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Alternating Implicit Block FDTD Method For Scalar Wave Equation

N. M. Nusi, M. Othman, M. Suleiman, F. Ismail and N. Alias

Abstract—In this paper, an alternating implicit block method for solving two dimensional scalar wave equation is presented. The new method consist of two stages for each time step implemented in alternating directions which are very simple in computation. To increase the speed of computation, a group of adjacent points is computed simultaneously. It is shown that the presented method increase the maximum time step size and more accurate than the conventional finite difference time domain (FDTD) method and other existing method of natural ordering.

Keywords—FDTD, Scalar wave equation, alternating direction implicit (ADI), alternating group explicit (AGE), asymmetric approximation.

I. Introduction

Finite difference time domain (FDTD) method is one of the most commonly numerical methods used for solving various type of electromagnetic problems [1]. Recently, a reduced scalar version of the FDTD method was developed by Aoyagi et. al [2] in source free region. In comparison with the FDTD method, the new version called the scalar wave equation finite difference time domain (WE-FDTD) requires less computation and storage. As both the FDTD and WE-FDTD methods are based on an explicit finite difference algorithm, the Courant-Friedrichs-Lewy (CFL) condition must be satisfied. A maximum time step size is limited by the minimum cell size in a computational domain. To overcome this problem, implicit methods must be employed which have no limit on the time-step size arising from the stability consideration. Alternating direction implicit (ADI) methods are mostly investigated because of their unconditionally stability and high efficiency for solving higher dimensional problems.

The ADI methods was first introduced by Peaceman, Rachford and Douglas in ([4],[3]) for heat equations in two dimensions and was later applied extensively in many numerical approximation problems. Since the ADI methods only need to solve a sequence of tridiagonal linear systems therefore the methods save much computer memory and CPU time. Due to high computational cost especially for large scale problems, Evans [3] has developed a group explicit strategy for solution to the partial differential equations. The strategy is based on stable fully explicit asymmetric schemes introduced by Saul'yev [5], which coupled in groups of adjacent points on the grid. The groups of points are solved in implicit equations which can be easily converted to explicit form. Combining the ADI methods with group explicit strategy would increase the efficiency such as discussed in ([6],[7]).

N.M. Nusi is with the Department of Engineering Science and Mathematics, College of Engineering, Universiti Tenaga Nasional, Putrajaya Campus, Jalan Ikram-UNITEN, 43000 Kajang, Selangor, Malaysia (mnoraini@uniten.edu.my).
M. Othman is with the Department of Communication Technology and Network, Faculty of Computer Science and Information technology, 43400 UPM Serdang, Selangor, Malaysia (mothman@fsktm.upm.edu.my)
M. Suleiman and F. Ismail are with the Department of mathematics, Faculty of Science, 43400 UPM Serdang, Selangor, Malaysia (msuleiman@math.upm.edu.my; fudziah@math.upm.edu.my)
N. Alias is with the Ibnu Sina Institute of Fundamental Studies, UTM Skudai, Johor, Malaysia

In this paper, we consider the following two-dimensional scalar wave equation problem on rectangular solution domain $\Omega = [0,1] \times [0,1]$

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad (1)$$

where c_0 is the speed of light in free space medium and u is field component function. In section II, we derive an alternating direction implicit method of (1) with truncation error $O(h^2 + k^2)$ and present a solving formula for group of adjacent points using asymmetric scheme. In section III, we provide the numerical experiment to illustrate the effectiveness of the the new method and compare it with some other existing methods of natural ordering strategy. The conclusions are given at the end section of the paper.

II. Mathematical Formulation

Introducing $v = \frac{\partial u}{\partial t}$, we can rewrite (1) as follows

$$\frac{\partial v}{\partial t} = c_0^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad (2)$$

where $(x, y) \in \Omega, t \in (0, T]$. The domain Ω is divided into a uniform grid size $h = \frac{1}{M}$ in both space directions and time increment $k = \frac{T}{N}$ which M and N are both positive integers. Grid points is denoted as $u^n_{i,j} = u(ih, jh, nk), v^n_{i,j} = v(ih, jh, nk)$ for $i, j = 0, 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$.

The solution of (2) using a simple alternating direction implicit method by Peaceman & Rachford [4] is obtained in two stages as follows

i) Stage 1

$$\begin{cases} \dfrac{v^{n+1/2}_{i,j} - v^n_{i,j}}{k} = c_0^2 \dfrac{\partial^2 u}{\partial x^2} & (3) \\ \dfrac{\nu^n_{i,j} + \nu^{n+1/2}_{i,j}}{4} = \dfrac{v^{n+1/2}_{i,j} - v^n_{i,j}}{k} & (4) \end{cases}$$

ii) Stage 2

$$\begin{cases} \dfrac{v^{n+1}_{i,j} - v^{n+1/2}_{i,j}}{k} = c_0^2 \dfrac{\partial^2 u}{\partial y^2} & (5) \\ \dfrac{\nu^{n+1}_{i,j} + \nu^{n+1/2}_{i,j}}{4} = \dfrac{v^{n+1}_{i,j} - v^{n+1/2}_{i,j}}{k} & (6) \end{cases}$$

where v is obtained using a simple weighted average between time level (n), $(n+1/2)$ and $(n+1)$. The time level $(n+1/2)$ is called an intermediate time level. The solution for each stage is computationally feasible as it only requires the solution of sets of tridiagonal equations along lines parallel to the x and y axes.

For the first stage, consider a group of two adjacent points along x axis i.e (i,j) and $(i+1,j)$. Then at each point, we approximate (3) using asymmetric discretization in [5] and combining with (4), we obtain a set of equations as follows

$$(4+\lambda^2)u_{i,j}^{n+1/2} - \lambda^2 u_{i+1,j}^{n+1/2} = (4-\lambda^2)u_{i,j}^{n} + \lambda^2 u_{i-1,j}^{n} + 2k\nu_{i,j}^{n} \quad (7)$$

$$(4+\lambda^2)u_{i+1,j}^{n+1/2} - \lambda^2 u_{i,j}^{n+1/2} = (4-\lambda^2)u_{i,j}^{n} + \lambda^2 u_{i+2,j}^{n} + 2k\nu_{i+1,j}^{n} \quad (8)$$

where $\lambda = c_0 k/h$. Equations (7-8) can be solved explicitly whose the implicit system is given by

$$\begin{bmatrix} 4+\lambda^2 & -\lambda^2 \\ -\lambda^2 & 4+\lambda^2 \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \end{bmatrix}^{n+1/2} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (9)$$

Therefore the solving formula for a group of two points sweeping in x direction at intermediate time level $n+1/2$ is given by

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j} \end{bmatrix}^{n+1/2} = \frac{1}{r} \begin{bmatrix} 4-\lambda^2 & 0 \\ 0 & 4-\lambda^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (10)$$

where $r = 16 + 8\lambda^2$ and

$$\alpha_1 = (4-\lambda^2)u_{i,j}^{n} + \lambda^2 u_{i-1,j}^{n} + 2k\nu_{i,j}^{n}$$
$$\alpha_2 = (4-\lambda^2)u_{i+1,j}^{n} + \lambda^2 u_{i+2,j}^{n} + 2k\nu_{i+1,j}^{n}$$

Similarly, for the next stage we can obtain the solving formula sweeping in y direction at time level $n+1$ for a group of two adjacent points i.e (i,j), $(i,j+1)$ as follows

$$\begin{bmatrix} u_{i,j} \\ u_{i,j+1} \end{bmatrix}^{n+1} = \frac{1}{r} \begin{bmatrix} 4-\lambda^2 & 0 \\ 0 & 4-\lambda^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad (11)$$

where

$$\beta_1 = (12-\lambda^2)u_{i,j}^{n+1/2} + \lambda^2 u_{i,j-1}^{n+1/2} - 8u_{i,j}^{n} - 2k\nu_{i,j}^{n}$$
$$\beta_2 = (12-\lambda^2)u_{i,j+1}^{n+1/2} + \lambda^2 u_{i,j+2}^{n+1/2} - 8u_{i,j+1}^{n} - 2k\nu_{i,j+1}^{n}$$

The solution vector v is given by

$$v_{i,j}^{n+1} = 4\left(\frac{u_{i,j}^{n+1} - 2u_{i,j}^{n+1/2} - u_{i,j}^{n}}{k}\right) + v_{i,j}^{n}$$

The local truncation error of the scheme in each stage is approximately of order $O(h^2+k^2)$.

III. Numerical Experiment

In this section, we present the numerical experiment of the proposed method for solving (1) on the solution region $\Omega = [0,1] \times [0,1]$ surrounded by PEC boundary conditions. The results by the proposed method are compared with the conventional FDTD and 4pEG-FDTD methods in ([1],[8]). The exact solution of the problem is given by

$$u(x,y,t) = \sqrt{2}\cos(\sqrt{2}\pi t)\sin[\pi(1-x)]\sin[\pi(1-y)]$$

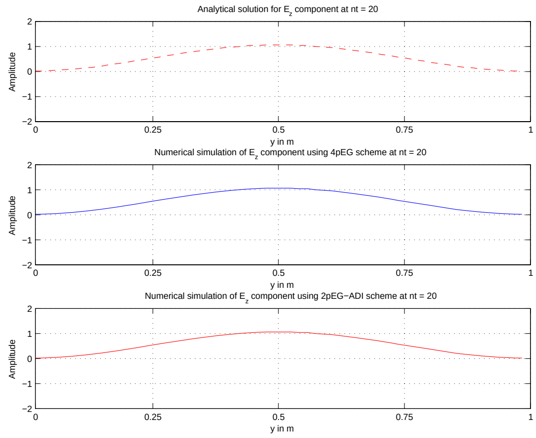

Fig. 1. Comparison result of E_z field after 20 time steps with $\lambda = 0.5$

TABLE I
PERFORMANCE OF FDTD, 4PEG AND 2PEG-ADI METHODS USING GRID CELL 81×81 AFTER 10 TIME STEPS

Scheme	k	M.E	A.A.E
FDTD	$2h$	-	-
	h	2.019e-2	8.424e-3
	$\frac{h}{2}$	7.661e-3	1.257e-4
	$\frac{h}{4}$	1.326e-3	5.536e-4
4pEG(N)	$2h$	9.014e-3	1.565e-3
	h	7.970e-4	2.559e-4
	$\frac{h}{2}$	1.907e-4	6.522e-5
	$\frac{h}{4}$	4.196e-5	1.991e-5
S-2pEG(N)	$2h$	1.360e-3	4.594e-4
	h	3.855e-4	1.272e-4
	$\frac{h}{2}$	9.302e-5	3.438e-5
	$\frac{h}{4}$	2.148e-5	9.954e-6

The experiment is carried out on 81×81 grid cell using different values of courant number λ to test the accuracy and stability of the presented method. The performance of the methods are given in the figure 1 and table I. From the results, it is apparent that the presented method is generally more accurate than the other existing methods (FDTD, 4pEG-FDTD). Furthermore the new method increase the maximum time step size, which is more stable than the conventional FDTD method.

IV. Conclusion

A new stable alternating direction implicit block with group explicit strategy is developed for the solution of scalar wave equation. This method has several advantages such as good parallelism and better accuracy than its conventional FDTD method.

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