

# An Approximation Method for Three Quark Systems in the Hyper-Spherical Approach

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**Abstract**—The bound state energy of three quark systems is studied in the framework of a non-relativistic spin independent phenomenological model. The hyper-spherical coordinates are considered for the solution of this system. According to Jacobi coordinate, we determined the bound state energy for (uud) and (ddu) quark systems, as quarks are flavorless mass, and it is restricted that choice potential at low and high range in nucleon bag for a bound state.

**Keywords**—Adiabatic expansion, grand angular momentum, binding energy, perturbation, baryons.

## I. INTRODUCTION

IN connection with baryons, the hyper-spherical adiabatic harmonic formalism is an appropriate method for calculation of the bound state for three body systems. We will be interested in the present paper in the low energy region of QCD theory, in which quarks interact strongly to form bound states known as proton or neutron that are three quarks (uud, ddu) systems. There is a potential model approach to this subject, although it's less fundamental. It is proved to be very useful even in non-relativistic approximation for bound states of quarks [1]-[6]. For the potential model of this study, the central conditions have been the flavor independent of the potential chosen and the existence of a confining term.

So, we study the bound state of a nucleon within a non-relativistic spin independent phenomenological model, with a potential that has two small and large ranges according to Coulomb and strong interaction at a nucleon bag. In order to understand and solve of the three quark bound state energy we used the Schrodinger equation. The idea is to show with a Coulomb potential at low range that quarks have asymptotic freedom, and at high range with a strong potential that quarks are confinement in a nucleon shell. These potentials have the great advantage that allowed us to obtain numerical solutions for three quark systems spectra. We introduce the hyper-spherical coordinates with considering of the perturbation for the three body problem. In order to do this, the Hamiltonian of the system in Jacobi coordinates ( $r, R$ ), interacting via the Coulomb and linear (strong) force shows in Fig. 1. The hyper-spherical method allows one to separate the hyper-radial motion from the angular part. As the physical boundary conditions for their solution can be easily formulated, and we can explain the behavior of potentials for small hyper-radii that leading to the numerical results as confirmed at discussion

and conclusion.

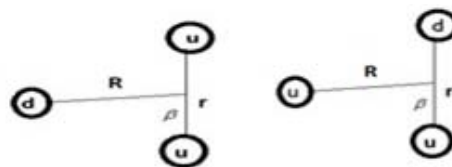


Fig. 1 The scheme of Jacobi coordinate used in three quark problems in the proton.

## II. DESCRIPTION OF THE THREE QUARK SYSTEMS

The hyper-spherical adiabatic approach will be reasonable to explain this three quark systems with equal mass. This method is based on expansion of the three body wave function into the surface functions.

The motion of three particles in their centre-of-mass (CM) system can be described as Jacobi-Coordinates ( $R, r$ ). According to the Hamiltonian operator  $H = T + V$ .

The kinetic-energy operator,  $T$ , is given by

$$T = -\frac{\hbar^2}{2m} \Delta_r - \frac{\hbar^2}{2M} \Delta_R \quad (1)$$

where  $\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$  is the reduced mass between two quarks

that are at  $r$  distance, and  $\frac{1}{M} = \frac{1}{M_1 + M_2} + \frac{1}{M_3}$  is the reduced

mass between 3th quark from two quark centre-of-mass.

Also  $\beta = \frac{m_2}{m_1 + m_2}$  where,  $m_i$  with its index refer to the mass of quarks.

Now, we introduce two kinds of potentials for the quarks in the baryons. The first one is the Coulomb potential that is large at short-range interaction when the quarks have asymptotic freedom, and the second one is the linear (strong) potential that is large at long-range interaction when the quarks are confinement in the nucleon bag. According to Fig. 2, in room 1, for the three-quark system the Hamiltonian has the form:

$$H_1 = -\frac{\hbar^2}{2m} \Delta_r - \frac{\hbar^2}{2M} \Delta_R + k \frac{q_1 q_2}{|r_2 - r_1|} + k \frac{q_1 q_3}{|r_1 - r_3|} + k \frac{q_2 q_3}{|r_2 - r_3|}, k = \frac{1}{4\pi\epsilon_0} \quad (2)$$

Or

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$$H_1 = \frac{\hbar^2}{2m} \left\{ -\Delta_r - \alpha^2 \Delta_R \right\} + k \frac{q_1 q_2}{r} + k \frac{q_1 q_3}{|R - \beta r|} + k \frac{q_2 q_3}{|R + \gamma r|} \quad (3)$$

where the Coulomb potential is dominant,  $\gamma = 1 - \beta$  and  $\alpha = \sqrt{\frac{m}{M}}$ . Here  $q_i$  denote the charge of quarks as,  $q_u = \frac{2}{3}e$ ,  $q_d = -\frac{1}{3}e$  and  $e$  is the electric charge unit. Also, in room 2, for the far away point (especially more than 0.1 Bohr radii) we use the linear potential, as we have:

$$H_2 = \frac{\hbar^2}{2m} \Delta_r - \frac{\hbar^2}{2M} \Delta_R + \kappa \left( |r_1 - r_2| + |r_1 - r_3| + |r_2 - r_3| \right) \quad (4)$$

or

$$H_2 = \frac{\hbar^2}{2m} \left\{ \Delta_r - \alpha^2 \Delta_R \right\} + \kappa \left( r + |R - \beta r| + |R + \gamma r| \right) \quad (5)$$

where  $\kappa$  is an electrical potential coefficient, effects inside the baryons bag and make the conditions close to the real conditions as it is only possible to determine from the numerical method. However, in this problem we select two quarks as a measurement for calculation of Bohr radii and energy unite.

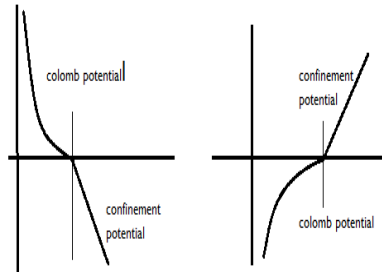


Fig. 2 The plan of our problem in hyper- spherical coordinate and two room positions determined

Now, we introduce dimensionless variables according to

$$\Delta_r = \frac{\partial^2}{\partial r^2}, \alpha^2 \Delta_R = \frac{\partial^2}{\partial \left( \frac{R}{a} \right)^2}, x = \frac{r}{a}, y = \frac{R}{a a} \quad (6)$$

Here  $a = \frac{a_0}{|\frac{2}{3} - \frac{1}{3}|}$  and  $a_0$  being the Bohr radius.

Then convert these variables with respect to the Hamiltonian hyper-spherical coordinates. By defining the hyper-radius and the hyper angle instead of the variables  $x$  and  $y$ , we have

$$|\vec{x}_i| = \rho \cos \omega_i, |\vec{y}_i| = \rho \sin \omega_i, \vec{X}_i = \frac{\vec{x}_i}{|\vec{x}_i|}, \vec{Y}_i = \frac{\vec{y}_i}{|\vec{y}_i|} \quad (7)$$

As notations  $X$  and  $Y$  are used for  $X_i = (\theta_{x_i}, \varphi_{x_i})$ ,  $Y_i = (\theta_{y_i}, \varphi_{y_i})$  respectively with  $0 \leq \rho \leq \infty$  and  $0 \leq \omega_i \leq \frac{\pi}{2}$ .

In the hyper spherical coordinate that are very useful for dealing with the three- body problem, we define  $(\rho, \Omega_i) = (\rho, \omega_i, \vec{x}_i, \vec{y}_i)$ . The kinetic energy of the Hamiltonian can be written as a Laplacian in a 6-dimensional space, due to the symmetry in the two Jacobi vectors, as:

$$T = \frac{\hbar^2}{2m} \left( -\rho^{-5/2} \frac{\partial^2}{\partial \rho^2} \rho^{5/2} + \frac{\Lambda^2}{\rho^2} \right) \quad (8)$$

where,

$$-\Delta_x - \Delta_y = -\rho^{-5/2} \frac{\partial^2}{\partial \rho^2} \rho^{5/2} + \frac{\Lambda^2}{\rho^2}$$

Also the grand angular momentum operators given by [2],

$$\Lambda^2 = -\frac{1}{\sin \omega_i \cos \omega_i} \frac{\partial^2}{\partial \omega_i^2} \sin \omega_i \cos \omega_i + \frac{L_{x_i}^2}{\cos^2 \omega_i} + \frac{L_{y_i}^2}{\sin^2 \omega_i} - \frac{1}{4}$$

where

$$L_{x_i} = x_i \times \frac{1}{i} \nabla_{x_i}, L_{y_i} = y_i \times \frac{1}{i} \nabla_{y_i}$$

Here,  $L_{x_i}$  and  $L_{y_i}$  are the orbital angular momentum operators corresponding to the variables  $x$  and  $y$ , as for the coulomb potential part we can write:

$$V_c = \left( \frac{q_1 q_2}{\rho \cos \omega_i} + \frac{q_1 q_3}{|\alpha \rho \sin \omega_i - \beta \rho \cos \omega_i|} + \frac{q_2 q_3}{|\alpha \rho \sin \omega_i + \gamma \rho \cos \omega_i|} \right) \quad (9)$$

and for the linear potential part

$$V_L = \kappa \left( \rho \cos \omega_i + |\alpha \rho \sin \omega_i - \beta \rho \cos \omega_i| + |\alpha \rho \sin \omega_i + \gamma \rho \cos \omega_i| \right) \quad (10)$$

The five angles symbolized by  $(\rho, \Omega_i) = (\rho, \omega_i, \vec{x}_i, \vec{y}_i)$  together with hyper radius  $\rho = \sqrt{x_i^2 + y_i^2}$ , (it is a measure for the three body system) provide a complete set of variables for describing the positions of all three quarks.

### III. THE HYPER-SPHERICAL ADIABATIC APPROACH

With respect to the hyper spherical adiabatic coordinates [7]-[10], the Schrodinger equation has the following form:

$$\frac{\hbar^2}{2m} \left( -\rho^{-5/2} \frac{\partial^2}{\partial \rho^2} \rho^{5/2} + \frac{\Lambda^2}{\rho^2} \right) \psi + (V_C + V_L) \psi = E \psi \quad (11)$$

We choice of the hyper spherical coordinates for parameterize the internal degrees of freedom of the system under study and the expansion of its wave function into a complete set of hyper angular adiabatic. In this approach the solution of the Schrodinger equation is divided in two steps, firstly one need to obtain the hyper angular adiabatic functions and its proper Eigen values, secondly one has to solve the infinite set of coupled one-dimensional differential equation for the hyper radial functions. The method consists in expanding the wave function of two particles in terms of a complete set of adiabatic Eigen function  $\psi(\rho, \Omega)$  which depend parametrically on the hyper -spherical radius. Here using surface functions as a basis for the solution of this Hamiltonian. The Eigen values, for this Hamiltonian with the fixed values of the hyper radius are:

$$H_\rho \Phi_n(\rho, \Omega) = U_n(\rho) \Phi_n(\rho, \Omega) \quad (12)$$

This equation is an angular differential part equation of the total Hamiltonian. Therefore, the Eigen value, known as Eigen potentials and its Eigen functions are called surface function. Since, surface functions form a complete set on the sphere of constant value of hyper radius; therefore we can expand the three bodies Eigen functions in the complete orthogonal set of surface function as bellows:

$$\Psi(\rho, \Omega) = \rho^{-5/2} \sum_{n=1}^{\infty} f_n(\rho) \Phi_n(\rho, \Omega) \quad (13)$$

By substituting this equation into (11), we gain coupled differential equations. Finally an infinite set of radial equations must be solved to obtain the energies, E. This radial equation is,

$$\left( -\frac{\partial^2}{\partial \rho^2} + U_n(\rho) - E \right) f_n(\rho) = \sum_{m=1}^{\infty} \left( Q_{nm} + 2P_{nm} \frac{\partial}{\partial \rho} \right) f_m(\rho) \quad (14)$$

The non-adiabatic coupling matrix element are given by,

$$Q_{nm}(\rho) = \int \Phi_n^*(\rho, \Omega) \frac{\partial^2}{\partial \rho^2} \Phi_m(\rho, \Omega) d\Omega \quad (15)$$

$$P_{nm}(\rho) = \int \Phi_n^*(\rho, \Omega) \frac{\partial}{\partial \rho} \Phi_m(\rho, \Omega) d\Omega \quad (16)$$

Here we introduce two kinds of approximations:

One is uncoupled adiabatic approximation (UAA):

$$\frac{\partial^2}{\partial \rho^2} f_n(\rho) = (U_n(\rho) - E^{UAA}) f_n(\rho). \quad (17)$$

Second is extreme adiabatic approximation (EAA):

$$\frac{\partial^2}{\partial \rho^2} f_n(\rho) = (U_n(\rho) - E^{EAA}) f_n(\rho). \quad (18)$$

Looking exactly as a radial two-body Schrodinger equation, where U act as an effective potential. For the calculation of the ground -state energies, we restrict our solution between  $E^{EAA}$  and  $E^{UAA}$  as:

$$E^{EAA} \leq E^{exact} \leq E^{UAA} \quad (19)$$

### IV. BEHAVIOR OF EIGEN POTENTIAL AND SURFACE FUNCTIONS FOR SMALL HYPER RADII

The main effort is to determine of the Eigen potential from (12). For small hyper radii, the grand angular momentum term  $\frac{\Lambda^2}{\rho^2}$  is dominated over the coulomb potential and the linear potential. Hence (12) is reduced to:

$$(\Lambda^2 - \rho^2 U_n(\rho)) \Phi_n(\rho, \Omega) = 0 \quad (20)$$

Comparing this relation with the Eigen value equation of the grand angular momentum operator

$$(\Lambda^2 - \ell(\ell+1)) Y_\ell(\Omega) = 0 \quad (21)$$

Give us,

$$U_n(\rho) \rightarrow \frac{\ell(\ell+1)}{\rho^2}$$

where

$$\Phi_n(\rho, \Omega) \rightarrow Y_\ell(\Omega)$$

Therefore

$$Y_\ell(\Omega) = N_\ell \cos^{l_{y_1}} \omega_1 \sin^{l_{y_2}} \omega_1 P_k^{(l_{y_1} + \frac{1}{2}, l_{y_2} + \frac{1}{2})}(\cos 2\omega_1) Y_{l_{y_1}, l_{y_2}}^{LM}(x_i, y_i)$$

$$N_\ell = \sqrt{\frac{2k! (l_x + l_y + 2k + 2)}{\Gamma(l_x + k + \frac{3}{2}) \Gamma(l_y + k + \frac{3}{2})}} \frac{(l_x + l_y + k + 2)}{\Gamma(l_x + k + \frac{3}{2}) \Gamma(l_y + k + \frac{3}{2})}$$

and

$$P_k^{(i,j)} = \frac{(x-1)^{-i} (x+1)^{-j}}{2^k k!} \frac{d^k}{dx^k} ((x-1)^{k+i} (x+1)^{k+j})$$

$$Y_{l_{x_i} l_{y_i}}^{LM}(x_i, y_i) = \sum_{m_{x_i} m_{y_i}} \langle l_{x_i} m_{x_i}, l_{y_i} m_{y_i} | LM \rangle Y_{l_{x_i} m_{x_i}}(x_i) Y_{l_{y_i} m_{y_i}}(y_i),$$

Here  $\langle l_{x_i} m_{x_i}, l_{y_i} m_{y_i} | LM \rangle$  are the Clebsch-Gordan coefficients and  $Y_{l_{x_i} m_{x_i}}, Y_{l_{y_i} m_{y_i}}$  is the usual spherical harmonic. The index  $[l]$  of the hyper-spherical harmonic denotes the set of quantum numbers  $[l] = \{k, l_x, l_y, L, M\}$ , as

the Eigen value is given by  $l = l_x + l_y + 2k + \frac{3}{2}$ . Therefore we

can consider second and third term in set (11), as a perturbative potential for grand angular momentum and finding the binding energy. As

$$H_\rho = \frac{\Lambda^2}{\rho^2} + \langle Y_\ell | V_C + V_L | Y_\ell \rangle \quad (22)$$

#### V.COMPUTATIONAL CALCULATION

Our main task is to find the bound state energy for this quark system in the nucleon. In the first step, we find the Eigen potential as Ueff and then substitute in (18). Now it is possible to solve this equation via numerical integration. In order to find our solution, we consider  $\kappa$  as a constant that make the below condition on the potentials. As for distance  $\rho \leq 0.1$  a the coulomb potential is dominantly than the linear potential and for  $\rho \geq 0.1$  a the linear potential is dominantly than the coulomb potential. And also for  $\rho=0.1$  a two portions are equal.

In this framework we treated the quark systems in the states with the total angular momentum  $L=0$  and used the numerical method based on the shooting method [10]. The procedure for numerical integration  $f_n(\rho)$  can be written as bellows:

- I. Specify Ueff, the step size, the initial position of hyper radii and the value of  $f_n(\rho_0)$  and  $(df/d\rho)_{\rho_0}$ .
- II. Have a trial guess at energy Eigen value E.
- III. A new integration "cycle" begins by using, (18) to compute  $(d^2 f/d\rho^2)_{\rho_0}$ .
- IV. For forward step, we compute  $f_n(\rho_0 + i\Delta\rho)$ ,  $(df/d\rho)_{\rho_0 + i\Delta\rho}$ ,  $(d^2 f/d\rho^2)_{\rho_0 + i\Delta\rho}$ .
- V. We choice E in a way that the boundary condition  $(\rho \rightarrow 1, f(\rho=1)=0)$  observe.

With respect to the properties of the three quark systems, we can determine the effective potentials with respect to the observed conditions for the nucleon quark systems in Tables I and II. Also we can see from Tables III and IV that solution obtained by shooting method can be determined the bound state energy in the nucleons for uud and udu (or ddu and dud). The corresponding Eigen potentials of three quark systems for  $L=0$  are shown in Figs. 3 and 5. Also, the radial wave function into hyper radii plot in Figs. 4 and 6 for finding bound state

energy in (uud, udu, ddu and dud) systems. The method of this research gives the lower bound of Eigen energy if we used the ground radial wave function in three quark systems with respect to shooting method.

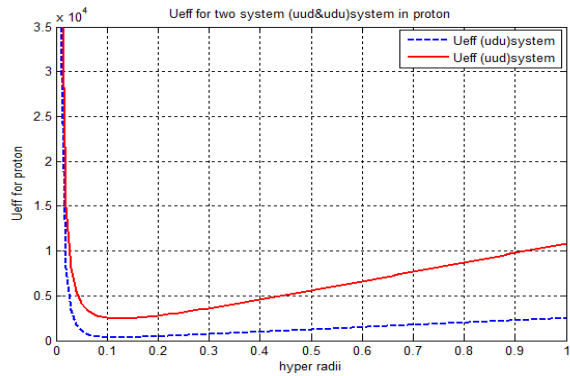


Fig. 3 Ueff for two kind of system (uu+d & ud+u) in proton

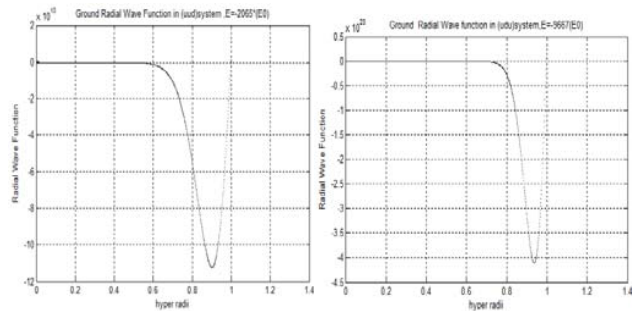


Fig. 4 The boundary condition observed at two systems for E defined in proton quark systems

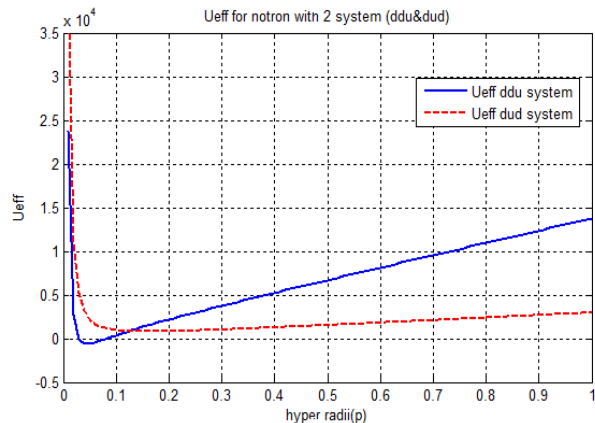


Fig. 5 Ueff for two kind of system (uu+d & ud+u) in proton

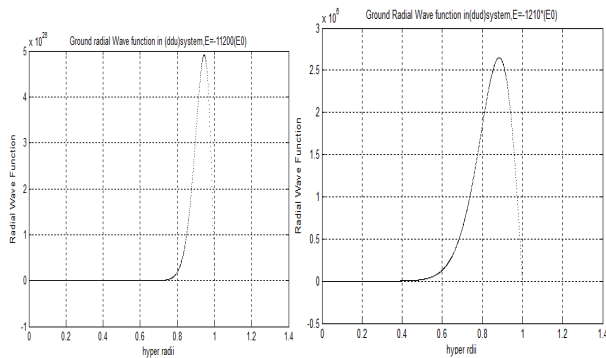


Fig. 6 The boundary condition observed at two systems for E defined in neutron quark systems

## VI. CONCLUSION

Within a nonrelativistic spin independent model with a global potential, we consider the bound state energy of three flavorless quark systems. The hyper- spherical method used for these systems and our calculations for the bound state energy is based on the shooting method. Although, the obtained results are not as good as the ones obtained for the global potential, but we think it give good estimate of the three quark system of nucleon with respect to the numerical calculations for two different potentials that one is the Coulomb potential at low- distance and the second is the linear potential at long- distance in the nucleon bag. At low-range distance, the linear potential is perturbation portion and for high- range distance at nucleon bag the Coulomb potential is perturbation portion. As it can be seen the numerical results obtained are only estimates for the three- quark system bound state.

TABLE I  
Ueff DETERMINED FOR TWO SYSTEMS IN PROTON

Proton	Ueff	observed condition
uud system	$U_{eff} = \frac{l(l+1)}{\rho^2} - (25.71)\frac{1}{\rho} + (1.78)\kappa\rho$	$\kappa=1444.38$
udu system	$U_{eff} = \frac{l(l+1)}{\rho^2} + (107.10)\frac{1}{\rho} + (15.65)\kappa\rho$	$\kappa=684.34$

TABLE II  
Ueff DETERMINED FOR TWO SYSTEMS IN NEUTRON

neutron	Ueff	observed condition
ddu system	$U_{eff} = \frac{l(l+1)}{\rho^2} - (125.71)\frac{1}{\rho} + (114.24)\kappa\rho$	$\kappa=110.04$
udd system	$U_{eff} = \frac{l(l+1)}{\rho^2} + (14.92)\frac{1}{\rho} + (6.02)\kappa\rho$	$\kappa=247.84$

TABLE III  
THE BOUND STATE ENERGY FOR TWO SYSTEMS IN PROTON

E(EAA)	uud system	udu system
proton	1.97 GeV	2.30 GeV

TABLE IV  
THE BOUND STATE ENERGY FOR TWO SYSTEMS IN NEUTRON

E(EAA)	ddu system	udu system
Neutron	0.66 GeV	0.28 GeV

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