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Alternative convergence analysis for a kind of singularly perturbed boundary value problems

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Abstract—A kind of singularly perturbed boundary value problems is under consideration. In order to obtain its approximation, simple upwind difference discretization is applied. We use a moving mesh iterative algorithm based on equi-distributing of the arc-length function of the current computed piecewise linear solution. First, a maximum norm a posteriori error estimate on an arbitrary mesh is derived using a different method from the one carried out by Chen [Advances in Computational Mathematics, 24(1-4) (2006), 197-212.]. Then, basing on the properties of discrete Green's function and the presented posteriori error estimate, we theoretically prove that the discrete solutions computed by the algorithm are first-order uniformly convergent with respect to the perturbation parameter ε.

Keywords—convergence analysis, Green's function, singularly perturbed, equi-distribution, moving mesh.

I. Introduction

NUMERICAL modeling of singularly perturbed problems is important and interesting in fluid mechanics, elastic mechanics, quantum mechanics, chemical reactions, and so on. The singularly perturbed problems are marked with a small perturbed parameter in the differential equation. The need for accurate solutions to them challenges numerical analysts to design new methods.

We consider a kind of nonconservative singularly perturbed two-point boundary value problems (the same form as in [1]) in fluid dynamics

$$Tu(x) := -\varepsilon u''(x) - p(x)u'(x) = f(x), x \in (0,1), \quad (1)$$
$$u(0) = 0, \quad u(1) = 0, \quad (2)$$

where ε is a constant of diffusion satisfying $0 < \varepsilon \leq 1$. We assume that $f(x)$ is sufficiently smooth. It is also assumed that $p(x) \in C^1[0,1]$ and that there are constants β and $\overline{\beta}$ such that

$$0 < \beta \leq p(x) \leq \overline{\beta}, \quad \text{and} \quad |p'(x)| \leq \overline{\beta}, \quad \forall x \in [0,1]. \quad (3)$$

For small values of ε, the problem (1)-(2) possesses a thin boundary layer of order $\mathcal{O}(\varepsilon)$ at $x = 0$; the solution u will vary rapidly in the layer region near the boundary. This boundary layer causes various difficulties in seeking the numerical solution of (1)-(2). It is well-known that conventional numerical methods for (1)-(2) can produce approximate solutions with oscillations that are unbounded when $\varepsilon \to 0$, and specialized methods, such as upwind scheme need to be adapted. One conclusion from the study in this paper is that the upwind scheme still works reasonably well if the grid is properly adapted so that the sharp boundary layer presented in the solution will be fully resolved.

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To obtain such a properly adapted grid, based on equidistributing a positive monitor function associated with the numerical solution (see [1], [3], [4], [5], [6], [7], [8] and so on), we consider the same moving mesh iterative algorithm as [9]. The mesh constructed by the algorithm has a fixed number of nodes and more nodes concentrate in the boundary layer. The simple upwind scheme is used in the algorithm. We show the computed solutions are ε-uniform convergence of first-order using the theory of discrete Green's functrion. The proof is different from and more general than the one in [2]. The result is supported by a lot of numerical experiments in [9].

Below a brief description of the moving mesh iterative algorithm is given.

step 1 Initialize mesh: The initial mesh $\omega = \{0, \frac{1}{N}, \frac{2}{N}, \cdots, 1\}$ is uniform.

step 2: For $k = 0, 1, \cdots$, assuming that the mesh $\{x_i^{(k)}\}$ is given, compute the discrete solution $\{u_i^{(k)}\}$ on $\{x_i^{(k)}\}$ satisfying

$$\begin{aligned} T^N u_i^{(k)} &:= -\varepsilon D D^- u_i^{(k)} - p_i D^+ u_i^{(k)} = f_i^{(k)}, \\ &\qquad\qquad 1 \leq i \leq N-1, \\ u_0^{(k)} &= 0, \quad u_N^{(k)} = 0, \end{aligned} \quad (4)$$

where T^N is the same simple upwind scheme as the one in [2], $p_i = p(x_i^{(k)})$, $f_i^{(k)} = f(x_i^{(k)})$, $h_i^{(k)} = x_i^{(k)} - x_{i-1}^{(k)}$, $\hbar_i^{(k)} = \frac{h^{(\ )} + h^{(\ )}_{+1}}{2}$ $\quad i = 1, \cdots, N$, and

$$D^+ u_i^{(k)} = \frac{u_{i+1}^{(k)} - u_i^{(k)}}{h_{i+1}^{(k)}}, \quad D^- u_i^{(k)} = \frac{u_i^{(k)} - u_{i-1}^{(k)}}{h_i^{(k)}},$$
$$D u_i^{(k)} = \frac{u_{i+1}^{(k)} - u_i^{(k)}}{\hbar_i^{(k)}}.$$

Let

$$\begin{aligned} l_i^{(k)} &= h_i^{(k)} \sqrt{1 + (D^- u_i^{(k)})^2} \\ &= \sqrt{(u_i^{(k)} - u_{i-1}^{(k)})^2 + (h_i^{(k)})^2}. \end{aligned} \quad (5)$$

be the arc-length between the points $(x_{i-1}^{(k)}, u_{i-1}^{(k)})$ and $(x_i^{(k)}, u_i^{(k)})$ in the piecewise linear computed solution $u^{(k)}(x)$. Then the total arc-length of the solution curve $u^{(k)}(x)$ is

$$\begin{aligned} L^{(k)} &:= \sum_{i=1}^{N} l_i^{(k)} = \sum_{i=1}^{N} h_i^{(k)} \sqrt{1 + (D^- u_i^{(k)})^2} \\ &= \sum_{i=1}^{N} \sqrt{(u_i^{(k)} - u_{i-1}^{(k)})^2 + (h_i^{(k)})^2}. \end{aligned} \quad (6)$$

step 3 Mesh test: Let $c_0 = 2$. If

$$\frac{\max_i l_i^{(k)}}{L^{(k)}} \leq \frac{c_0}{N} \quad (7)$$

then go to step 5, else continue to step 4.
step 4: Generate a new mesh by equi-distributing arc-length of current computed solution:
Choose new points $0 = x_0^{(k+1)} < x_1^{(k+1)} < \cdots < x_N^{(k+1)} = 1$, such that the distance from $(x_{i-1}^{(k+1)}, u^{(k)}(x_{i-1}^{(k+1)}))$ to $(x_i^{(k+1)}, u^{(k)}(x_i^{(k+1)}))$, $(i = 1, \cdots, N)$, measured along the solution curve $u^{(k)}(x)$, is $L^{(k)}/N$. Then return to step 2.
step 5: $\{x_0^*, x_1^*, \cdots, x_N^*\} = \{x_0^{(k+1)}, x_1^{(k+1)}, \cdots, x_N^{(k+1)}\}$, $\{u_0^*, u_1^*, \cdots, u_N^*\} = \{u_0^{(k)}, u_1^{(k)}, \cdots, u_N^{(k)}\}$. Stop the algorithm.

The rest of the paper is organized as follows. Stability properties of differential operators are given in section II. In section III, We present the convergence analysis of the algorithm.

Notation:
In the estimates, we use the maximum norm by $||v(x)||_\infty = ess \sup_{x\in[0,1]} |v(x)|$. Throughout the paper, C, sometimes subscripted, denotes a generic positive constant that is independent of ε and any mesh used.

II. Stability properties of the differential operator

We shall use the result of Lemma 2.1 in [2], which can be stated as follows.

Lemma 2.1: If $p(x) \in C^1([0,1])$ satisfies (3) and $f(x) = -F'(x)$, where $F(x)$ is a bounded piecewise continuous function, then there exists a unique weak solution $u(x) \in C[0,1]$ of (1)-(2) and

$$||u(x)||_\infty \le C\,||T\,u(x)||_*\,, \tag{8}$$

where

$$\begin{aligned} ||f(x)||_* &= \min_{F:\,F'=f} ||F(x)||_\infty \\ &= \min_{c\,\in\,R} \left\| \int_x^1 f(t)dt + c \right\|_\infty . \end{aligned} \tag{9}$$

Lemma 2.2: For any $v(x), w(x) \in H^1(0,1)$ such that $v(0) = w(0)$, $v(1) = w(1)$ and

$$T\,v(x) - T\,w(x) = -F'(x),$$

where $F(x)$ is a bounded piecewise continuous function, we have

$$||v(x) - w(x)||_\infty \le C\,||T\,v(x) - T\,w(x)||_*\,. \tag{10}$$

Proof: Using the equality $T\,v(x) - T\,w(x) = T\,[v(x) - w(x)]$ and Lemma 2.1 , we easily get the desired result. ■

Corollary 2.1: Let $u(x)$ be the solution of (1)-(2), u_i^N be the solution of (4) on an arbitrary nonuniform mesh and U^I be its piecewise linear interpolant, we have

$$||u(x) - U^I(x)||_\infty \le C_1\,||T\,u(x) - T\,U^I(x)||_*\,. \tag{11}$$

Remark 2.1: In the analysis, we use a piecewise linear interpolation of numerical solutions instead of a piecewise quadratic interpolation in [2].

III. Convergence analysis of the iterative algorithm

A. A posterior error estimate on an arbitrary mesh

Here, we present another error estimate method, different from the one in [2] but with the same accuracy and easier to extend to the more general case than the problem (1)-(2).

First, we introduce the continuous and discrete operators and functions

$$A\,v(x) := \varepsilon\,v(x)' + p(x)\,v(x) + \int_x^1 p'(s)\,v(s)ds,$$

$$F(x) := \int_x^1 f(s)ds;$$

$$A^N\,v_i := \varepsilon\,D^- v_i + \frac{\hbar_i}{h_{i+1}} p_i\,v_i + \sum_{k=i}^{N-1} \hbar_k\,D^+ p_k v_{k+1},$$

$$F_i^N := \sum_{k=i}^{N-1} \hbar_k f_k.$$

Note that $T\,v(x) = -(A\,v(x))'$, $f(x) = -F'(x)$ on $(0,1)$ and $T^N\,v_i = -DA^N\,v_i$, $f_i = -DF_i^N$ on ω. Thus,

$$\begin{array}{l} A\,u(x) - F(x) \equiv \alpha \quad \text{on} \quad (0,1), \\ A^N\,u_i^N - F_i^N \equiv a \quad \text{on} \quad \omega \end{array} \tag{12}$$

with constants α and a.

Let U^I be the piecewise linear interpolant of the solution u_i^N to (4). By the inequality (11) and the definition of the norm (9), we have

$$\begin{aligned} ||u(x) - U^I(x)||_\infty &\le C_1\,||T(u(x) - U^I(x))||_* \\ &= C_1 \min_{c\in R} ||A(u(x) - U^I(x)) + c||_\infty . \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\min_{c\in R} ||A(u(x) - U^I(x)) + c||_\infty \\ &\quad \le ||A(u(x) - U^I(x)) + a - \alpha||_\infty , \end{aligned}$$

where a and α are the constants from (12). Also, for any $x \in (x_{i-1}, x_i) \subset (0,1) \setminus \omega$, we have

$$\begin{aligned} &A(u(x) - U^I(x)) + a - \alpha \\ &\quad = A^N\,u_i^N - F_i^N - A\,U^I(x) + F(x). \end{aligned}$$

Next, we need to bound the error $A^N\,u_i^N - F_i^N - A\,u^N(x_{i-1/2}) + F(x_{i-1/2})$, where $x_{i-1/2} = (x_i + x_{i-1})/2 \in (x_{i-1}, x_i)$.

Since $(U^I(x))' = D^- u_i^N$, $\forall x \in (x_{i-1}, x_i)$, using the definitions of A and A^N and integrating by parts, we have

$$\begin{aligned} &A^N\,u_i^N - A\,U^I(x_{i-1/2}) \\ &\quad = \textstyle\sum_{k=i}^{N-1} \hbar_k\,D^+ p_k\,u_{k+1}^N - \int_{x\ -1\ 2}^1 p'(s)\,U^I(s)ds \\ &\qquad + \frac{\hbar}{h\ +1} p_i\,u_i^N - p(x_{i-1/2}) U^I(x_{i-1/2}). \end{aligned}$$

For the terms on the right hand side of the above equation, we obtain the bounds

$$\begin{aligned} &\left| \frac{h}{2}\,D^+ p_k\,u_{k+1}^N - \int_{x\ -1\ 2}^x p'(s)\,U^I(s)ds \right| \\ &\quad \le Ch_k ||p'(x)||_\infty |u_{k+1}^N - u_k^N|, \end{aligned}$$

$$\left|\frac{h_{k+1}}{2} D^+p_k\, u_{k+1}^N - \int_{x_k}^{x_{k+1/2}} p'(s)\, U^I(s)ds\right| \le Ch_{k+1}||p'(x)||_\infty |u_{k+1}^N - u_k^N|,$$
$$\left|\frac{\hbar_i}{h_{i+1}} p_i\, u_i^N - p(x_{i-1/2}) u^N(x_{i-1/2})\right| \le C||p(x)||_\infty \max_{k=1,\cdots,N} h_k |u_k^N|,$$

where the relation $h_k \ge Q$ for all k (see [4], Theorem 3.1) is used in the last estimate.

Note that the matrix associated with T^N is an M-matrix. By the comparison principle, we see that for $k = 0, \cdots, N$, the inequality $|u_k^N| \le C \max_{k=1,2,\cdots,N-1} |f_k|/\beta$ is held. It is obvious that

$$\left|\frac{\hbar_i}{h_{i+1}} p_i\, u_i^N - p(x_{i-1/2}) u^N(x_{i-1/2})\right| \le C_7||p(x)||_\infty \max_{k=1,\cdots,N} h_k.$$

Thus

$$|A^N u_i^N - A\, U^I(x)| \le C_6||p'(x)||_\infty \max_{k=0,\cdots,N-1} |u_{k+1}^N - u_k^N| + C_7||p(x)||_\infty \max_{k=0,\cdots,N-1} h_{k+1}.$$

It remains to bound $F_i^N - F(x_{i-1/2}) = \sum_{k=i}^{N-1} \hbar_k f_k - \int_{x_{i-1/2}}^1 f(s)ds$. Combining the following two estimate

$$\left|\frac{h_k}{2} f_k - \int_{x_{k-1/2}}^{x_k} f(s)ds\right| \le Ch_k^2\, ||f'(x)||_\infty,$$
$$\left|\frac{h_{k+1}}{2} f_k - \int_{x_k}^{x_{k+1/2}} f(s)ds\right| \le Ch_{k+1}^2\, ||f'(x)||_\infty,$$

we obtain

$$|F_i^N - F(x_{i-1/2})| \le C_8||f'(x)||_\infty \max_{k=0,\cdots,N-1} h_{k+1}.$$

After a minor amount of collecting terms, we find that

$$||T\,(u(x) - U^I(x))||_* \le C_2 \max_{k=0,\cdots,N-1} |u_{k+1}^N - u_k^N| + C_3 \max_{k=0,\cdots,N-1} h_{k+1} \tag{13}$$

with the constants $C_2 = C_6||p'(x)||_\infty, \quad C_3 = C_7||p(x)||_\infty + C_8||f'(x)||_\infty$.

Finally, using (11) and (13), we get the main result of this section.

Theorem 3.1: Let $u(x)$ be the solution of (1)-(2), u_i^N be the solution of (4) on an arbitrary nonuniform mesh and U^I be its piecewise linear interpolant. We have

$$\begin{aligned} ||U^I(x) - u(x)||_\infty &\le C_1\,(C_2 \max_{k=1,\cdots,N} |u_k^N - u_{k-1}^N| + C_3 \max_{k=1,\cdots,N} h_k) \\ &\le C_4 \max_{k=1,\cdots,N} \sqrt{(u_k^N - u_{k-1}^N)^2 + h_k^2}. \end{aligned}$$

B. Accuracy of the numerical solution computed when the algorithm terminates

First, we estimate the bound on the length of the polygonal solution curve, which is crucial for the following convergence analysis.

Lemma 3.1: Let $\{u_i^N\}$ be the solution of (4) on an arbitrary mesh $\{x_i\}$. Let L^N be the total arc-length along the solution curve $u^N(x)$. Then

$$1 \le L^N \le C_5\,.$$

Proof: The process is similar to but different from the proof of Lemma 3.2 in [2]. From (6), we know $L^N = \sum_{i=1}^N h_i \sqrt{1 + |D^- u_i^N|^2} \ge \sum_{i=1}^N h_i = 1$. Next we only need to prove the upper bound of L^N.

For $j = 1, \cdots, N-1$, we define the discrete Green's function $G(x_i, x_j)$ with respect to the operator T^N (with the Dirichlet boundary condition) defined in (4) associated with the point x_j by

$$\begin{cases} T^N G(x_i, x_j) = \frac{\delta_{ij}}{\hbar_i}, & i = 1, \cdots, N-1, \\ G(0, x_j) = G(1, x_j) = 0, \end{cases} \tag{14}$$

where the Kronecker function δ_{ij} is 1 if $i = j$ and 0 otherwise. Then for each i, we have

$$u_i^N = \sum_{j=1}^{N-1} \hbar_i\, G(x_i, x_j)\, f_j. \tag{15}$$

It is clear that T^N is an M-matrix. Apply the M-matrix theory and the comparison function $v = 0$ and

$$w = \frac{2}{\beta} \begin{cases} 1, & \text{if} \quad 0 \le i \le j \le N \\ \prod_{k=j+1}^{i} \left(1 + \frac{\beta h_k}{2\varepsilon}\right)^{-1}, & \text{if} \quad 0 \le j < i \le N \end{cases}$$

to conclude that

$$0 \le G(x_i, x_j) \le 2/\beta, \quad 1 \le i \le N, 1 \le j \le N-1. \tag{16}$$

Fix $j \in \{1, 2, \cdots, N-1\}$, from (14) and (16) we have

$$T^N G(x_i, x_j) = 0, \quad i = 1, \cdots, j-1,$$
$$G(0, x_j) = 0, \quad G(x_j, x_j) \ge 0.$$

$G(x_i, x_j)$ is an increasing function of i for $i = 1, \cdots, j$. So it follows from (16) that

$$\sum_{i=1}^{j} |G(x_i, x_j) - G(x_{i-1}, x_j)| = G(x_j, x_j) \le 2/\beta. \tag{17}$$

Similarly $G(x_i, x_j)$ is a decreasing function of i for $i = j+1, \cdots, N$. So

$$\sum_{i=j+1}^{N} |G(x_i, x_j) - G(x_{i-1}, x_j)| = G(x_j, x_j) \le 2/\beta. \tag{18}$$

The inequality (17) plus (18) yields

$$\sum_{i=1}^{N} |G(x_i, x_j) - G(x_{i-1}, x_j)| \le 4/\beta. \tag{19}$$

Because

$$\begin{aligned} L^N = \sum_{i=1}^N h_i \sqrt{1 + |D^- u_i^N|^2} &\le \sum_{i=1}^N h_i\,(1 + |D^- u_i^N|) \\ &= 1 + \sum_{i=1}^N |u_i^N - u_{i-1}^N|, \end{aligned}$$

Then, using (15) and (19), we get

$$\begin{aligned} L^N &\leq 1 + \sum_{i=1}^{N}\sum_{j=1}^{N-1} \hbar_j |f_j| \cdot |G(x_i, x_j) - G(x_{i-1}, x_j)| \\ &\leq 1 + ||f||_\infty \sum_{j=1}^{N-1} \hbar_j \sum_{i=1}^{N} |G(x_i, x_j) - G(x_{i-1}, x_j)| \\ &\leq 1 + \frac{4||f||_\infty}{\beta} \\ &=: C_5. \end{aligned}$$

Thus, we complete the proof.

After that, we obtain the following convergence result:

Theorem 3.2: Suppose that the algorithm (the simple upwind difference approximation is applied) reaches its stopping criterion and halts. Let the final mesh generated be $\{x_i^*\}$. Let $u(x)$ be the solution of (1)-(2). Let $\{u_i^*\}$ be the discrete solution for (4) computed on this mesh. Let U^I be the piecewise linear interpolation of $\{(x_i^*, u_i^*)\}$. Then we have

$$\max_{0 \leq i \leq N} |u(x_i) - u_i^*| \leq CN^{-1}.$$

Proof: Let

$$\begin{aligned} l_i^* &= \sqrt{(u_i^* - u_{i-1}^*)^2 + (x_i^* - x_{i-1}^*)^2} \\ &= \sqrt{(u_i^* - u_{i-1}^*)^2 + (h_i^*)^2} \end{aligned}$$

be the arc-length between successive knots in the polygonal-computed solution on the final generated mesh and L^N is the total arc-length along the computed solution curve. By Lemma 3.1, $1 \leq L^N \leq C_5$ is held. Therefore

$$\begin{aligned} |u(x_i) - u_i^*| &\leq ||U^I(x) - u(x)||_\infty \\ &\leq C_4 \max_{i=1,\cdots,N} \sqrt{(u_i^* - u_{i-1}^*)^2 + (h_i^*)^2} \\ &\leq C_4 \max_{1 \leq i \leq N} l_i^* \\ &\leq C_4 \frac{c_0}{N} L^N \\ &\leq C_4\, c_0\, C_5\, N^{-1}, \end{aligned}$$

where Theorem 3.1, (7) and the definition of l_i^* are used.

Remark 3.1: Theorem 3.2 suggests that our moving mesh iterative algorithm can generate first-order uniformly convergent approximations when the simple upwind scheme is applied, which coincide with the results in [2].

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References

[1] G. Beckett and J. A. Mackenzie, *Convergence analysis of finite difference approximations on equidistributed grids to a singularly perturbed boundary value problem*, Appl. Numer. Math., 35 (2000), 87-109.

[2] Y. Chen, *Uniform convergence analysis of finite difference approximations for singularly perturbed problems on an adapted grid*, Advances in Computational Mathematics, 24(1-4) (2006), 197-212.

[3] N. Kopteva, *Maximum norm a posteriori error estimates for a one-dimensional convection-diffusion problem*, SIAM J. Numer. Anal., 39 (2001), 423-441.

[4] N. Kopteva and M. Stynes, *A robust adaptive method for a quasi-linear one dimensional convection-diffusion problem*, SIAM J. Numer. Anal., 39(4) (2001), 1446-1467.

[5] T. Linss, *Uniforming pointwise convergence of finite difference schemes using grid equidistribution*, Computing, 66 (2001), 27-39.

[6] J. Mackenzie, *Uniform convergence analysis of an upwind finite-difference approximation of a convection-diffusion boundary value problem on an adaptive grid*, IMA J. Numer. Anal., 19 (1999), 233-249.

[7] Y. Qiu and D. M. Sloan, *Analysis of difference approximations to a singularly perturbed two-point boundary value problem on an adaptively generated grid*, J. Comput. Appl. Math., 101 (1999), 1-25.

[8] Y. Qiu, D. M. Sloan, and T. Tang, *Numerical solution of a singularly perturbed two-point boundary value problem using equidistribution: analysis of convergence*, J. Comput. Appl. Math., 116 (2000), 121-143.

[9] J. Yang and Y. Chen, *A moving mesh method for a singularly perturbed convection-diffusion boundary value problem (in Chinese)*, Natural science journal of xiangtan university, 26(3) (2004), 24-29.