x, t)} u(x, t) = \frac{1}{\Gamma(\gamma(x, t))} \int_0^t (t-s)^{\gamma(x, t)-1} u(x, s) ds,$$

where  $t > 0$  and  $\Gamma(\cdot)$  is Gamma function.

Based on the above definition, variable-order fractional integration has a following useful property:

$$I_t^{\gamma(x, t)} t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma(x, t)+1)} t^{\beta+\gamma(x, t)}, & \beta > -1, \\ 0, & \text{otherwise.} \end{cases}$$

*Definition 2:* The fractional derivative of  $u(x, t)$  in the Caputo sense is defined as [55], [56]

$$\begin{aligned} {}_0 D_t^{\gamma(x, t)} u(x, t) &= I_t^{q-\gamma(x, t)} D_t^q u(x, t) \\ &= \frac{1}{\Gamma(q-\gamma(x, t))} \int_0^t (t-s)^{q-\gamma(x, t)-1} \frac{\partial^q u(x, s)}{\partial s^q} ds, \end{aligned}$$

for  $q-1 < \gamma(x, t) \leq q$ ,  $t > 0$  and  $q \in \mathbb{N}$ .

It has a following useful property:

$${}_0 D_t^{\gamma(x, t)} t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma(x, t)+1)} t^{\beta-\gamma(x, t)}, & q \leq \beta \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

## III. LEGENDRE-LAGUERRE FUNCTIONS

In this paper, we need to introduce two-variable functions to deal with variable-order time fractional partial differential equations. Consider

$$\begin{aligned} \psi_{mn}(x, t) &= P_m(x) L_n(t), \quad (x, t) \in \Omega = [0, 1] \times [0, \infty), \\ m &= 0, 1, \dots, M, \quad n = 0, 1, \dots, N. \end{aligned}$$

where the shifted Legendre polynomials are defined on the interval  $[0, 1]$  and the Laguerre polynomials are defined on the interval  $[0, \infty]$ . So that, shifted Legendre and Laguerre

polynomials are denoted by  $P_m(x)$ ,  $m = 0, 1, \dots, M$ , and  $L_n(t)$ ,  $n = 0, 1, \dots, N$ , respectively [57], [58].

The Legendre-Laguerre functions are orthogonal with respect to the weight function  $w(x, t) = e^{-t}$  in the interval  $\Omega$  with the orthogonal property

$$\int_0^\infty \int_0^1 w(x, t) \psi_{mn}(x, t) \psi_{ij}(x, t) dx dt = \frac{1}{2m+1} \delta_{mi} \delta_{nj}. \quad (1)$$

where  $\delta_{mi}$  and  $\delta_{nj}$  are the Kronecker functions. A function  $f(x, t)$ , which is integrable in  $\Omega$  can be expanded as

$$f(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} \psi_{mn}(x, t),$$

where

$$f_{mn} = (2m+1) \int_0^\infty \int_0^1 w(x, t) f(x, t) \psi_{mn}(x, t) dx dt. \quad (2)$$

Then, we have truncated series for  $f$  as

$$f(x, t) \simeq \sum_{m=0}^M \sum_{n=0}^N f_{mn} \psi_{mn}(x, t) = P^T(x) FL(t), \quad (3)$$

where

$$F = \begin{bmatrix} f_{00} & f_{01} & \cdots & f_{0N} \\ f_{10} & f_{11} & \cdots & f_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{M0} & f_{M1} & \cdots & f_{MN} \end{bmatrix},$$

$$\begin{aligned} P(x) &= [P_0(x), P_1(x), \dots, P_M(x)]^T \\ L(t) &= [L_0(t), L_1(t), \dots, L_N(t)]^T. \end{aligned} \quad (4)$$

#### IV. PSEUDO-OPERATIONAL MATRICES OF LEGENDRE AND LAGUERRE POLYNOMIALS

In this section, we introduce the integral pseudo-operational matrix of the integer and variable fractional order for Legendre and Laguerre polynomials.

##### A. Integral Pseudo-Operational Matrix of the Integer Order

To calculate the integral pseudo-operational matrix of the integer order of Legendre polynomials use the Taylor polynomials, which defined as follows [59]

$$T_i(x) = x^i, \quad i = 0, 1, \dots, M.$$

The following relation holds among these polynomials and Legendre polynomials:

$$P(x) = D_1 T(x), \quad (5)$$

where

$$T(x) = [1, x, x^2, \dots, x^M]^T,$$

$$D_1 = [d_{ij}^1]_{(M+1) \times (M+1)},$$

$$d_{ij}^1 = \begin{cases} \frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}, & i \geq j, \\ 0 & \text{otherwise}, \end{cases} \quad i, j = 0, 1, \dots, M.$$

$D_1$  is the transformation matrix of the Legendre polynomials to the Taylor polynomials. Then, by integrating  $P(x)$ , we obtain the pseudo-operational matrix of Legendre polynomials

$$\begin{aligned} \int_0^x P(s) ds &= \int_0^x D_1 T(s) ds = D_1 \int_0^x T(s) ds \\ &= x D_1 H_1 T(x) = x D_1 H_1 D_1^{-1} P(x) \\ &= x Q_1 P(x), \end{aligned}$$

where  $Q_1 = D_1 H_1 D_1^{-1}$  is the pseudo-operational matrix of the Legendre polynomials and

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{M+1} \end{bmatrix}.$$

Also, we can write  $L(t)$  in the matrix form as follows

$$L(t) = D_2 T(t), \quad (6)$$

where

$$T(t) = [1, t, t^2, \dots, t^N]^T,$$

$$D_2 = [d_{ij}^2]_{(N+1) \times (N+1)},$$

$$d_{ij}^2 = \begin{cases} \frac{(-1)^j(i)!}{(i-j)!(j!)^2}, & i \geq j, \\ 0 & \text{otherwise.} \end{cases} \quad i, j = 0, 1, \dots, N.$$

$D_2$  is the transformation matrix of the Laguerre polynomials to the Taylor polynomials. Then, by integrating  $L(t)$ , we achieve the pseudo-operational matrix of integer integration of Laguerre polynomials

$$\begin{aligned} \int_0^t L(s) ds &= \int_0^t D_2 T(s) ds = D_2 \int_0^t T(s) ds \\ &= t D_2 H_2 T(t) = t D_2 H_2 D_2^{-1} L(t) = t Q_2 L(t), \end{aligned}$$

where  $Q_2 = D_2 H_2 D_2^{-1}$  is the pseudo-operational matrix of the Laguerre polynomials and

$$H_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N+1} \end{bmatrix}.$$

##### B. Integral Pseudo-Operational Matrix of the Variable Fractional Order

In this section, the pseudo-operational matrix of variable-order fractional integration of Laguerre polynomials by use of some properties of Riemann-Liouville fractional integral and Taylor polynomials is derived. First, we obtain the pseudo-operational matrix of variable-order fractional integration with order  $\gamma(x, t) > 0$  of Taylor polynomials as

$$I_t^{\gamma(x, t)} T(t) = t^{\gamma(x, t)} \theta_N^{\gamma(x, t)} T(t), \quad (7)$$

where

$$\theta_N^{\gamma(x,t)} = \begin{bmatrix} \frac{\Gamma(1)}{\Gamma(1+\gamma(x,t))} & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2+\gamma(x,t))} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma(N+1)}{\Gamma(N+1+\gamma(x,t))} \end{bmatrix}.$$

Also, to deal with the problem, we need to

$$I_t^{\gamma(x,t)} t T(t) = t^{1+\gamma(x,t)} \hat{\theta}_N^{\gamma(x,t)} T(t), \quad (8)$$

where

$$\hat{\theta}_N^{\gamma(x,t)} = \begin{bmatrix} \frac{\Gamma(2)}{\Gamma(2+\gamma(x,t))} & 0 & 0 & 0 \\ 0 & \frac{\Gamma(3)}{\Gamma(3+\gamma(x,t))} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma(N+2)}{\Gamma(N+2+\gamma(x,t))} \end{bmatrix}.$$

**Theorem 1:** Let  $L(t)$  be the Laguerre polynomials vector defined in (4) and  $q-1 < \gamma(x,t) \leq q \in \mathbb{Z}^+$ . The pseudo-operational matrix of variable-order fractional integration of Laguerre polynomials expressed as

$$I_t^{\gamma(x,t)} L(t) = t^{\gamma(x,t)} \xi_N^{\gamma(x,t)} L(t), \quad (9)$$

where  $\xi_N^{\gamma(x,t)} = D_2 \theta_N^{\gamma(x,t)} D_2^{-1}$ .

**Proof:** By using the pseudo-operational matrix of variable-order fractional integration of Taylor polynomials in (7) and transformation matrix of the Laguerre polynomials to the Taylor polynomials in (6), we have

$$\begin{aligned} I_t^{\gamma(x,t)} L(t) &= I_t^{\gamma(x,t)} D_2 T(t) = t^{\gamma(x,t)} D_2 \theta_N^{\gamma(x,t)} T(t) \\ &= t^{\gamma(x,t)} D_2 \theta_N^{\gamma(x,t)} D_2^{-1} L(t) \\ &= t^{\gamma(x,t)} \xi_N^{\gamma(x,t)} L(t), \end{aligned}$$

$\xi_N^{\gamma(x,t)}$  is called the pseudo-operational matrix of variable-order fractional integration for the Laguerre polynomials.

## V. APPLICATIONS AND RESULTS

This section is devoted to the study of variable-order time fractional partial differential equations as

$$F(D_t^{\gamma(x,t)} u(x,t), \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial^2 u(x,t)}{\partial t^2}), \quad (10)$$

$$\frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}, u(x,t)) = g(x,t),$$

$$q-1 < \gamma(x,t) \leq q, \quad 0 \leq x \leq 1, \quad t > 0,$$

with initial conditions

$$u(x,0) = f_0(x), \quad \frac{\partial u(x,0)}{\partial t} = f_1(x),$$

and boundary conditions

$$u(0,t) = \varphi_0(t), \quad u(1,t) = \varphi_1(t).$$

So that,  $u(x,t)$  is an unknown function, the known functions  $f_0(x)$ ,  $f_1(x)$ ,  $\varphi_0(t)$ ,  $\varphi_1(t)$ , and  $g(x,t)$  are defined on interval  $\Omega$ . Also,  $q = \max_{(x,t) \in \Omega} \{\gamma(x,t)\}$  and  $q \in \mathbb{Z}^+$ .

For this problem assume that, the highest order of derivative

respect to  $x$  and  $t$  is 2. Therefore, we obtain the following approximate functions as

$$\frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} \simeq P^T(x) U L(t), \quad (11)$$

where unknown matrix  $U$  define as follows

$$U = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{M0} & u_{M1} & \cdots & u_{MN} \end{bmatrix}.$$

Therefore, to approximate other functions, we use the integral pseudo-operational matrix of the integer and variable fractional order. By integrating of the above equation with respect to  $t$  and substituting initial condition into it, we obtain

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \simeq t P^T(x) U Q_2 L(t) + f_1''(x), \quad (12)$$

integrating (12) with respect to  $t$

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq t^2 P^T(x) U Q_2 \hat{Q}_2 L(t) + t f_1''(x) + f_0''(x), \quad (13)$$

where

$$\begin{aligned} \int_0^t s L(s) ds &= \int_0^t s D_2 T(s) ds = D_2 \int_0^t s T(s) ds \\ &= t^2 D_2 \hat{H}_2 T(t) = t^2 D_2 \hat{H}_2 D_2^{-1} L(t) \\ &= t^2 \hat{Q}_2 L(t), \end{aligned} \quad (14)$$

and

$$\hat{H}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N+2} \end{bmatrix}.$$

Now, by integrating (13) of order 2 with respect to  $x$ , we get

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &\simeq x t^2 P^T(x) Q_1^T U Q_2 \hat{Q}_2 L(t) \\ &+ t(f_1'(x) - f_1'(0)) + (f_0'(x) - f_0'(0)) + \frac{\partial u(0,t)}{\partial x} \end{aligned} \quad (15)$$

and

$$\begin{aligned} u(x,t) &\simeq x^2 t^2 P^T(x) \hat{Q}_1^T Q_1^T U Q_2 \hat{Q}_2 L(t) \\ &+ t(f_1(x) - f_1(0) - x f_1'(0)) \\ &+ (f_0(x) - f_0(0) - x f_0'(0)) \\ &+ x \frac{\partial u(0,t)}{\partial x} + \varphi_0(t), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \int_0^x s P(s) ds &= \int_0^x s D_1 T(s) ds = D_1 \int_0^x s T(s) ds \\ &= x^2 D_1 \hat{H}_1 T(x) = x^2 D_1 \hat{H}_1 D_1^{-1} P(x) \\ &= x^2 \hat{Q}_1 P(x), \end{aligned} \quad (17)$$

and

$$\hat{H}_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{M+2} \end{bmatrix}.$$

To calculate the above equations, integrating (15) with respect to  $x$  from 0 to 1

$$\begin{aligned} u(1, t) - u(0, t) &\simeq t^2 \left( \int_0^1 x P^T(x) dx \right) Q_1^T U Q_2 \hat{Q}_2 L(t) \\ &+ t(f_1(1) - f_1(0) - f_1'(0)) \\ &+ (f_0(1) - f_0(0) - f_0'(0)) + \frac{\partial u(0, t)}{\partial x}, \end{aligned}$$

from (5), we get

$$\int_0^1 x P^T(x) dx = \int_0^1 x T^T(x) D_1^T dx = S^T D_1^T,$$

where

$$S = \left[ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{M+2} \right]^T.$$

Then,

$$\begin{aligned} \frac{\partial u(0, t)}{\partial x} &\simeq \varphi_1(t) - \varphi_0(t) - t^2 S^T D_1^T Q_1^T U Q_2 \hat{Q}_2 L(t) \\ &- t(f_1(1) - f_1(0) - f_1'(0)) \\ &- (f_0(1) - f_0(0) - f_0'(0)). \end{aligned} \quad (18)$$

Substituting (18) in (15) and (16), we obtain the approximation of  $u(x, t)$  and  $\frac{\partial u(x, t)}{\partial x}$  by use of Legendre and Laguerre polynomials. Also, we need to calculate the following expression, by integrating (12) of order 2 with respect to  $x$  and using initial and boundary conditions

$$\frac{\partial^3 u(x, t)}{\partial x \partial t^2} \simeq x P^T(x) Q_1^T U L(t) + \frac{\partial^3 u(0, t)}{\partial x \partial t^2}, \quad (19)$$

$$\begin{aligned} \frac{\partial^2 u(x, t)}{\partial t^2} &\simeq x^2 P^T(x) \hat{Q}_1^T Q_1^T U L(t) + x \frac{\partial^3 u(0, t)}{\partial x \partial t^2} \\ &+ \varphi_0''(t). \end{aligned} \quad (20)$$

$\frac{\partial^3 u(0, t)}{\partial x \partial t^2}$  is an unknown function. By integrating (19) from 0 to 1 with respect to  $x$ , we get

$$\frac{\partial^3 u(0, t)}{\partial x \partial t^2} \simeq \varphi_1''(t) - \varphi_0''(t) - S^T D_1^T Q_1^T U L(t).$$

Then

$$\begin{aligned} \frac{\partial^2 u(x, t)}{\partial t^2} &\simeq x^2 P^T(x) \hat{Q}_1^T Q_1^T U L(t) \\ &+ x(\varphi_1''(t) - \varphi_0''(t) - S^T D_1^T Q_1^T U L(t)) \\ &+ \varphi_0''(t). \end{aligned} \quad (21)$$

By integrating (21) with respect to  $t$ , we obtain

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &\simeq x^2 t P^T(x) \hat{Q}_1^T Q_1^T U Q_2 L(t) \\ &+ x(\varphi_1'(t) - \varphi_0'(t) - t S^T D_1^T Q_1^T U Q_2 L(t)) \\ &+ \varphi_0'(t) + f_1(x). \end{aligned} \quad (22)$$

In addition, it is necessary to calculate fractional derivatives of  $u(x, t)$  by applying the integral pseudo-operational matrix of variable fractional order and Riemann-Liouville fractional integral properties.

For  $0 < \gamma(x, t) \leq 1$ , by integrating (22) with respect to  $t$  of order  $\gamma(x, t)$  and taking initial and boundary conditions, we have

$$\begin{aligned} D_t^{\gamma(x, t)} u(x, t) &= I_t^{1-\gamma(x, t)} \left( \frac{\partial u(x, t)}{\partial t} \right) \\ &\simeq x^2 t^{2-\gamma(x, t)} P^T(x) \hat{Q}_1^T Q_1^T U Q_2 \hat{\xi}_N^{1-\gamma(x, t)} L(t) \\ &+ x I_t^{1-\gamma(x, t)} (\varphi_1'(t) - \varphi_0'(t)) \\ &- x t^{2-\gamma(x, t)} S^T D_1^T Q_1^T U Q_2 \hat{\xi}_N^{1-\gamma(x, t)} L(t) \\ &+ \frac{\Gamma(1)}{\Gamma(2-\gamma(x, t))} t^{1-\gamma(x, t)} f_1(x) + I_t^{1-\gamma(x, t)} (\varphi_0'(t)), \end{aligned} \quad (23)$$

so that

$$I_t^{1-\gamma(x, t)} (t L(t)) \simeq t^{2-\gamma(x, t)} \hat{\xi}_N^{1-\gamma(x, t)} L(t),$$

according to (8), we obtain

$$\hat{\xi}_N^{1-\gamma(x, t)} = D_2 \hat{\theta}_N^{1-\gamma(x, t)} D_2^{-1}.$$

Also, for  $1 < \gamma(x, t) \leq 2$ ,

$$\begin{aligned} D_t^{\gamma(x, t)} u(x, t) &= I_t^{2-\gamma(x, t)} \left( \frac{\partial^2 u(x, t)}{\partial t^2} \right) \\ &\simeq x^2 t^{2-\gamma(x, t)} P^T(x) \hat{Q}_1^T Q_1^T U \hat{\xi}_N^{2-\gamma(x, t)} L(t) \\ &+ x I_t^{2-\gamma(x, t)} (\varphi_1''(t) - \varphi_0''(t)) \\ &- x t^{2-\gamma(x, t)} S^T D_1^T Q_1^T U \hat{\xi}_N^{2-\gamma(x, t)} L(t) \\ &+ I_t^{2-\gamma(x, t)} (\varphi_0''(t)). \end{aligned} \quad (24)$$

We obtain an algebraic equation by substituting the above approximate functions in (10) and nodal points of Newton-Cotes [60]. Then, we get unknown matrix  $U$  by solving a system of algebraic equation and using Newton's iterative method. Ultimately, by substituting  $U$  in (16), we achieve the approximate solution of the problem.

## VI. ERROR ANALYSIS

In this section, we analyze the upper bound of error for the numerical method and present error analysis based on the residual function.

### A. Upper Bound of Error

We indicate that Legendre-Laguerre expansion of a continuous function  $f(x, t)$  converges uniformly. But before that, we present the upper bound for its error by the following theorem. Let  $P_{M, N}$  consists of all polynomials of degree at most  $M$  for variable  $x$  and degree at most  $N$  for variable  $t$ . Thus, for  $f \in C(\Omega)$ , there exists unique  $p_{M, N} \in P_{M, N}$  such that

$$\|f(x, t) - f_{M, N}(x, t)\|_{L_w^2(\Omega)} \leq \|f(x, t) - p_{M, N}(x, t)\|_{L_w^2(\Omega)}. \quad (25)$$

Also, we define

$$L_w^2(\Omega) = \{\vartheta : \vartheta \text{ is measurable on } \Omega \text{ and } \|\vartheta\|_w < \infty\},$$



equipped with the following inner product and norm

$$\langle \vartheta, \rho \rangle_w = \int_{\Omega} \vartheta(x, t) \rho(x, t) w(x, t) dx dt, \quad \|\vartheta\|_w = \langle \vartheta, \vartheta \rangle_w.$$

**Definition 3:** Let  $f(x, t)$  be a function of two real variables which is continuous at a certain point  $(x_0, t_0)$  and such that all its partial derivatives are also continuous at that point. Then the Taylor series expansion of  $f(x, y)$  about the point  $(x_0, y_0)$  can be obtained as [60], [61]

$$f(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \frac{\partial^n}{\partial t^n} \left( \frac{\partial^m f}{\partial x^m} \right) \Big|_{(x_0, t_0)} (x - x_0)^m (t - t_0)^n.$$

We can write

$$\begin{aligned} f(x, t) &= \sum_{m=0}^M \sum_{n=0}^N \frac{1}{m!n!} \frac{\partial^n}{\partial t^n} \left( \frac{\partial^m f}{\partial x^m} \right) \Big|_{(x_0, t_0)} (x - x_0)^m (t - t_0)^n \\ &+ R_{MN}(x, t). \end{aligned} \quad (26)$$

If all partial derivatives of  $f$  of order  $M + N + 2$  are exist, then

$$\begin{aligned} |R_{MN}(x, t)| &\leq \frac{(x - x_0)^{M+1} (t - t_0)^{N+1}}{(M + 1)!(N + 1)!} \\ &\times \sup_{(x, t) \in \Omega} \left| \frac{\partial^{M+N+2} f}{\partial x^{M+1} \partial t^{N+1}}(x, t) \right|. \end{aligned} \quad (27)$$

**Theorem 2:** Suppose that the real sufficiently smooth function  $f$ , is expanded by the Legendre-Laguerre functions in  $\Omega$ , as

$$f_{M,N}(x, t) \simeq \sum_{m=0}^M \sum_{n=0}^N \bar{f}_{mn} \psi_{mn}(x, t) = \bar{F}^T \psi_{MN}(x, t),$$

where

$$\begin{aligned} \psi_{MN}(x, t) &= [\psi_{00}(x, t), \psi_{01}(x, t), \dots, \psi_{0N}(x, t), \dots \\ &\quad, \psi_{M0}(x, t), \psi_{M1}(x, t), \dots, \psi_{MN}(x, t)]^T, \\ \bar{F} &= [\bar{f}_{00}, \bar{f}_{01}, \dots, \bar{f}_{0N}, \dots, \bar{f}_{M0}, \bar{f}_{M1}, \dots, \bar{f}_{MN}]^T. \end{aligned}$$

If the bounded on the right hand side of (27) in magnitude by

$$C_{MN} = \sup_{(x, t) \in \Omega} \left| \frac{\partial^{M+N+2} f}{\partial x^{M+1} \partial t^{N+1}}(x, t) \right|,$$

we can estimate the upper bound of error as

$$\|f(x, t) - f_{M,N}(x, t)\|_{L_w^2(\Omega)} \leq \frac{C_{MN} \sqrt{(2N + 2)!}}{(M + 1)!(N + 1)! \sqrt{(2M + 3)!}}. \quad (28)$$

In addition, let

$$\tilde{f}_{M,N}(x, t) \simeq \tilde{F}^T \psi_{MN}(x, t),$$

be the approximate solution obtained by the proposed method in Section V, where

$$\tilde{F} = [\tilde{f}_{00}, \tilde{f}_{01}, \dots, \tilde{f}_{0N}, \dots, \tilde{f}_{M0}, \tilde{f}_{M1}, \dots, \tilde{f}_{MN}]^T.$$

Then, we have

$$\begin{aligned} &\|f(x, t) - \tilde{f}_{M,N}(x, t)\|_{L_w^2(\Omega)} \\ &\leq \frac{C_{MN} \sqrt{(2N + 2)!}}{(M + 1)!(N + 1)! \sqrt{(2M + 3)!}} \\ &+ \theta_{MN} \|\tilde{F} - \bar{F}\|_2, \end{aligned} \quad (29)$$

where

$$\theta_{MN} = \sqrt{\sum_{m=0}^M \frac{N + 1}{2m + 1}},$$

and the norm  $\|\cdot\|_2$  is the usual Euclidean norm of vectors.

**Proof:** We define

$$p_{MN}(x, t) = \sum_{m=0}^M \sum_{n=0}^N \frac{1}{m!n!} \frac{\partial^n}{\partial t^n} \left( \frac{\partial^m f}{\partial x^m} \right) \Big|_{(0,0)} x^m t^n, \quad (30)$$

by using (26) and (27) about  $(x_0, t_0) = (0, 0)$ , we have

$$|f(x, t) - p_{MN}(x, t)| \leq \frac{C_{MN}}{(M + 1)!(N + 1)!} x^{M+1} t^{N+1}. \quad (31)$$

Applying above equation, we get

$$\begin{aligned} &\|f(x, t) - f_{M,N}(x, t)\|_{L_w^2(\Omega)}^2 \\ &= \int_0^\infty \int_0^1 |f(x, t) - \bar{F}^T \psi_{MN}(x, t)|^2 e^{-t} dx dt \\ &\leq \int_0^\infty \int_0^1 |f(x, t) - p_{M,N}(x, t)|^2 e^{-t} dx dt \\ &\leq \int_0^\infty \int_0^1 \left| \frac{C_{MN}}{(M + 1)!(N + 1)!} x^{M+1} t^{N+1} \right|^2 e^{-t} dx dt \\ &= C_{MN}^2 \int_0^\infty \int_0^1 \frac{x^{2M+2} t^{2N+2}}{((M + 1)!(N + 1)!)^2} e^{-t} dx dt \\ &= \frac{C_{MN}^2 (2N + 2)!}{((M + 1)!)^2 ((N + 1)!)^2 (2M + 3)!}, \end{aligned} \quad (32)$$

by taking the square roots of both sides, we obtain the upper bound of the error. Also, one can easily find that

$$\begin{aligned} &\|f(x, t) - \tilde{f}_{M,N}(x, t)\|_{L_w^2(\Omega)} \\ &\leq \|f(x, t) - f_{M,N}(x, t)\|_{L_w^2(\Omega)} \\ &+ \|f_{M,N}(x, t) - \tilde{f}_{M,N}(x, t)\|_{L_w^2(\Omega)}. \end{aligned} \quad (33)$$

We then have,

$$\begin{aligned}
 & \|f_{M,N}(x,t) - \tilde{f}_{M,N}(x,t)\|_{L_w^2(\Omega)} \quad (34) \\
 &= \left( \int_0^\infty \int_0^1 |f_{M,N}(x,t) - \tilde{f}_{M,N}(x,t)|^2 e^{-t} dx dt \right)^{\frac{1}{2}} \\
 &\leq \left( \int_0^\infty \int_0^1 \left| \sum_{m=0}^M \sum_{n=0}^N (\bar{F} - \tilde{F}) \psi_{MN}(x,t) \right|^2 e^{-t} dx dt \right)^{\frac{1}{2}} \\
 &\leq \left( \int_0^\infty \int_0^1 \left[ \sum_{m=0}^M \sum_{n=0}^N |\bar{F} - \tilde{F}|^2 \right] \right. \\
 &\quad \times \left. \left[ \sum_{m=0}^M \sum_{n=0}^N |\psi_{MN}(x,t)|^2 \right] e^{-t} dx dt \right)^{\frac{1}{2}} \\
 &= \left( \sum_{m=0}^M \sum_{n=0}^N |\bar{F} - \tilde{F}|^2 \right)^{\frac{1}{2}} \\
 &\quad \times \left( \sum_{m=0}^M \sum_{n=0}^N \int_0^\infty \int_0^1 |\psi_{MN}(x,t)|^2 e^{-t} dx dt \right)^{\frac{1}{2}} \\
 &= \|\bar{F} - \tilde{F}\|_2 \left( \sum_{m=0}^M \frac{N+1}{2m+1} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Consequently, from (32)-(34), we obtain

$$\begin{aligned}
 & \|f(x,t) - \tilde{f}_{M,N}(x,t)\|_{L_w^2(\Omega)} \quad (35) \\
 &\leq \frac{C_{MN} \sqrt{(2N+2)!}}{(M+1)!(N+1)! \sqrt{(2M+3)}} \\
 &\quad + \left( \sum_{m=0}^M \frac{N+1}{2m+1} \right)^{\frac{1}{2}} \|\bar{F} - \tilde{F}\|_2.
 \end{aligned}$$

Above theorem demonstrates that with increasing the terms of Legendre-Laguerre functions the error tends to zero.

### B. Residual Error

The error function of the approximate solution  $u_{MN}(x,t)$  for  $u(x,t)$ , where  $u(x,t)$  is the exact solution of (14), is define as follow:

$$e_{MN}(x,t) = u_{MN}(x,t) - u(x,t).$$

According to the problem,  $u_{MN}(x,t)$  satisfies in

$$\begin{aligned}
 & F(D_t^{\gamma(x,t)} u_{MN}(x,t), \frac{\partial^2 u_{MN}(x,t)}{\partial x^2}, \frac{\partial^2 u_{MN}(x,t)}{\partial t^2}, \\
 & \frac{\partial u_{MN}(x,t)}{\partial x}, \frac{\partial u_{MN}(x,t)}{\partial t}, u_{MN}(x,t)) - g(x,t) \\
 &= r_{MN}(x,t),
 \end{aligned}$$

where  $r_{MN}(x,t)$  is the residual function. Also, in order to achieve the approximate error  $\tilde{e}_{MN}(x,t)$  to the error function  $e_{MN}(x,t)$  using the techniques of Section V, as

$$\begin{aligned}
 & F(D_t^{\gamma(x,t)} e_{MN}(x,t), \frac{\partial^2 e_{MN}(x,t)}{\partial x^2}, \frac{\partial^2 e_{MN}(x,t)}{\partial t^2}, \\
 & \frac{\partial e_{MN}(x,t)}{\partial x}, \frac{\partial e_{MN}(x,t)}{\partial t}, e_{MN}(x,t)) \\
 &= r_{MN}(x,t),
 \end{aligned}$$

with initial and boundary conditions

$$\begin{aligned}
 e_{MN}(x,0) &= e_0(x), & \frac{\partial e_{MN}(x,0)}{\partial t} &= e_1(x), \\
 e_{MN}(0,t) &= \phi_0(t), & e_{MN}(1,t) &= \phi_1(t),
 \end{aligned}$$

where  $e_0(x)$ ,  $e_1(x)$ ,  $\phi_0(t)$  and  $\phi_1(t)$  are known functions. Therefore, the approximate solution is obtain

$$\tilde{u}_{MN}(x,t) = u_{MN}(x,t) + \tilde{e}_{MN}(x,t). \quad (36)$$

Ultimately, the general error of the problem is

$$E(x,t) = u(x,t) - \tilde{u}_{MN}(x,t). \quad (37)$$

## VII. ILLUSTRATIVE EXAMPLES

In order to test the validity of the present method, some examples are solved and the numerical results are compared with their exact solution and other methods. The computations associated with the examples were performed using MATLAB.

*Example 1:* Consider the time variable fractional order mobile-immobile advection-dispersion model is as follows [40], [47]

$$\frac{\partial u(x,t)}{\partial t} + D_t^{\gamma(x,t)} u(x,t) = -\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (38)$$

with the following initial condition

$$u(x,0) = 5x(1-x), \quad 0 \leq x \leq 1,$$

and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq 1,$$

with

$$\begin{aligned}
 f(x,t) &= 5x(1-x) + \frac{5x(1-x)t^{1-\gamma(x,t)}}{\Gamma(2-\gamma(x,t))} \\
 &\quad + 10(t+1) + 5(t+1)(1-2x),
 \end{aligned}$$

and

$$\gamma(x,t) = 0.8 + 0.005 \cos(xt) \sin(x).$$

The exact solution of this example is  $u(x,t) = 5(t+1)x(1-x)$ . Now, let us find the approximate solution given by Legendre-Laguerre functions. Let

$$\frac{\partial u^3(x,t)}{\partial x^2 \partial t} \simeq P^T(x) U L(t), \quad (39)$$

by integrating (39) with respect to  $t$ , we get

$$\frac{\partial u^2(x,t)}{\partial x^2} \simeq t P^T(x) U Q_2 L(t) - 10. \quad (40)$$

By integrating (40) with respect to  $x$  of order 2 and using initial and boundary conditions, we obtain

$$\begin{aligned}
 \frac{\partial u(x,t)}{\partial x} &\simeq x t P^T(x) Q_1^T U Q_2 L(t) + 5 - 10x \quad (41) \\
 &\quad - t S^T D_1^T Q_1^T U Q_2 L(t), \\
 u(x,t) &\simeq x^2 t P^T(x) \hat{Q}_1^T Q_1^T U Q_2 L(t) + 5x - 5x^2 \\
 &\quad - t x S^T D_1^T Q_1^T U Q_2 L(t).
 \end{aligned}$$

Moreover, by integrating (39) of order 2 with respect to  $x$ , we obtain

$$\frac{\partial u(x,t)}{\partial t} \simeq x^2 P^T(x) \hat{Q}_1^T Q_1^T U L(t) - x S^T D_x^T Q_1^T U L(t). \quad (42)$$

With a view to  $0 < \gamma(x,t) \leq 1$ , we integrating (42) of variable fractional with respect to  $t$ ,

$$D_t^{\gamma(x,t)} u(x,t) \simeq x^2 t^{1-\gamma(x,t)} P^T(x) \hat{Q}_1^T Q_1^T U \xi_N^{1-\gamma(x,t)} L(t) - x t^{1-\gamma(x,t)} S^T D_1^T Q_1^T U \xi_N^{1-\gamma(x,t)} L(t). \quad (43)$$

By replacing above approximation in (38) and using collocation points, we get the system of algebraic equations. We take  $M = N = 1$ , obtains

$$U = \begin{bmatrix} -10 & -4.58922 \times 10^{-16} \\ -6.17113 \times 10^{-16} & 1.45846 \times 10^{-15} \end{bmatrix},$$

then, with regards to (41) have

$$\begin{aligned} u(x,t) &\simeq (3.33 \times 10^{-16} xt + 2.43 \times 10^{-16} xt^2 - 5t \\ &+ 5.18 \times 10^{-16} t^2 - 5)x^2 \\ &\times (2.75 \times 10^{-16} t^2 + 5t + 5)x. \end{aligned} \quad (44)$$

In view of the error introduced in section 6, we have error problem as

$$\begin{cases} \frac{\partial e_{11}(x,t)}{\partial t} + D_t^{\gamma(x,t)} e_{11}(x,t) + \frac{\partial e_{11}(x,t)}{\partial x} - \frac{\partial^2 e_{11}(x,t)}{\partial x^2} \\ = r_{MN}(x,t), \\ e_{11}(x,0) = 0, \\ e_{11}(0,t) = 0, \\ e_{11}(1,t) = 4.8978931 \times 10^{-40} t, \end{cases}$$

where

By solving the above problem, we get the absolute error  $\tilde{e}_{11}(x,t)$  as

$$\begin{aligned} \tilde{e}_{11}(x,t) &= (5.62 \times 10^{-63} xt + 1.00 \times 10^{-63} xt^2)x^2 \\ &+ 5.92 \times 10^{-65} xt^2. \end{aligned}$$

So, by using (36) the approximate solution is

$$\begin{aligned} \tilde{u}_{11}(x,t) &= (3.33 \times 10^{-16} xt + 2.43 \times 10^{-16} xt^2 - 5t \\ &+ 5.18 \times 10^{-16} t^2 - 5)x^2 + (2.75 \times 10^{-16} t^2 + 5t + 5)x \\ &+ (5.62 \times 10^{-63} xt + 1.00 \times 10^{-63} xt^2)x^2 \\ &+ 5.92 \times 10^{-65} xt^2. \end{aligned}$$

Also, Table I shows the maximum absolute errors obtained for various values of  $t$  with  $M = N = 1$ . In Table II, the absolute error obtained between the approximate solution and the exact solution with that CPU time (in seconds). From the comparison in Table II, it is displayed that the present method more accurate than the method in [40]. Also, due to the errors table and figures in [47], the present method more accurate compared with that method.

**Example 2:** Consider the time variable fractional order mobile-immobile advection-dispersion model is as follows [40], [41], [47]

$$\frac{\partial u(x,t)}{\partial t} + D_t^{\gamma(x,t)} u(x,t) = -\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$

with initial condition  $u(x,0) = 10x^2(1-x)^2$ ,  $0 \leq x \leq 1$  and boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 \leq t \leq T,$$

with

$$\begin{aligned} f(x,t) &= 10(1 + \frac{t^{1-\gamma(x,t)}}{\Gamma(2-\gamma(x,t))})x^2(1-x)^2 \\ &+ 10(4x^3 - 6x^2 + 2x - 12x^2 + 12x - 2)(t+1), \end{aligned}$$

and

$$\gamma(x,t) = 1 - 0.5 \exp(-xt).$$

The exact solution of this example is  $u(x,t) = 10(t+1)x^2(1-x)^2$ . To solve the problem by proposed method, we take  $M = 2, N = 1$ , obtain

$$U = \begin{bmatrix} 4.792 \times 10^{-16} & 3.204 \times 10^{-16} & 20 \\ -7.313 \times 10^{-16} & -5.958 \times 10^{-16} & 1.084 \times 10^{-14} \end{bmatrix},$$

then,

$$\begin{aligned} u(x,t) &= x^2(10t + 10tx^2 + 5.52 \times 10^{-15} xt^2 \\ &- 2.71 \times 10^{-15} x^2 t^2 - 20xt - 2.67 \times 10^{-15} t^2 \\ &+ 10) + x(-1.33 \times 10^{-16} t^2 + 8.01 \times 10^{-16} t) \\ &- 20x^3 + 10x^4. \end{aligned}$$

We see that  $u(x,t)$  is a good approximation with the exact solution by using a few terms of Legendre-Laguerre functions. In Table III, the absolute error obtained between the approximate solution and the exact solution with that CPU time (in seconds), which demonstrates that the proposed method is more accurate in comparison to the methods in [40], [41], [47]. Also, Table IV shows the maximum absolute error obtained between the approximate solutions and the exact solution for various values of  $t$ .

**Example 3:** Consider the following linear variable-order time fractional partial differential equations

$$\begin{aligned} D_t^{\gamma(x,t)} u(x,t) + \frac{\partial u}{\partial x}(x,t) - x \frac{\partial^2 u}{\partial x^2}(x,t) &= f(x,t), \\ 0 < \gamma(x,t) &\leq 1, \end{aligned}$$

with initial condition  $u(x,0) = \exp(x)$ ,  $0 \leq x \leq 1$  and boundary conditions

$$u(0,t) = (t^2+1)(1-t), \quad u(1,t) = (t^2+1)(\exp(1)-t), \quad t > 0,$$

with

$$\begin{aligned} f(x,t) &= \frac{2t^{2-\gamma(x,t)}}{\Gamma(3-\gamma(x,t))} - \frac{t^{1-\gamma(x,t)}}{\Gamma(2-\gamma(x,t))} \\ &- \frac{6t^{3-\gamma(x,t)}}{\Gamma(4-\gamma(x,t))} + (t^2+1)(1-x)\exp(x). \end{aligned}$$

The exact solution of this problem is  $u(x,t) = (t^2+1)(\exp(x)-t)$ . Table V, presents the absolute errors between the approximate solutions and the exact solution for various functions of  $\gamma(x,t)$  with various values of  $M, N$ . From Table V, we can see clearly that the error gets more and more small with increasing  $M$ . Fig. 1, illustrates the absolute error and approximate solution obtained by the proposed method for  $M = 4, N = 3$ .



$$\begin{aligned}
r_{MN}(x, t) = & 2.091 \times 10^{-15}t + 4.861 \times 10^{-16}xt^2 + 1.370 \times 10^{-14}x^2t + 3.331 \times 10^{-15}x^3 \\
& + (-5.516 \times 10^{-16}x + 1.037 \times 10^{-14}x^2 + 3.331 \times 10^{-16}x^3) \frac{t^{0.2-0.005 \cos(xt) \sin(x)}}{\Gamma(1.2-0.005 \cos(xt) \sin(x))} \\
& - 4.861 \times 10^{-16}x^3t - 2.430 \times 10^{-16}x^2t^2 - 8.083 \times 10^{-16}x^2 \\
& - 4.167 \times 10^{-15}xt - 4.861 \times 10^{-14}x^2t^2 + 6.665 \times 10^{-14}x^2t \\
& + 2.010 \times 10^{-15}xt^2 + 4.751 \times 10^{-16}x - 1.313 \times 10^{-15}t^2.
\end{aligned}$$

TABLE I

MAXIMUM ABSOLUTE ERRORS WITH VARIOUS VALUES OF $t$ FOR EXAMPLE 1				
$M = 1, N = 1$	$t = 1$	$t = 10$	$t = 100$	$t = 1000$
$x \in [0, 1]$	$3.91 \times 10^{-17}$	$3.47 \times 10^{-15}$	$4.20 \times 10^{-13}$	$4.27 \times 10^{-11}$

TABLE II

COMPARISON OF THE ABSOLUTE ERROR FOR  $\gamma(x, t) = 0.8 + 0.005 \cos(xt) \sin(x)$  AND  $t = 1$  WITH METHOD IN [40] OF EXAMPLE 1

$x_i$	Present Method $M = 1, N = 1$	Method in [40] $N = 13$
0.1	$1.71 \times 10^{-17}$	0
0.2	$2.90 \times 10^{-17}$	$2.2205 \times 10^{-16}$
0.3	$3.61 \times 10^{-17}$	$4.4409 \times 10^{-16}$
0.4	$3.91 \times 10^{-17}$	0
0.5	$3.85 \times 10^{-17}$	0
0.6	$3.48 \times 10^{-17}$	$4.4409 \times 10^{-16}$
0.7	$2.86 \times 10^{-17}$	0
0.8	$2.03 \times 10^{-17}$	0
0.9	$1.06 \times 10^{-17}$	$6.6613 \times 10^{-16}$
Cpu	$1.69 \times 10^{-2}$	-

*Example 4:* As a final example, consider the variable-order fractional SG equations is as follows

$$\begin{aligned}
D_t^{\gamma(x,t)} u(x, t) &= \frac{\partial^2 u}{\partial x^2}(x, t) - \sin(u(x, t)) + f(x, t), \\
1 &< \gamma(x, t) \leq 2,
\end{aligned}$$

with the following initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \quad 0 < x < 1,$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = t^2 \sin(1), \quad 0 \leq t \leq 1,$$

with

$$f(x, t) = \left( \frac{2t^{2-\gamma(x,t)}}{\Gamma(3-\gamma(x,t))} + t^2 \right) \sin(x) + \sin(t^2 \sin(x)).$$

The exact solution of this example is  $u(x, t) = t^2 \sin(x)$ . In order to show the accuracy of the method, the absolute errors between exact and numerical solutions for various values of  $M, N$  with  $\gamma(x, t) = 2 - 0.2 \sin(x) \exp(-t)$  are shown in Table VI. From Table VI, we can see clearly that the error gets more and more small with increasing  $M$ . Authors in [28] expressed this problem for  $\gamma(x, t) = \alpha$ , which  $\alpha$  is a real number. Also, we presented absolute errors between exact and numerical solutions for  $\gamma(x, t) = 1.15, 1.85, 2$  with various values of  $M, N$  in Table VII and Fig. ?? . Due to the errors table in [28], the present method more accurate compared with that method. With regards to these table and figure, it is seen that the approximate solutions converge to the exact solution.

## VIII. CONCLUSION

In the present work, applied the collocation method to approximate the solution of variable-order time fractional partial differential equations. One significant advantage of this method is that with using a few terms of Legendre-Laguerre functions, approximate solution converges to the exact solution and with increasing the amount of  $M$  and  $N$  the accuracy is increased sufficiently. Also, we introduced a new technique to obtain the operational matrices with the least error, which is called the pseudo-operational matrix. According to these pseudo-operational matrix, we can be transformed our problem to a nonlinear system of algebraic equations. As seen from the numerical examples, the results demonstrate that this method is more accurate than some existing methods.

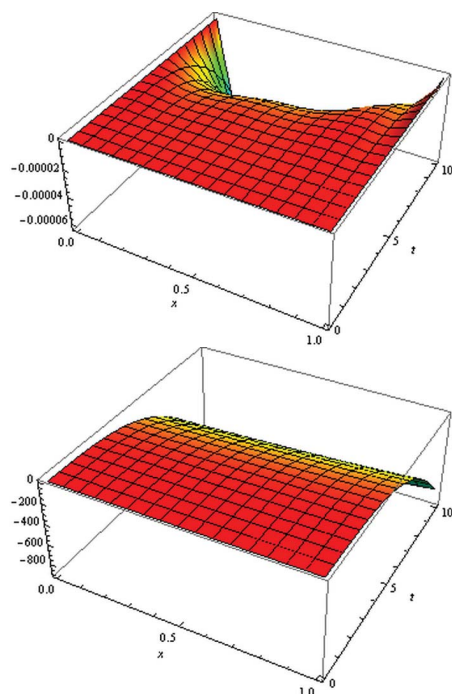


Fig. 1 Absolute error and approximate solution for  $\gamma(x, t) = 1 - 0.2 \exp(-xt)$  with  $M = 4, N = 3$  of Example 3

TABLE III  
ABSOLUTE ERRORS FOR  $\gamma(x, t) = 1 - 0.5 \exp(-xt)$  WITH  $t = 1$  OF EXAMPLE 2

$(x_i, t_i)$	Present Method	Method in [40]	Method in [41]	Method in [47]
	$M = 2, N = 1$	$N = 20$	$N = 100$	$M = N = 10$
0.1	$4.84 \times 10^{-18}$	$1.5629 \times 10^{-4}$	$6.3759 \times 10^{-5}$	$4.7781 \times 10^{-15}$
0.2	$2.71 \times 10^{-17}$	$1.4006 \times 10^{-3}$	$4.9040 \times 10^{-6}$	$1.1399 \times 10^{-16}$
0.3	$6.46 \times 10^{-17}$	$2.9751 \times 10^{-3}$	$5.9837 \times 10^{-5}$	$2.7608 \times 10^{-16}$
0.4	$1.04 \times 10^{-17}$	$4.2976 \times 10^{-3}$	$7.7810 \times 10^{-6}$	$7.0724 \times 10^{-16}$
0.5	$1.31 \times 10^{-16}$	$4.9721 \times 10^{-3}$	$5.8089 \times 10^{-5}$	$1.0486 \times 10^{-17}$
0.6	$1.33 \times 10^{-16}$	$4.8034 \times 10^{-3}$	$8.2984 \times 10^{-6}$	$2.3397 \times 10^{-16}$
0.7	$1.09 \times 10^{-16}$	$3.8152 \times 10^{-3}$	$5.8931 \times 10^{-5}$	$2.3096 \times 10^{-16}$
0.8	$6.44 \times 10^{-17}$	$2.2746 \times 10^{-3}$	$5.9463 \times 10^{-6}$	$8.4324 \times 10^{-17}$
0.9	$1.86 \times 10^{-17}$	$7.2075 \times 10^{-4}$	$6.2975 \times 10^{-5}$	$7.9527 \times 10^{-17}$
Cpu	$5.76 \times 10^{-2}$	-	-	-

TABLE IV  
MAXIMUM ABSOLUTE ERRORS WITH VARIOUS VALUES OF  $t$  OF EXAMPLE 2

$M = 2, N = 1$	$t = 1$	$t = 10$	$t = 100$	$t = 1000$
$x \in [0, 1]$	$1.55 \times 10^{-16}$	$1.78 \times 10^{-14}$	$2.11 \times 10^{-12}$	$2.14 \times 10^{-10}$

TABLE V  
ABSOLUTE ERRORS WITH VARIOUS FUNCTIONS OF  $\gamma(x, t)$  FOR EXAMPLE 3

$(x_i, t_i)$	$\gamma(x, t) = 1 - 0.5 \exp(-xt)$		$\gamma(x, t) = 1 - \exp(-xt)$		$\gamma(x, t) = 0.5$	
	$M = 3, N = 3$	$M = 6, N = 3$	$M = 3, N = 3$	$M = 6, N = 3$	$M = 3, N = 3$	$M = 6, N = 3$
(0.1, 0.1)	$3.06 \times 10^{-8}$	$9.50 \times 10^{-10}$	$4.16 \times 10^{-8}$	$8.85 \times 10^{-10}$	$1.46 \times 10^{-8}$	$9.52 \times 10^{-9}$
(0.2, 0.2)	$3.24 \times 10^{-8}$	$6.41 \times 10^{-9}$	$7.75 \times 10^{-8}$	$6.19 \times 10^{-9}$	$2.16 \times 10^{-8}$	$6.41 \times 10^{-9}$
(0.3, 0.3)	$6.39 \times 10^{-9}$	$1.82 \times 10^{-8}$	$9.00 \times 10^{-8}$	$1.79 \times 10^{-8}$	$1.08 \times 10^{-7}$	$1.83 \times 10^{-8}$
(0.4, 0.4)	$9.25 \times 10^{-8}$	$3.60 \times 10^{-8}$	$2.50 \times 10^{-7}$	$3.57 \times 10^{-8}$	$5.68 \times 10^{-8}$	$3.61 \times 10^{-8}$
(0.5, 0.5)	$2.51 \times 10^{-7}$	$5.72 \times 10^{-8}$	$4.73 \times 10^{-7}$	$5.70 \times 10^{-8}$	$6.42 \times 10^{-8}$	$5.72 \times 10^{-8}$
(0.6, 0.6)	$4.75 \times 10^{-8}$	$7.72 \times 10^{-8}$	$3.24 \times 10^{-7}$	$7.73 \times 10^{-8}$	$1.59 \times 10^{-7}$	$7.72 \times 10^{-8}$
(0.7, 0.7)	$6.56 \times 10^{-7}$	$8.99 \times 10^{-8}$	$3.51 \times 10^{-7}$	$9.02 \times 10^{-8}$	$8.59 \times 10^{-7}$	$9.00 \times 10^{-8}$
(0.8, 0.8)	$9.43 \times 10^{-7}$	$8.76 \times 10^{-8}$	$6.56 \times 10^{-7}$	$8.79 \times 10^{-8}$	$1.11 \times 10^{-6}$	$8.76 \times 10^{-8}$
(0.9, 0.9)	$4.38 \times 10^{-7}$	$6.12 \times 10^{-8}$	$6.31 \times 10^{-7}$	$6.13 \times 10^{-8}$	$3.37 \times 10^{-7}$	$6.12 \times 10^{-8}$

TABLE VI  
ABSOLUTE ERROR WITH DIFFERENT VALUES OF  $M, N$  WITH  $\gamma(x, t) = 2 - 0.2 \sin(x) \exp(-t)$  FOR EXAMPLE 4

$(x_i, t_i)$	$M = 2, N = 2$	$M = 3, N = 2$	$M = 5, N = 2$
(0, 0)	0	0	0
(0.1, 0.1)	$4.75 \times 10^{-6}$	$1.96 \times 10^{-9}$	$4.62 \times 10^{-11}$
(0.2, 0.2)	$2.66 \times 10^{-5}$	$1.17 \times 10^{-8}$	$3.42 \times 10^{-10}$
(0.3, 0.3)	$7.55 \times 10^{-5}$	$2.57 \times 10^{-8}$	$9.55 \times 10^{-10}$
(0.4, 0.4)	$5.80 \times 10^{-5}$	$8.67 \times 10^{-9}$	$1.95 \times 10^{-9}$
(0.5, 0.5)	$4.52 \times 10^{-5}$	$4.94 \times 10^{-8}$	$3.25 \times 10^{-9}$
(0.6, 0.6)	$1.84 \times 10^{-4}$	$3.71 \times 10^{-8}$	$4.47 \times 10^{-9}$
(0.7, 0.7)	$2.65 \times 10^{-4}$	$2.69 \times 10^{-7}$	$5.41 \times 10^{-9}$
(0.8, 0.8)	$2.38 \times 10^{-4}$	$3.29 \times 10^{-7}$	$5.80 \times 10^{-9}$
(0.9, 0.9)	$1.52 \times 10^{-4}$	$1.67 \times 10^{-7}$	$3.84 \times 10^{-9}$
(1, 1)	$1.77 \times 10^{-18}$	$1.77 \times 10^{-18}$	$1.77 \times 10^{-18}$

TABLE VII  
ABSOLUTE ERRORS WITH VARIOUS VALUES OF  $\gamma(x, t)$  AND  $M$  FOR  $N = 2$  OF EXAMPLE 4

$(x_i, t_i)$	$\gamma(x, t) = 1.15$		$\gamma(x, t) = 1.85$		$\gamma(x, t) = 2$	
	$M = 2$	$M = 3$	$M = 2$	$M = 3$	$M = 2$	$M = 3$
(0.1, 0.1)	$1.19 \times 10^{-7}$	$1.39 \times 10^{-9}$	$1.37 \times 10^{-7}$	$1.94 \times 10^{-9}$	$1.47 \times 10^{-7}$	$1.83 \times 10^{-9}$
(0.3, 0.3)	$2.20 \times 10^{-6}$	$4.75 \times 10^{-8}$	$2.07 \times 10^{-6}$	$2.87 \times 10^{-8}$	$1.99 \times 10^{-6}$	$2.57 \times 10^{-8}$
(0.5, 0.5)	$2.00 \times 10^{-7}$	$2.33 \times 10^{-8}$	$1.27 \times 10^{-7}$	$3.78 \times 10^{-8}$	$1.04 \times 10^{-7}$	$5.31 \times 10^{-8}$
(0.7, 0.7)	$1.15 \times 10^{-5}$	$3.66 \times 10^{-7}$	$1.16 \times 10^{-5}$	$2.89 \times 10^{-7}$	$1.17 \times 10^{-5}$	$2.63 \times 10^{-8}$
(0.9, 0.9)	$7.69 \times 10^{-6}$	$1.21 \times 10^{-7}$	$7.56 \times 10^{-6}$	$1.54 \times 10^{-7}$	$7.43 \times 10^{-6}$	$1.72 \times 10^{-7}$

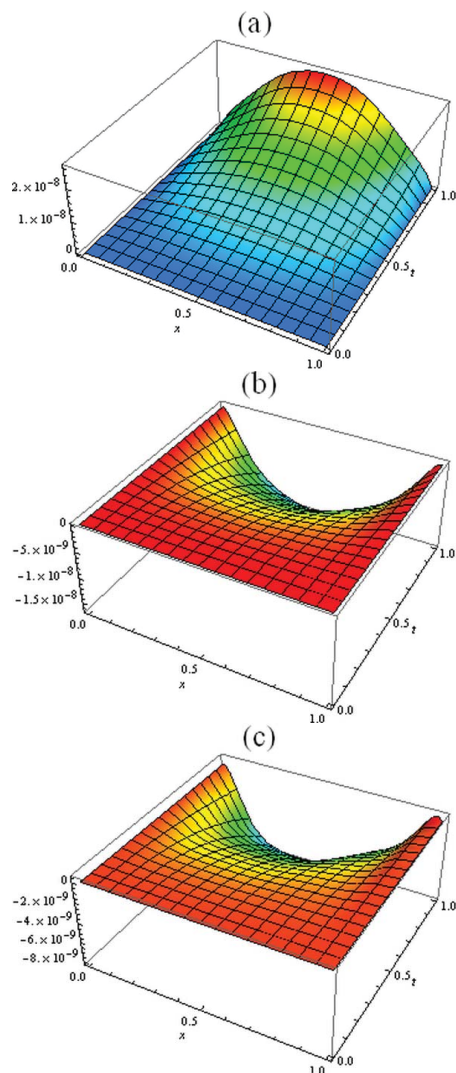


Fig. 2 Errors between the exact and approximation solutions for (a)  $\gamma(x, t) = 1.15$ , (b)  $\gamma(x, t) = 1.85$ , (c)  $\gamma(x, t) = 2$  with  $M = 5, N = 2$  for Example 4

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