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Almost periodic solution for an impulsive neural networks with distributed delays

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Abstract—By using the estimation of the Cauchy matrix of linear impulsive differential equations and Banach fixed point theorem as well as Gronwall-Bellman's inequality, some sufficient conditions are obtained for the existence and exponential stability of almost periodic solution for an impulsive neural networks with distributed delays. An example is presented to illustrate the feasibility and effectiveness of the results.

Keywords—Almost periodic solution; Exponential stability; Neural networks; Impulses.

I. Introduction

In the past few years, different types of recurrent neural networks have been extensively studied due to their promising potential for applications in the areas of signal and image processing, associative memories and pattern classification, parallel computation and optimization problems. One of the most popular models in the literature of recurrent neural network is the following shunting inhibitory cellular neural networks (SICNNs) with delays:

$$\begin{cases} \dot{x}_{ij}(t) &= -a_{ij}(t)x_{ij}(t) \\ & \quad - \sum\limits_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f_{ij}(x_{kl}(t))x_{ij}(t) \\ & \quad +L_{ij}(t), t\geq 0, \\ x_{ij}(t) &= \varphi_{ij}(t), t\in[-\tau,0], \\ & \quad i=1,2\dots,n, j=1,2\dots,m, \end{cases}$$

where $C_{ij}(t)$ denotes the cell at the (i,j) position of the lattice at the t; the r-neighborhood $N_r(i,j)$ of $C_{ij}(t)$ is

$$N_r(i,j)=\{C^{kl}(t):\max(|k-i|,|l-j|)\leq r, 1\leq k\leq n, \\ 1\leq l\leq m\},$$

$x_{ij}(t)$ is the activity of the cell $C_{ij}(t)$; $L_{ij}(t)$ is the external inputs to $C_{ij}(t)$; $a_{ij}(t)>0$ represents the passive decay rate of the cell activity; $C_{ij}^{kl}(t)\geq 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}; the activity functions $f_{ij}(\cdot)$ are continuous functions representing the output or firing rate of the cell $C^{kl}(t)$, $\varphi_{ij}(t)$ are the initial functions.

Since Bouzerdout and Pinter in [1-3] described SICNNs as a new cellular neural networks, SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. It is well known that studies on neural dynamic systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, almost periodic oscillatory properties, chaos and bifurcation. In applications, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. If we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general. Recently, a lot of sufficient conditions have been given for almost periodic oscillation of SICNNs with constant time delays or time-varying delays in the literature, see [4-10] and the references cited therein.

On the other hand, impulsive effects widely exist in many dynamical systems involving such areas as population dynamics, automatic control, neural networks and so on. For example, in implementation of electronic networks in which state is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change or other sudden noise, that is, does exhibit impulsive effects. For significance of impulsive effects, one can see [11-18].

Let $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=[0,\infty)$, $\Omega\subset\mathbb{R}$, $\Omega\neq\emptyset$ and $\mathbb{B}=\{\{\tau_k\}\in\mathbb{R}:\tau_k<\tau_{k+1}, k\in\mathbb{Z}, \lim\limits_{k\to\pm\infty}\tau_k=\pm\infty\}$ denote the set of all sequences that are unbounded and strictly increasing.

To the best of our knowledge, the almost periodic oscillatory behavior is seldom considered for Cohen-Grossberg SICNNs with continuously distributed delays and impulses, which is described by the following integro-differential equations:

$$\begin{cases} x_{ij}'(t) &= -a_{ij}(x_{ij}(t))\Big[b_{ij}(x_{ij}(t)) \\ & \quad + \sum\limits_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(t)w_{ij}\Big(\int_0^{+\infty}k_{ij}(s)\times \\ & \quad x_{gl}(t-s)ds\Big)x_{ij}(t)-I_{ij}(t)\Big], \\ & \quad t\neq\tau_k, k\in\mathbb{Z}, \\ \Delta x_{ij}(\tau_k) &= \alpha_{ijk}x_{ij}(\tau_k)+\gamma_{ijk}, \\ & \quad i=1,2,\dots,n,\ j=1,2,\dots,m,\ k\in\mathbb{Z}, \end{cases} \tag{1}$$

where $a_{ij}(x_{ij}(t))$ and $b_{ij}(x_{ij}(t))$ represent an amplification function at time t and an appropriately behaved function at time t, respectively; $w_{ij}\in C(\mathbb{R},\mathbb{R}^+)$ denote the normal and the delayed activation functions; $\{\tau_k\}\in\mathbb{B}$, with the constants $\alpha_{ijk}\in\mathbb{R}$, $\gamma_{ijk}\in\mathbb{R}$, $k\in\mathbb{Z}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$.

Let $t_0\in\mathbb{R}$. Introduce the following notation:

$PC(t_0)$ is the space of all functions $\phi:[-\infty,t_0]\to\Omega$ having points of discontinuity at $\theta_1,\theta_2,\ldots\in(-\infty,t_0)$ of the first kind and left continuous at these points.

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For $J \subset \mathbb{R}$, $PC(J,\mathbb{R})$ is the space of all piecewise continuous functions from J to $\mathbb{R}$ with points of discontinuity of the first kind τ_k, at which it is left continuous.

Let $x(t) = x(t, t_0, x_0), x = (x_{11}, \cdots, x_{ij}, \cdots, x_{mn})^T$, $x_0 = (x_{011}, \cdots, x_{0ij}, \cdots, x_{0mn})^T$. The system (1) is supplemented with initial values problem given by

$$x(t_0 + 0, t_0, x_0) = x_0.$$

The rest of this paper is organized as follows: In Section 2, we will introduce some necessary notations, definitions and lemmas which will be used in the paper. In Section 3, some sufficient conditions are derived ensuring the existence and exponential stability of the almost periodic solution. An example is given to illustrate the effectiveness of our results in section 4.

II. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which will be used in what follows.

Definition 2.1([19]) Let $x(t) \in C(\mathbb{R},\mathbb{R})$ be continuous in t. $x(t)$ is said to be almost periodic in the sense of Bohr on $\mathbb{R}$, if for any $\epsilon > 0$, the set $T(x, \epsilon) = \{\tau : |x(t+\tau) - x(t)| < \epsilon, \forall t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $|x(t + \tau) - x(t)| < \epsilon$, $\forall t \in \mathbb{R}$.

Definition 2.2([20]) A sequence $x : \mathbb{Z} \to \mathbb{R}$ is called an almost periodic sequence if the ϵ- translation set of x:

$$T\{\epsilon, x\} := \{\tau \in \mathbb{Z} : |x(n + \tau) - x(n)| < \epsilon \ for\ all\ n \in \mathbb{Z}\}$$

is a relatively dense set in $\mathbb{Z}$ for all $\epsilon > 0$, that is, for any given $\epsilon > 0$, there exists an integer $l > 0$ such that each discrete interval of length l contains a integer $\tau = \tau(\epsilon) \in T\{\epsilon, x\}$ such that

$$|x(n + \tau) - x(n)| < \epsilon \ for\ all\ n \in \mathbb{Z},$$

τ is called the ϵ- translation number of $x(n)$.

Definition 2.3([21]) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$ there exists a relatively dense set of ϵ-almost periods common for any sequences.

Definition 2.4([21]) The function $x(t) \in PC(\mathbb{R},\mathbb{R})$ is said to be almost periodic, if the following hold:

(a) The set of sequences $\{\tau_k^j\}, \tau_k^j = \tau_{k+j} - \tau_k, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic.

(b) For any $\epsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $x(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \epsilon$.

(c) For any $\epsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|x(t + \tau) - x(t)| < \epsilon$ for all $t \in \mathbb{R}$ satisfying the condition $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$.
The elements of T are called ϵ-almost periods.

Throughout this paper, we assume that

(H_1) $a_{ij}(\cdot) \in C(\mathbb{R},\mathbb{R}^+)$ and there exist positive constants $\underline{a}_{ij}$ and $\overline{a}_{ij}$ such that $0 < \underline{a}_{ij} \le a_{ij}(\cdot) \le \overline{a}_{ij}$ and $\overline{a}_{ij} \le \underline{a}_{ij} e, i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

(H_2) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_{k\in\mathbb{Z}} \tau_k^1 = \theta > 0$.

(H_3) The sequence $\{\alpha_{ijk}\}$ is almost periodic and $\frac{\overline{a}_{ij}}{\underline{a}_{ij}} - 1 \le \alpha_{ijk} \le \frac{\underline{a}_{ij}}{\overline{a}_{ij}} e^2 - 1,\ k \in \mathbb{Z},\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

(H_4) The sequence $\{\gamma_{ijk}\}$ is almost periodic and $\gamma = \sup_{k\in\mathbb{Z}} |\gamma_{ijk}|$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

(H_5) $b_{ij}(\cdot) \in C^1(\mathbb{R},\mathbb{R})$ and $b_{ij}(0) = 0$. There exist positive constants $\underline{b}'_{ij}$, $\overline{b}'_{ij}$ and L_b such that $0 < \underline{b}'_{ij} \le b'_{ij}(t) \le \overline{b}'_{ij}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,$ and for $u,\ v \in \mathbb{R}$, $\max_{1\le i\le n, 1\le j\le m} |b_{ij}(u) - b_{ij}(v)| \le L_b|u - v|$.

(H_6) The functions $C_{ij}^{gl}(t)$, $I_{ij}(t)$ are almost periodic in the sense of Bohr and $|I_{ij}(t)| < \infty$, $t \in \mathbb{R}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$

(H_7) The functions w_{ij} are almost periodic in the sense of Bohr and $w_{ij}(0) = 0$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$ There exist positive constant L_w such that for $u,\ v \in \mathbb{R}$, $\max_{1\le i\le n, 1\le j\le m} |w_{ij}(u) - w_{ij}(v)| \le L_w|u - v|$.

(H_8) The delay kernels $k_{ij} \in C(\mathbb{R},\mathbb{R})$ and there exist positive constants $\overline{k}_{ij}$ such that

$$\int_0^{+\infty} |k_{ij}(s)|\, ds \le \overline{k}_{ij}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

Now, we shall transform system (1) and state some notations, which will be used in later sections.

From (H_1), the antiderivative of $1/a_{ij}(x_{ij})$ exists. We choose an antiderivative $h_{ij}(x_{ij})$ of $1/a_{ij}(x_{ij})$ that satisfies $h_{ij}(0) = 0$. Obviously, $(d/dx_{ij})h_{ij}(x_{ij}) = 1/a_{ij}(x_{ij})$. By $a_{ij}(x_{ij}) > 0$, we obtain that $h_{ij}(x_{ij})$ is strictly monotone increasing about x_{ij}. In view of derivative theorem for inverse function, the inverse function $h_{ij}^{-1}(x_{ij})$ of $h_{ij}(x_{ij})$ is differentiable and $(d/dx_{ij})h_{ij}^{-1}(x_{ij}) = a_{ij}(h_{ij}^{-1}(x_{ij}))$. By (H_5), composition function $b_{ij}(h_{ij}^{-1}(z))$ is differentiable. Denote $u_{ij}(t) = h_{ij}(x_{ij}(t))$. It is easy to see that $u'_{ij}(t) = x'_{ij}(t)/a_i(x_{ij}(t))$ and $x_{ij}(t) = h_{ij}^{-1}(u_{ij}(t))$. Substituting these equalities into system (1), we get

$$\begin{cases} u'_{ij}(t) = -b_{ij}(h_{ij}^{-1}(u_{ij}(t))) \\ \qquad - \sum_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(t) w_{ij}\bigg(\int_0^{+\infty} k_{ij}(s)\times \\ \qquad h_{gl}^{-1}(u_{gl}(t-s))\, ds \bigg) h_{ij}^{-1}(u_{ij}(t)) \\ \qquad +I_{ij}(t), \quad t \neq \tau_k, \\ \Delta u_{ij}(\tau_k) = h_{ij}((1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}) - u_{ij}(\tau_k) \\ \qquad := r_{ij}(u_{ij}(\tau_k)), \end{cases} \tag{2}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

If $u_{ij}(t) \neq 0$ for all $t \in \mathbb{R}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,$ from the definitions of $h_{ij}(z)$ and $h_{ij}^{-1}(z)$, using Lagrange mean-value theorem, we have

$$\begin{aligned} & r_{ij}(u_{ij}(\tau_k)) \\ = {} & h_{ij}((1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}) - u_{ij}(\tau_k) \\ = {} & \frac{h_{ij}((1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk})}{(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}} \end{aligned}$$

$$
\begin{aligned}
&\times[(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk}]-u_{ij}(\tau_k)\\
&=\frac{(1+\alpha_{ijk})h_{ij}((1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk})}{(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk}}\\
&\times\frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)}u_{ij}(\tau_k)-u_{ij}(\tau_k)\\
&+\frac{h_{ij}((1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk})}{(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk}}\gamma_{ijk}\\
&=\left[\frac{(1+\alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})}-1\right]u_{ij}(\tau_k)+\frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})}\\
&=\mu_{ijk}u_{ij}(\tau_k)+\nu_{ijk},
\end{aligned}
$$

where

$$
\begin{aligned}
a_{ij}(\xi_{ijk}) &= \frac{1}{h'_{ij}(\xi_{ijk})}\\
&= \frac{(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk}}{h_{ij}((1+\alpha_{ik})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk})}, \quad (3)
\end{aligned}
$$

$$
\begin{aligned}
a_{ij}(\eta_{ijk}) &= a_{ij}(h_{ij}^{-1}(\zeta_{ijk}))=(h_{ij}^{-1})'(\zeta_{ijk})\\
&= \frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)}, \quad (4)
\end{aligned}
$$

in which $\eta_{ijk} = h_{ij}^{-1}(\zeta_{ijk})$, ξ_{ijk} is between 0 and $(1+\alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k))+\gamma_{ijk}$, η_{ijk} is between 0 and $h_{ij}^{-1}(u_{ij}(\tau_k))$, and

$$\mu_{ijk}=\frac{(1+\alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})}-1, \quad (5)$$

$$\nu_{ijk}=\frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})}, \quad (6)$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ k\in\mathbb{Z}$.

Then system (2) can be rewritten as

$$
\begin{cases}
u'_{ij}(t) = -e_{ij}(u_{ij}(t))u_{ij}(t)\\
\quad -\sum\limits_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(t)w_{ij}\bigg(\int_0^{+\infty}k_{ij}(s)\times\\
\quad h_{gl}^{-1}(u_{gl}(t-s))\,ds\bigg)h_{ij}^{-1}(u_{ij}(t))\\
\quad +I_{ij}(t),\quad t\neq\tau_k,\\
\Delta u_{ij}(\tau_k)=\mu_{ijk}u_{ij}(\tau_k)+\nu_{ijk},\\
\quad i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ k\in\mathbb{Z},
\end{cases} \quad (7)
$$

where $e_{ij}(u_{ij}(t)) := b_{ij}(h_{ij}^{-1}(u_{ij}(t)))/u_{ij}(t)$ for all $t\in\mathbb{R}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

System (1) has an almost periodic solution which is globally exponentially stable if and only if system (7) has an almost periodic solution which is globally exponentially stable.

Let $E_{ij}(t) = e_{ij}(u_{ij}(t)) = \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}$, $i = 1,2,\ldots,n,\ j=1,2,\ldots,m$.

Together with the system (7) we consider the linear system

$$
\begin{cases}
u'_{ij}(t)=-E_{ij}(t)u_{ij}(t), & t\neq\tau_k,\\
\Delta u_{ij}(\tau_k)=\mu_{ijk}u_{ij}(\tau_k), & k\in\mathbb{Z},
\end{cases} \quad (8)
$$

where $t\in\mathbb{R}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

Now let us consider the equations

$$u'_{ij}(t)=-E_{ij}(t)u_{ij}(t),\quad \tau_{k-1}<t\le\tau_k,\quad \{\tau_k\}\in\mathbb{B}$$

and their solutions

$$u_{ij}(t)=u_{ij}(s)\exp\bigg\{-\int_s^t E_{ij}(\sigma)\,d\sigma\bigg\}$$

for $\tau_{k-1}<s<t\le\tau_k$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

Then, recall [22], the Cauchy matrix of the linear system (8) is

$$
W_{ij}(t,s)=\begin{cases}
\exp\bigg\{-\int_s^t E_{ij}(\sigma)\,d\sigma\bigg\}, \tau_{k-1}<s<t<\tau_k;\\
\prod\limits_{l=m}^{k+1}(1+\mu_{ijl})\exp\bigg\{-\int_s^t E_{ij}(\sigma)\,d\sigma\bigg\},\\
\qquad \tau_{m-1}<s\le\tau_m<\tau_k<t\le\tau_{k+1},
\end{cases}
$$

and the solutions of system (8) are in the form

$$
\begin{aligned}
u_{ij}(t;t_0;u_{ij}(t_0)) = W_{ij}(t,t_0)u_{ij}(t_0),\ t_0\in\mathbb{R},\\
i=1,2,\ldots,n,\ j=1,2,\ldots,m.
\end{aligned}
$$

Lemma 2.1 If the conditions $(H_1)-(H_2)$, (H_5) and the following condition:

(**H**) $u_{ij}(t)\in PC(\mathbb{R},\mathbb{R})$ is almost periodic function satisfying

$$0<M\le\inf_{t\in\mathbb{R}}|u_{ij}(t)|\le\sup_{t\in\mathbb{R}}|u_{ij}(t)|\le N,$$

where M, N are positive constants, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

hold. Then $E_{ij}(t) = e_{ij}(u_{ij}(t))$ is almost periodic, $i = 1,2,\ldots,n,\ j=1,2,\ldots,m$.

Proof: From the definition of $e_{ij}(u_{ij}(t))$, $E_{ij}(t) = e_{ij}(u_{ij}(t)) = \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

Case I: $\frac{dE_{ij}(t)}{du_{ij}} = \frac{de_{ij}(u_{ij}(t))}{du_{ij}} = 0$, then $E_{ij}(t)=C_i$, where C_{ij} is a constant, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$. Hence $E_{ij}(t)\in C(\mathbb{R},\mathbb{R})$ is almost periodic, $i = 1,2,\ldots,n,\ j=1,2,\ldots,m$.

Case II: $\frac{dE_{ij}(t)}{du_{ij}} = \frac{de_{ij}(u_{ij}(t))}{du_{ij}} \neq 0$, then $E_{ij}(t)$ depends on $u_{ij}(t)$, and $E_{ij}(t)\in PC(\mathbb{R},\mathbb{R})$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

We now check the three conditions given in Definition 2.4.

(a) With (H_2), it is trivially satisfied.

(b) For any $\epsilon>0$, if t' and t'' belong to one and the same interval of continuity $E_{ij}(t)$ (t' and t'' also belong to one and the same interval of continuity $u_{ij}(t)$), by condition (b) of Definition 2.4, there exists a positive numbers δ_{ij}, such that $|t'-t''|<\delta_i$ implies $|u_{ij}(t')-u_{ij}(t'')| < \frac{M^2\epsilon}{2L_b\bar{a}_{ij}N}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$. We take $\delta = \min\limits_{1\le i\le n,1\le j\le m}\{\delta_{ij}\}$, when $|t'-t''|<\delta$, for $1\le i\le n, 1\le$

$j \le m$,

$$\begin{aligned}
&|E_{ij}(t') - E_{ij}(t'')| \\
&= \left|\frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t')))}{u_{ij}(t')} - \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))}{u_{ij}(t'')}\right| \\
&= \left|\frac{u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))}{u_{ij}(t')u_{ij}(t'')}\right| \\
&\le \frac{|u_{ij}(t'')b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t')))|}{M^2} \\
&\quad + \frac{|u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - u_{ij}(t')b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))|}{M^2} \\
&\le \frac{L_b|h_{ij}^{-1}(u_{ij}(t'))||u_{ij}(t') - u_{ij}(t'')|}{M^2} \\
&\quad + \frac{N|b_{ij}(h_{ij}^{-1}(u_{ij}(t'))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t'')))|}{M^2} \\
&\le \frac{L_b\overline{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\
&\quad + \frac{L_bN|h_{ij}^{-1}(u_{ij}(t')) - h_{ij}^{-1}(u_{ij}(t''))|}{M^2} \\
&\le \frac{L_b\overline{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\
&\quad + \frac{L_b\overline{a}_{ij}N|u_{ij}(t') - u_{ij}(t'')|}{M^2} \\
&= \frac{2L_b\overline{a}_{ij}N}{M^2}|u_{ij}(t') - u_{ij}(t'')| < \epsilon,
\end{aligned}$$

so condition (b) of Definition 2.4 is satisfied.

(c) Let $P_{ij} = \frac{M^2}{2L_b\overline{a}_{ij}N} + 1$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. It is obvious that $\frac{M^2}{2L_b\overline{a}_{ij}NP_{ij}} < 1$. For any $\epsilon > 0$, with the almost periodicity of $u_{ij}(t)$, by Definition 2.4, there exists a relatively dense set T such that if $\tau \in T$, then$|u_{ij}(t+\tau) - u_{ij}(t)| < \frac{M^2\epsilon}{2L_b\overline{a}_{ij}NP_{ij}}$ for all $t \in \mathbb{R}$ satisfying $|t - \tau_k| > \frac{M^2\epsilon}{2L_b\overline{a}_{ij}NP_{ij}}$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.
Then for all $t \in \mathbb{R}$ satisfying the condition $|t-\tau_k| > \epsilon > \frac{M^2\epsilon}{2L_b\overline{a}_{ij}NP_{ij}}$, we obtain

$$\begin{aligned}
&|E_{ij}(t+\tau) - E_{ij}(t)| \\
&= \left|\frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau)))}{u_{ij}(t+\tau)} - \frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}\right| \\
&= \left|\frac{u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau)))}{u_{ij}(t)u_{ij}(t+\tau)} - \frac{u_{ij}(t+\tau)b_i(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)u_{ij}(t+\tau)}\right| \\
&\le \frac{1}{M^2}|u_{ij}(t)b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau))) \\
&\quad - u_{ij}(t+\tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau)))| \\
&\quad + \frac{1}{M^2}|u_{ij}(t+\tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau))) \\
&\quad - u_{ij}(t+\tau)b_{ij}(h_{ij}^{-1}(u_{ij}(t)))| \\
&\le \frac{L_b\overline{a}_{ij}N|u_{ij}(t+\tau) - u_{ij}(t)|}{M^2} \\
&\quad + \frac{N|b_{ij}(h_{ij}^{-1}(u_{ij}(t+\tau))) - b_{ij}(h_{ij}^{-1}(u_{ij}(t)))|}{M^2} \\
&\le \frac{L_b\overline{a}_{ij}N|u_{ij}(t+\tau) - u_{ij}(t)|}{M^2} \\
&\quad + \frac{L_bN|h_{ij}^{-1}(u_{ij}(t+\tau)) - h_{ij}^{-1}(u_{ij}(t))|}{M^2} \\
&\le \frac{L_b\overline{a}_{ij}N|u_{ij}(t+\tau) - u_{ij}(t)|}{M^2} \\
&\quad + \frac{L_b\overline{a}_{ij}N|u_{ij}(t+\tau) - u_{ij}(t)|}{M^2} \\
&= \frac{2L_b\overline{a}_{ij}N}{M^2}|u_{ij}(t+\tau) - u_{ij}(t)| \\
&< \frac{\epsilon}{P_{ij}} < \epsilon,
\end{aligned}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$, hence the condition ($c$) of Definition 2.4 is satisfied.

From Definition 2.4, $E_{ij}(t) \in PC(\mathbb{R}, \mathbb{R})$ is almost periodic, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. This completes the proof. ■

Now from [21], we have

Lemma 2.2 If the conditions $(H_1) - (H_7)$ and (**H**) hold, then for each $\epsilon > 0$, there exist ϵ_1, $0 < \epsilon_1 < \epsilon$, relatively dense sets T of real numbers and Q of whole numbers, such that the following relations are fulfilled:

(a) $|E_{ij}(t+\tau) - E_{ij}(t)| < \epsilon$, $\ t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(b) $|C_{ij}^{gl}(t+\tau) - C_{ij}^{gl}(t)| < \epsilon$, $\ t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(c) $|I_{ij}(t+\tau) - I_{ij}(t)| < \epsilon$, $\ t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(d) $|w_{ij}(t+\tau) - w_{ij}(t)| < \epsilon$, $\ t \in \mathbb{R}, \tau \in T, |t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(e) $|\alpha_{ij(k+q)} - \alpha_{ijk}| < \epsilon$, $\ q \in Q$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(f) $|\gamma_{ij(k+q)} - \gamma_{ijk}| < \epsilon$, $\ q \in Q$, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$;

(g) $|\tau_k^q - \tau| < \epsilon_1$, $\ q \in Q, \tau \in T, k \in \mathbb{Z}$.

Lemma 2.3 If the conditions $(H_1) - (H_4)$ and (**H**) hold, then the sequences $\{\mu_{ijk}\}$ and $\{\nu_{ijk}\}$ is almost periodic, $k \in \mathbb{Z}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

Proof: For convenience, let

$$\begin{aligned}
F_{ij} &= \max\left\{\frac{2\overline{a}_{ij}N(\overline{a}_{ij} + 2\underline{a}_{ij})}{\underline{a}_{ij}^2M^2}, \frac{(\overline{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}^2M^2}\right\}, \\
&\quad \frac{\overline{a}_{ij}(\overline{a}_{ij}N^2 + \underline{a}_{ij}N^2 + \overline{a}_{ij}M^2)}{\underline{a}_{ij}^2M^2}, \\
L_{ij} &= \max\left\{\frac{2\overline{a}_{ij}N\gamma(\overline{a}_{ij} + \underline{a}_{ij})}{\underline{a}_{ij}^3M_{ij}^2}, \frac{\overline{a}_{ij}N^2\gamma(\overline{a}_{ij} + \underline{a}_{ij})}{\underline{a}_{ij}^3M^2}, \right. \\
&\quad \left. \frac{\overline{a}_{ij}N\gamma + \underline{a}_{ij}N\gamma + \overline{a}_{ij}\underline{a}_{ij}M^2}{\underline{a}_{ij}^3M^2}\right\},
\end{aligned}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $F = \max\limits_{1\le i\le n, 1\le j\le m}\{F_{ij}\}$, $L = \max\limits_{1\le i\le n, 1\le j\le m}\{L_{ij}\}$ and $H = \max\{F, L\}$.

For any $\epsilon > 0$, since $u_{ij}(t) \in PC(\mathbb{R}, \mathbb{R})$ is almost periodic function, by Definition 2.4, there exists $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $u_{ij}(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|u_{ij}(t') - u_{ij}(t'')| < \frac{\epsilon}{4H}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

With the left continuousness of τ_k, take numbers ϵ_0, $\tau_k' < \tau_k$ such that τ_k' and τ_k belong to one and the same interval of continuity of $u_{ij}(t)$, and $0 < \epsilon_0 < \tau_k - \tau_k' < \min\{\delta, \frac{\theta}{2}, \frac{\epsilon}{12H}\}$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

From Lemma 2.2, for $\epsilon_0 < \frac{\epsilon}{12H}$, because the sequences $\{\tau_k^j\}$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbb{B}$ is uniformly almost periodic, for $q \in Q$ (without loss of generality, assuming $q \geq 0$), let $\tau = \inf_{k\in\mathbb{Z}}\{\tau_k^q\}$, implying $\tau \in T$ and $\tau_k + \tau \leq \tau_{k+q}$, where the sets T and Q are determined in Lemma 2.2 such that $0 \leq \tau_k^q - \tau = \tau_{k+q} - \tau_k - \tau < \epsilon_0 < \min\{\frac{\epsilon}{12H}, \delta, \frac{\theta}{2}\}$. Then $\tau_k + \tau$ and τ_{k+q}, $\tau_k' + \tau$ and $\tau_k + \tau$ belong to one and the same interval of continuity of $u_{ij}(t)$, respectively, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

By ($\boldsymbol{H}$) and Lemma 2.2, we have

$$\begin{aligned}
&|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)|\\
&\leq |u_{ij}(\tau_k) - u_{ij}(\tau_k + \tau)| + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})|\\
&\leq |u_{ij}(\tau_k') - u_{ij}(\tau_k)| + |u_{ij}(\tau_k') - u_{ij}(\tau_k + \tau)|\\
&\quad + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})|\\
&\leq |u_{ij}(\tau_k') - u_{ij}(\tau_k)| + |u_{ij}(\tau_k' + \tau) - u_{ij}(\tau_k')|\\
&\quad + |u_{ij}(\tau_k' + \tau) - u_{ij}(\tau_k + \tau)|\\
&\quad + |u_{ij}(\tau_k + \tau) - u_{ij}(\tau_{k+q})|\\
&< \frac{\epsilon}{4H} + \frac{\epsilon}{12H} + \frac{\epsilon}{4H} + \frac{\epsilon}{4H} = \frac{5\epsilon}{6H},
\end{aligned} \tag{9}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

Let $G_{ijk} = (1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}$, then $G_{ijk} = x_{ij}(\tau_k^+) = h_{ij}^{-1}(u_{ij}(\tau_k^+))$, $M \leq |h_{ij}(G_{ijk})| = |u_{ij}(\tau_k^+)| \leq N$, $|G_{ijk}| = |h_{ij}^{-1}(u_{ij}(\tau_k^+))| \leq \overline{a}_{ij}|u_{ij}(\tau_k^+)| \leq \overline{a}_{ij}N$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$. Then

$$\begin{aligned}
&|G_{ij(k+q)} - G_{ijk}|\\
&= \big|\big[(1 + \alpha_{ij(k+q)})h_{ij}^{-1}(u_{ij}(\tau_{(k+q)})) + \gamma_{ij(k+q)}\big]\\
&\quad - \big[(1 + \alpha_{ijk})h_{ij}^{-1}(u_{ij}(\tau_k)) + \gamma_{ijk}\big]\big|\\
&\leq \big|h_{ij}^{-1}(u_{ij}(\tau_{(k+q)})) - h_{ij}^{-1}(u_{ij}(\tau_k))\big|\\
&\quad + \big|\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_{(k+q)})) - \alpha_{ijk}h_{ij}^{-1}(u_{ij}(\tau_k))\big|\\
&\quad + |\gamma_{ij(k+q)} - \gamma_{ijk}|\\
&\leq \overline{a}_{ij}\big|u_{ij}(\tau_{(k+q)}) - u_{ij}(\tau_k)\big| + \big|\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_{(k+q)}))\\
&\quad - \alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_k))\big|\\
&\quad + \big|\alpha_{ij(k+q)}h_{ij}^{-1}(u_{ij}(\tau_k)) - \alpha_{ijk}h_{ij}^{-1}(u_{ij}(\tau_k))\big|\\
&\quad + |\gamma_{ij(k+q)} - \gamma_{ijk}|\\
&\leq 2\overline{a}_{ij}|u_{ij}(\tau_{(k+q)}) - u_{ij}(\tau_k)|\\
&\quad + |h_{ij}^{-1}(u_{ij}(\tau_k))||\alpha_{ij(k+q)} - \alpha_{ijk}| + |\gamma_{ij(k+q)} - \gamma_{ijk}|\\
&\leq 2\overline{a}_{ij}|u_{ij}(\tau_{(k+q)}) - u_{ij}(\tau_k)|\\
&\quad + \overline{a}_{ij}|u_{ij}(\tau_k)||\alpha_{ij(k+q)} - \alpha_{ijk}| + |\gamma_{ij(k+q)} - \gamma_{ijk}|\\
&\leq 2\overline{a}_{ij}|u_{ij}(\tau_{(k+q)}) - u_{ij}(\tau_k)| + \overline{a}_{ij}N|\alpha_{ij(k+q)} - \alpha_{ijk}|\\
&\quad + |\gamma_{ij(k+q)} - \gamma_{ijk}|,
\end{aligned} \tag{10}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

By (3), (4) and (10), we obtain

$$\begin{aligned}
&|a_{ij}(\xi_{ij(k+q)}) - a_{ij}(\xi_{ijk})|\\
&= \left|\frac{G_{ij(k+q)}}{h_{ij}(G_{ij(k+q)})} - \frac{G_{ijk}}{h_{ij}(G_{ijk})}\right|\\
&= \frac{\big|h_{ij}(G_{ijk})G_{ij(k+q)} - G_{ijk}h_{ij}(G_{ij(k+q)})\big|}{\big|h_{ij}(G_{ij(k+q)})h_{ij}(G_{ijk})\big|}\\
&\leq \frac{\big|h_{ij}(G_{ijk})G_{ij(k+q)} - h_{ij}(G_{ij(k+q)})G_{ij(k+q)}\big|}{M^2}\\
&\quad + \frac{\big|h_{ij}(G_{ij(k+q)})G_{ij(k+q)} - G_{ijk}h_{ij}(G_{ij(k+q)})\big|}{M^2}\\
&\leq \frac{1}{M^2}|G_{ij(k+q)}|\big|h_{ij}(G_{i(k+q)}) - h_{ij}(G_{ijk})\big|\\
&\quad + \frac{1}{M^2}|h_{ij}(G_{ij(k+q)})||G_{ij(k+q)} - G_{ijk}|\\
&\leq \frac{\frac{\overline{a}_{ij}N}{\underline{a}_{ij}}|G_{ij(k+q)} - G_{ijk}| + N|G_{ij(k+q)} - G_{ijk}|}{M^2}\\
&\leq (\frac{(\overline{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|G_{ij(k+q)} - G_{ijk}|\\
&\leq \frac{2\overline{a}_{ij}(\overline{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|u_{ij}(\tau_{(k+q)}) - u_{ij}(\tau_k)|\\
&\quad + \frac{\overline{a}_{ij}(\overline{a}_{ij} + \underline{a}_{ij})N^2}{\underline{a}_{ij}M^2}|\alpha_{ij(k+q)} - \alpha_{ijk}|\\
&\quad + \frac{(\overline{a}_{ij} + \underline{a}_{ij})N}{\underline{a}_{ij}M^2}|\gamma_{ij(k+q)} - \gamma_{ijk}|
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&|a_{ij}(\eta_{ij(k+q)}) - a_{ij}(\eta_{ijk})|\\
&= \left|\frac{h_{ij}^{-1}(u_{ij}(\tau_{k+q}))}{u_{ij}(\tau_{k+q})} - \frac{h_{ij}^{-1}(u_{ij}(\tau_k))}{u_{ij}(\tau_k)}\right|\\
&= \frac{\big|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_k) - u_{ij}(\tau_{k+q})h_{ij}^{-1}(u_{ij}(\tau_k))\big|}{|u_{ij}(\tau_{k+q})u_{ij}(\tau_k)|}\\
&\leq \frac{\big|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_k) - h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_{k+q})\big|}{M^2}\\
&\quad + \frac{\big|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))u_{ij}(\tau_{k+q}) - u_{ij}(\tau_{k+q})h_{ij}^{-1}(u_{ij}(\tau_k))\big|}{M^2}\\
&\leq \frac{\big|h_{ij}^{-1}(u_{ij}(\tau_{k+q}))\big||u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)|}{M^2}\\
&\quad + \frac{|u_{ij}(\tau_{k+q})|\big|h_{ij}^{-1}(u_{ij}(\tau_{k+q})) - h_{ij}^{-1}(u_{ij}(\tau_k))\big|}{M^2}\\
&\leq \frac{2\overline{a}_{ij}N}{M^2}|u_{ij}(\tau_{k+q}) - u_{ij}(\tau_k)|,
\end{aligned} \tag{12}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ k \in \mathbb{Z}$.

Finally, with (5), (6) and (9) – (12), for each $q \in Q$,

$$\begin{aligned}
&|\mu_{ij(k+q)} - \mu_{ijk}|\\
&= \Big|(\frac{(1 + \alpha_{ij(k+q)})a_{ij}(\eta_{ij(k+q)})}{a_{ij}(\xi_{ij(k+q)})} - 1)\\
&\quad - (\frac{(1 + \alpha_{ijk})a_{ij}(\eta_{ijk})}{a_{ij}(\xi_{ijk})} - 1)\Big|\\
&\leq \frac{1}{\underline{a}_{ij}^2}\big|1 + \alpha_{ij(k+q)}\big|\big|a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ijk})\\
&\quad - a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})\big|\\
&\quad + \frac{\big|a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})\big|\big|\alpha_{ij(k+q)} - \alpha_{ijk}\big|}{\underline{a}_{ij}^2}\\
&\leq \frac{\big|a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ijk}) - a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ij(k+q)})\big|}{\overline{a}_{ij}\underline{a}_{ij}}
\end{aligned}$$

$$+\frac{\left|a_{ij}(\eta_{ij(k+q)})a_{ij}(\xi_{ij(k+q)})-a_{ij}(\xi_{ij(k+q)})a_{ij}(\eta_{ijk})\right|}{\overline{a}_{ij}\underline{a}_{ij}} +\frac{\overline{a}_{ij}^2|\alpha_{ij(k+q)}-\alpha_{ijk}|}{\underline{a}_{ij}^2}$$

$$\le \frac{\left|a_{ij}(\xi_{ij(k+q)})-a_{ij}(\xi_{ijk})\right|+\left|a_{ij}(\eta_{ij(k+q)})-a_{ij}(\eta_{ijk})\right|}{\underline{a}_{ij}} +\frac{\overline{a}_{ij}^2|\alpha_{ij(k+q)}-\alpha_{ijk}|}{\underline{a}_{ij}^2}$$

$$\le \frac{2\overline{a}_{ij}N(\overline{a}_{ij}+2\underline{a}_{ij})}{\underline{a}_{ij}^2M^2}|u_{ij}(\tau_{(k+q)})-u_{ij}(\tau_k)| +\frac{\overline{a}_{ij}(\overline{a}_{ij}N^2+\underline{a}_{ij}N^2+\overline{a}_{ij}M^2)}{\underline{a}_{ij}^2M^2}|\alpha_{ij(k+q)}-\alpha_{ijk}| +\frac{(\overline{a}_{ij}+\underline{a}_{ij})N}{\underline{a}_{ij}^2M^2}|\gamma_{ij(k+q)}-\gamma_{ijk}|$$

$$< H\frac{5\epsilon}{6H}+H\frac{\epsilon}{12H}+H\frac{\epsilon}{12H}=\epsilon,$$

and

$$|\nu_{ij(k+q)}-\nu_{ijk}| = \left|\frac{\gamma_{ij(k+q)}}{a_{ij}(\xi_{ij(k+q)})}-\frac{\gamma_{ijk}}{a_{ij}(\xi_{ijk})}\right|$$

$$= \frac{\left|\gamma_{ij(k+q)}a_{ij}(\xi_{ijk})-\gamma_{ijk}a_{ij}(\xi_{ij(k+q)})\right|}{\left|a_{ij}(\xi_{ij(k+q)})a_{ij}(\xi_{ijk})\right|}$$

$$\le \frac{1}{\underline{a}_{ij}^2}\left|\gamma_{ij(k+q)}a_{ij}(\xi_{ijk})-\gamma_{ij(k+q)}a_{ij}(\xi_{ij(k+q)})\right| +\frac{1}{\underline{a}_{ij}^2}\left|\gamma_{ij(k+q)}a_{ij}(\xi_{ij(k+q)})-\gamma_{ijk}a_{ij}(\xi_{ij(k+q)})\right|$$

$$\le \frac{\gamma\left|a_{ij}(\xi_{ij(k+q)})-a_{ij}(\xi_{ijk})\right|+\overline{a}_{ij}|\gamma_{ij(k+q)}-\gamma_{ijk}|}{\underline{a}_{ij}^2}$$

$$\le \frac{2\overline{a}_{ij}N\gamma(\overline{a}_{ij}+\underline{a}_{ij})}{\underline{a}_{ij}^3M^2}|u_{ij}(\tau_{(k+q)})-u_{ij}(\tau_k)| +\frac{\overline{a}_{ij}N^2\gamma(\overline{a}_{ij}+\underline{a}_{ij})}{\underline{a}_{ij}^3M^2}|\alpha_{ij(k+q)}-\alpha_{ijk}| +\frac{(\overline{a}_{ij}N\gamma+\underline{a}_{ij}N\gamma+\overline{a}_{ij}\underline{a}_{ij}M^2)}{\underline{a}_{ij}^3M^2}|\gamma_{ij(k+q)}-\gamma_{ijk}|$$

$$< H\frac{5\epsilon}{6H}+H\frac{\epsilon}{12H}+H\frac{\epsilon}{12H}=\epsilon,$$

where $i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ k\in\mathbb{Z}$. Since Q is relatively dense set of whole numbers, hence the sequences $\{\mu_{ijk}\}$ and $\{\nu_{ijk}\}$ is almost periodic, $k\in\mathbb{Z}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$. This completes the proof. ■

Lemma 2.4([21]) Let $\{\tau_k\}\in\mathbb{B}$ and the condition (H_2) hold. Then for $l>0$ there exists a positive integer A such that on each interval of length l, we have no more than A elements of the sequence $\{\tau_k\}$, i.e.,

$$i(s,t)\le A(t-s)+A,$$

where $i(s,t)$ is the number of the points τ_k in the interval (s,t).

Lemma 2.5 If the conditions $(H_1)-(H_3)$, (H_5), (**H**) and the following condition:

(H_9) $\alpha_{ij}=\underline{a}_{ij}\underline{b}'_{ij}-2A>0$, where constant A is determined in Lemma 2.4, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

hold, then:

(i) For the Cauchy matrix $W_{ij}(t,s)$ of system (8), there exist positive numbers α_{ij} and β_{ij} such that

$$e^{-\beta_{ij}(t-s)}\le W_{ij}(t,s)\le e^{2A}e^{-\alpha_{ij}(t-s)},$$

where $\beta_{ij}=\overline{a}_{ij}\overline{b}'_{ij}$, $t\ge s$, $t,s\in\mathbb{R}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

(ii) For any $\epsilon>0$, $t\ge s$, $t,s\in\mathbb{R}$, $|t-\tau_k|>\epsilon$, $|s-\tau_k|>\epsilon$, $k\in\mathbb{Z}$ there exists a relatively dense set T of the function $E_{ij}(t)$ and a positive constant Γ such that for $\tau\in T$ it follows that

$$|W_{ij}(t+\tau,s+\tau)-W_{ij}(t,s)|\le\epsilon\Gamma e^{-\frac{\alpha_{ij}}{2}(t-s)},$$
$$t\ge s,\ t,s\in\mathbb{R},\ i=1,2,\ldots,n,\ j=1,2,\ldots,m.$$

Proof: Because the proof of the second part of this lemma is similar to Lemma 3 in [23], hence we will only prove the first part of the lemma.

From the definition of $E_{ij}(t)$, using Lagrange mean-value theorem, one gets

$$E_{ij}(t)=\frac{b_{ij}(h_{ij}^{-1}(u_{ij}(t)))}{u_{ij}(t)}=\frac{b'_{ij}(o_i)h_{ij}^{-1}(u_{ij}(t))}{u_{ij}(t)} = b'_{ij}(o_{ij})a_{ij}(\rho_{ij}),$$

where o_i is between 0 and $h_{ij}^{-1}(u_{ij}(t))$, ρ_{ij} is between 0 and $u_{ij}(t)$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$.

Thus

$$\underline{a}_{ij}\underline{b}'_{ij}\le E_{ij}(t)=b'_{ij}(o_{ij})a_{ij}(\rho_{ij})\le\overline{a}_{ij}\overline{b}'_{ij}=\beta_{ij},\quad i=1,2,\ldots,n,\ j=1,2,\ldots,m. \tag{13}$$

Since the sequence $\{\mu_{ijk}\}$ is almost periodic, then it is bounded. From (H_3) and (5) it follows that $1\le 1+\mu_{ijk}\le e^2$ for $i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ k\in\mathbb{Z}$.

With the presentation of $W_{ij}(t,s)$, the last inequality and (13) it follows that

$$\begin{aligned} e^{-\beta_{ij}(t-s)} &\le W_{ij}(t,s)\le(1+\mu_{ijk})^{i(s,t)}e^{-\underline{a}_{ij}\underline{b}'_{ij}(t-s)}\\ &\le (1+\mu_{ijk})^{A(t-s)+A}e^{-\underline{a}_{ij}\underline{b}'_{ij}(t-s)}\\ &= e^{2A}e^{(2A-\underline{a}_{ij}\underline{b}'_{ij})(t-s)}\\ &= e^{2A}e^{-\alpha_{ij}(t-s)}, \end{aligned}$$

where $t\ge s$, $t,\ s\in\mathbb{R}$, $k\in\mathbb{Z}$, $i=1,2,\ldots,n,\ j=1,2,\ldots,m$. This completes the proof. ■

For convenience, we introduce the notation:

$$\overline{f}=\sup_{t\in\mathbb{R}}|f(t)|,\quad \underline{f}=\inf_{t\in\mathbb{R}}|f(t)|.$$

III. Main results

Let

$$M = \min_{1\le i\le n, 1\le j\le m}\left\{\frac{\Theta_{ij}}{\beta_{ij}} - \frac{\gamma e^{2A}}{1-e^{-\alpha_{ij}}}\right\},$$
$$K = \max_{1\le i\le n, 1\le j\le m}\left\{\frac{\overline{I}_{ij}e^{2A}}{\alpha_{ij}} + \frac{\gamma e^{2A}}{1-e^{-\alpha_{ij}}}\right\},$$
$$r = \max_{1\le i\le n, 1\le j\le m}\left\{\frac{e^{2A}}{\alpha_{ij}}\left[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}\overline{a}_{ij}^2\right]\right\},$$
$$\lambda = \inf_{1\le i\le n, 1\le j\le m}\alpha_{ij},$$

where $\Theta_{ij} = \inf_{t\in\mathbb{R}}\Big\{-\sum_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(t)w_{ij}\Big(\int_0^{+\infty}k_{ij}(\sigma)\times h_{ij}^{-1}(\varphi_{gl}(t-\sigma))\,d\sigma\Big)h_{ij}^{-1}(\varphi_{ij}(t)) + I_{ij}(t)\Big\}$.

Theorem 3.1 Assume that $(H_1)-(H_9)$ and (**H**) hold. If $M>0$, $r<1$ and $\frac{K}{1-r}<1$, then there exists a unique nonzero almost periodic solution of (1).

Proof: Set $\mathbb{X} = \{\varphi(t)\in PC(\mathbb{R},\mathbb{R}^n) : \varphi(t) = (\varphi_{11}(t),\ldots,\varphi_{ij}(t),\ldots,\varphi_{nm}(t))^T$, where $\varphi_{ij}(t)$ is a almost periodic function satisfying $0 < M \le \inf_{t\in\mathbb{R}}|\varphi_{ij}(t)| \le \sup_{t\in\mathbb{R}}|\varphi_{ij}(t)| \le N = \frac{K}{1-r},\ i=1,2,\ldots,n,\ j=1,2,\ldots,m$ $\}$ with the norm

$$\|\varphi\| = \max_{1\le i\le n, 1\le j\le m}\{\sup_{t\in\mathbb{R}}|\varphi_{ij}(t)|\},$$

then $(\mathbb{X},\|\cdot\|)$ is a Banach space.

Define an operator Φ on $\mathbb{X}$ by

$$(\Phi\varphi)(t) = ((\Phi_{11}\varphi)(t),...,(\Phi_{ij}\varphi)(t),...,(\Phi_{nm}\varphi)(t))^T,\ t\in\mathbb{R},$$

where

$$\begin{aligned}&(\Phi_{ij}\varphi)(t)\\ &= \int_{-\infty}^t W_{ij}(t,s)\Big[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times\\ &w_{ij}\Big(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{ij}(s-\sigma))\,d\sigma\Big)h_{ij}^{-1}(\varphi_{gl}(s))\\ &+I_{ij}(s)\Big]\,ds + \sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk},\\ &i=1,2,\ldots,n,\ j=1,2,\ldots,m. \end{aligned}\tag{14}$$

Set $\mathbb{X}^*$ be a subset of $\mathbb{X}$ defined by

$$\mathbb{X}^* = \{\varphi\in\mathbb{X} : \|\varphi-\varphi_0\| \le \frac{rK}{1-r}\},$$

where

$$\varphi_0 = (\varphi_{011},...,\varphi_{0ij},...,\varphi_{0nm})^T$$

and

$$\varphi_{0ij} = \int_{-\infty}^t W_{ij}(t,s)I_{ij}(s)\,ds + \sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk},\quad i=1,2,\ldots,n,\ j=1,2,\ldots,m.$$

Then, it follows Lemma 2.5 that

$$\begin{aligned}\|\varphi_0\| &= \max_{1\le i\le n, 1\le j\le m}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^t W_{ij}(t,s)I_{ij}(s)\,ds\\ &\quad+\sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\Big|\Big\}\\ &\le \max_{1\le i\le n, 1\le j\le m}\Big\{\sup_{t\in\mathbb{R}}\Big[\int_{-\infty}^t |W_{ij}(t,s)||I_{ij}(s)|\,ds\\ &\quad+\sum_{\tau_k<t}|W_{ij}(t,\tau_k)||\nu_{ijk}|\Big]\Big\}\\ &\le \max_{1\le i\le n, 1\le j\le m}\Big\{\sup_{t\in\mathbb{R}}\Big[\int_{-\infty}^t e^{2A}e^{-\alpha_{ij}(t-s)}\overline{I}_{ij}\,ds\\ &\quad+\sum_{\tau_k<t}e^{2A}e^{-\alpha_{ij}(t-\tau_k)}\nu_{ijk}\Big]\Big\}\\ &\le \max_{1\le i\le n, 1\le j\le m}\Big\{\frac{\overline{I_{ij}}e^{2A}}{\alpha_{ij}} + \frac{\gamma e^{2A}}{1-e^{-\alpha_{ij}}}\Big\} = K. \end{aligned}\tag{15}$$

Then for arbitrary $\varphi\in\mathbb{X}^*$, from (14) and (15) we have

$$\|\varphi\| \le \|\varphi-\varphi_0\| + \|\varphi_0\| \le \frac{rK}{1-r} + K = \frac{K}{1-r}.$$

Now we prove that Φ is self-mapping from $\mathbb{X}^*$ to $\mathbb{X}^*$.

Firstly, we shall show that for arbitrary $\varphi\in\mathbb{X}^*$, then $\Phi\varphi\in\mathbb{X}^*$. In fact

$$\begin{aligned}&\|\Phi\varphi-\varphi_0\|\\ &= \max_{1\le i\le n, 1\le j\le m}\Big\{\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^t W_{ij}(t,s)\Big[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\\ &\times w_{ij}\Big(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\Big)\times\\ &h_{ij}^{-1}(\varphi_{ij}(s))\Big]\,ds\Big|\Big\}\\ &\le \max_{1\le i\le n, 1\le j\le m}\Big\{\sup_{t\in\mathbb{R}}\Big[\int_{-\infty}^t e^{2A}e^{-\alpha_{ij}(t-s)}\times\\ &\Big[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}\overline{a}_{ij}\|\varphi\|\Big]\overline{a}_{ij}\|\varphi\|\,ds\Big]\Big\}\\ &\le \max_{1\le i\le n, 1\le j\le m}\Big\{\frac{e^{2A}}{\alpha_{ij}}\Big[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}\overline{a}_{ij}^2\Big]\Big\}\|\varphi\|\\ &= r\|\varphi\| \le \frac{rK}{1-r}. \end{aligned}\tag{16}$$

Moreover, we get

$$\begin{aligned}&\sup_{t\in\mathbb{R}}|(\Phi_{ij}\varphi)(t)|\\ &= \sup_{t\in\mathbb{R}}\Big\{\Big|\int_{-\infty}^t W_{ij}(t,s)\Big[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times\\ &w_{ij}\Big(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\Big)h_{ij}^{-1}(\varphi_{ij}(s))\\ &+I_{ij}(s)\Big]\,ds + \sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\Big|\Big\}\end{aligned}$$

$$\leq \max_{1\leq i\leq n,1\leq j\leq m}\left\{\frac{e^{2A}}{\alpha_{ij}}\left[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}\overline{a}_{ij}\|\varphi\|\right]\times \overline{a}_{ij}\|\varphi\|\right\}+K$$
$$\leq \frac{rK}{1-r}+K=\frac{K}{1-r}=N \quad (17)$$

and

$$\begin{aligned}
&\inf_{t\in\mathbb{R}}|(\Phi_{ij}\varphi)(t)|\\
&= \inf_{t\in\mathbb{R}}\left\{\left|\int_{-\infty}^{t}W_{ij}(t,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)\times h_{ij}^{-1}(\varphi_{ij}(s))+I_{ij}(s)\right]ds+\sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\right|\right\}\\
&\geq \inf_{t\in\mathbb{R}}\left\{\left|\int_{-\infty}^{t}W_{ij}(t,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)\times h_{ij}^{-1}(\varphi_{ij}(s))+I_{ij}(s)\right]ds\right|-\left|\sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\right|\right\}\\
&\geq \inf_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}W_{ij}(t,s)\Theta_{ij}\,ds\right\}-\sup_{t\in\mathbb{R},k\in\mathbb{Z}}\left\{\left|\sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\right|\right\}\\
&\geq \inf_{t\in\mathbb{R}}\left\{\int_{-\infty}^{t}e^{-\beta_{ij}(t-s)}\Theta_{ij}\,ds\right\}-\sup_{k\in\mathbb{Z}}|\gamma_{ijk}|\frac{e^{2A}}{1-e^{-\alpha_{ij}}}\\
&\geq \min_{1\leq i,j\leq n}\left\{\frac{\Theta_{ij}}{\beta_{ij}}-\frac{\gamma e^{2A}}{1-e^{-\alpha_{ij}}}\right\}=M,\\
&i=1,2,\dots,n,\ j=1,2,\dots,m. \qquad (18)
\end{aligned}$$

Now, we shall prove that $\Phi\varphi$ is almost periodic. In fact, let $\tau\in T$, $q\in Q$, where the sets T and Q are determined in Lemma 2.3. By Lemma 2.5, we have

$$\begin{aligned}
&|(\Phi_{ij}\varphi)(t+\tau)-(\Phi_{ij}\varphi)(t)|\\
&=\left|\int_{-\infty}^{t+\tau}W_{ij}(t+\tau,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)\times h_{ij}^{-1}(\varphi_{ij}(s))+I_{ij}(s)\right]ds+\sum_{\tau_k<t+\tau}W_{ij}(t+\tau,\tau_k)\nu_{ijk}\right.\\
&\left.-\int_{-\infty}^{t}W_{ij}(t,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)h_{ij}^{-1}(\varphi_{ij}(s))+I_{ij}(s)\right]ds-\sum_{\tau_k<t}W_{ij}(t,\tau_k)\nu_{ijk}\right|\\
&\leq \int_{-\infty}^{t}|W_{ij}(t+\tau,s+\tau)-W_{ij}(t,s)|\left|-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s+\tau)w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)\times h_{ij}^{-1}(\varphi_{gl}(s+\tau-\sigma))\,d\sigma\right)h_{ij}^{-1}(\varphi_{ij}(s+\tau))+I_{ij}(s+\tau)\right|ds\\
&+\int_{-\infty}^{t}|W_{ij}(t,s)|\left|-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s+\tau)w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)\times h_{ij}^{-1}(\varphi_{gl}(s+\tau-\sigma))\,d\sigma\right)h_{ij}^{-1}(\varphi_{ij}(s+\tau))\right.\\
&\left.+\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)\times h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)h_{ij}^{-1}(\varphi_{ij}(s))+I_{ij}(s+\tau)-I_{ij}(s)\right|ds\\
&+\sum_{\tau_k<t}|W_{ij}(t+\tau,\tau_{k+q})-W_{ij}(t,\tau_k)||\nu_{ij(k+q)}|\\
&+\sum_{\tau_k<t}|W_{ij}(t,\tau_k)||\nu_{ij(k+q)}-\nu_{ijk}|\\
&\leq C\epsilon,\ i=1,2,\dots,n,\ j=1,2,\dots,m, \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
C=\max_{1\leq i,j\leq n}\Bigg\{&\frac{1}{\alpha_{ij}}\sum_{C^{gl}\in N_r(i,j)}(2\Gamma\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}+L_w\overline{k}_{ij}e^{2A})\overline{a}_{ij}^2N+\frac{e^{2A}+2\Gamma\overline{I}_{ij}}{\alpha_{ij}}\\
&+\frac{e^{2A}}{\alpha_{ij}}\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}L_w\overline{k}_{ij}\overline{a}_{ij}^2+\frac{\gamma\Gamma}{1-e^{-\frac{\alpha_{ij}}{2}}}+\frac{e^{2A}}{1-e^{-\alpha_{ij}}}\Bigg\}.
\end{aligned}$$

It follows from $(16)-(19)$ that $\Phi\varphi\in\mathbb{X}^*$.

For arbitrary $\varphi\in\mathbb{X}^*$, $\psi\in\mathbb{X}^*$, we can get

$$\begin{aligned}
&\|\Phi\varphi-\Phi\psi\|\\
&=\max_{1\leq i\leq n,1\leq j\leq m}\left\{\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{t}W_{ij}(t,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\varphi_{gl}(s-\sigma))\,d\sigma\right)h_{ij}^{-1}(\varphi_{ij}(s))\,ds\right.\right.\right.\\
&\left.\left.\left.-\int_{-\infty}^{t}W_{ij}(t,s)\left[-\sum_{C^{gl}\in N_r(i,j)}C_{ij}^{gl}(s)\times w_{ij}\left(\int_0^{+\infty}k_{ij}(\sigma)h_{ij}^{-1}(\psi_{gl}(s-\sigma))\,d\sigma\right)\times h_{ij}^{-1}(\psi_{ij}(s))\,d\sigma\right]\right]ds\right|\right\}\\
&\leq \max_{1\leq i\leq n,1\leq j\leq m}\left\{\frac{e^{2A}}{\alpha_{ij}}\left[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}\overline{L}_w\overline{k}_{ij}\overline{a}_{ij}^2\right]\right\}
\end{aligned}$$

$$\times \max_{1\le i\le n,1\le j\le m}\left\{\sup_{t\in\mathbb{R}}|\varphi_{ij}(t)-\psi_{ij}(t)|\right\} = r\|\varphi-\psi\| < \|\varphi-\psi\|. \quad (20)$$

Then from (20), it follows that Φ is a contraction operator in $\mathbb{X}^*$. So, Φ has exactly a unique nonzero fixed point φ^* in $\mathbb{X}^*$ such that $\Phi\varphi^* = \varphi^*$. It is easy to verify that φ^* satisfies (7). Thus, system (1) has exactly one nonzero almost periodic solution. This completes the proof. ■

Theorem 3.2 Assume that the conditions in Theorem 3.1 hold. If $r < \lambda$, then the unique nonzero almost periodic solution of (1) is exponentially stable.

Proof: Let $x(t)$ be arbitrary solution of (7) with the initial condition $x(t_0+0,t_0,x_0) = x_0$, and $y(t) = (y_{11}(t),\cdots,y_{ij}(t),\cdots,y_{mn}(t))^T$ be the unique almost periodic solution of (7) with the initial condition $y(t_0+0,t_0,y_0) = y_0$. Then from (14), we have

$$\begin{aligned} & x(t)-y(t) \\ = \; & W(t,t_0)(x_0-y_0) \\ & +\int_{t_0}^{t} W(t,s)\Big[-\sum_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(s)\times \\ & w_{ij}\Big(\int_0^{+\infty} k_{ij}(\sigma)h_{ij}^{-1}(x_{gl}(s-\sigma))\,d\sigma\Big)h_{ij}^{-1}(x_{ij}(s))\,ds \\ & +\sum_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(s)w_{ij}\Big(\int_0^{+\infty} k_{ij}(\sigma)\times \\ & h_{ij}^{-1}(y_{gl}(s-\sigma))\,d\sigma\Big)h_{ij}^{-1}(y_{ij}(s))\,d\sigma\Big]\,ds. \end{aligned} \quad (21)$$

It follows from Lemma 2.5, (20) and (21) that

$$\begin{aligned} & \|x-y\| \\ \le \; & e^{2A}e^{-\lambda(t-t_0)}\|x_0-y_0\| \\ & +\max_{1\le i\le n,1\le j\le m}\left\{\frac{e^{2A}}{\alpha_{ij}}\Big[\sum_{C^{gl}\in N_r(i,j)}\overline{C}_{ij}^{gl}\overline{L}_w\overline{k}_{ij}\overline{a}_{ij}^2\Big]\right\} \\ & \int_{t_0}^{t} e^{-\lambda(t-t_0)}\|x(s)-y(s)\|ds \\ \le \; & e^{2A}e^{-\lambda(t-t_0)}\|x_0-y_0\| \\ & +r\int_{t_0}^{t} e^{-\lambda(t-s)}\|x(s)-y(s)\|ds, \end{aligned}$$

that is

$$\|x-y\|e^{\lambda t} \le e^{2A}e^{\lambda t_0}\|x_0-y_0\| + r\int_{t_0}^{t} e^{\lambda s}\|x(s)-y(s)\|ds.$$

By Gronwall-Bellman's Lemma, we have

$$\|x-y\| \le e^{2A}\|x_0-y_0\|e^{(r-\lambda)(t-t_0)}.$$

So, the almost periodic solution $y(t)$ is exponentially stable since $r-\lambda<0$. Thus the unique almost periodic solution of (1) is exponentially stable. This completes the proof. ■

IV. An Example

Consider the following CGSICNNs:

$$\begin{cases} x'_{ij}(t) = -a_{ij}(x_{ij}(t))\Big[b_{ij}(x_{ij}(t)) + \sum_{C^{gl}\in N_r(i,j)} C_{ij}^{gl}(t) \\ \qquad \times w_{ij}\Big(\int_0^{+\infty} k_{ij}(s)x_{gl}(t-s)\,ds\Big)x_{ij}(t) \\ \qquad -I_{ij}(t)\Big], \quad t\ne\tau_k, \\ \Delta x_{ij}(\tau_k) = \alpha_{ijk}x_{ij}(\tau_k) + \frac{1-e^{4-4e^4}}{30e^8}, \\ \qquad i=1,2,\ldots,n,\ j=1,2,\ldots,m,\ k\in\mathbb{Z}, \end{cases} \quad (22)$$

where

$$(a_{ij})_{2\times2} = \begin{bmatrix} 3+\sin u & 3+\cos u \\ 3-\sin u & 3-\cos u \end{bmatrix},$$

$$(b_{ij})_{2\times2} = \begin{bmatrix} 0.5e^4u & e^4u \\ 1.5e^4u & 2e^4u \end{bmatrix},$$

$$(C_{ij})_{2\times2} = \begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5 \end{bmatrix},$$

$$(I_{ij})_{2\times2} = \begin{bmatrix} 0.45+0.05\sin t & 0.45-0.05\sin t \\ 0.45+0.05\cos t & 0.45-0.05\cos t \end{bmatrix},$$

$$\omega_{ij} = \frac{1}{32}|u|,\ K_{ij}(t) = e^{-e^4 t},\ i,j=1,2,$$

and (H_2) holds with $A=2$. Obviously,

$$\begin{aligned} & \bar{a}_{ij}=4,\ \underline{a}_{ij}=2,\ i,j=1,2;\ \bar{b}'_{11}=\underline{b}'_{11}=0.5e^4, \\ & \bar{b}'_{12}=\underline{b}'_{12}=e^4,\ \bar{b}'_{21}=\underline{b}'_{21}=1.5e^4,\ \bar{b}'_{22}=\underline{b}'_{22}=2e^4, \\ & \Sigma_{C^{gl}\in N_1(i,j)}\overline{C}_{11}^{gl}=\Sigma_{C^{gl}\in N_1(i,j)}\overline{C}_{12}^{gl}=\Sigma_{C^{gl}\in N_1(i,j)}\overline{C}_{21}^{gl} \\ & =\Sigma_{C^{gl}\in N_1(i,j)}\overline{C}_{22}^{gl}=4,\ L_b=2e^4, L_\omega=\frac{1}{32}, \\ & \bar{I}_{11}=\bar{I}_{12}=\bar{I}_{21}=\bar{I}_{22}=0.5,\ \underline{I}_{11}=\underline{I}_{12}=\underline{I}_{21}=\underline{I}_{22} \\ & =0.4, 1\le\alpha_{ijk}\le\frac{e^2}{2}-1,\ k\in\mathbb{Z}, \alpha_{11}=\underline{a}_{11}\underline{b}'_{11}-2A \\ & =e^4-4, \alpha_{12}=\underline{a}_{12}\underline{b}'_{12}-2A=2e^4-4,\ \alpha_{21}=\underline{a}_{21}\underline{b}'_{21} \\ & -2A=3e^4-4, \alpha_{22}=\underline{a}_{22}\underline{b}'_{22}-2A=4e^4-4, \\ & \beta_{11}=\bar{a}_{11}\bar{b}'_{11}=2e^4, \beta_{12}=\bar{a}_{12}\bar{b}'_{12}=4e^4, \\ & \beta_{21}=\bar{a}_{21}\bar{b}'_{21}=6e^4, \beta_{22}=\bar{a}_{22}\bar{b}'_{22}=8e^4. \end{aligned}$$

By a direct calculation, we get

$$\min_{1\le i,j\le2}\Theta_{ij}=0.4-\frac{2}{e^4},$$

$$M=\frac{0.4-\frac{2}{e^4}}{8e^4}-\frac{e^4}{1-e^{4-4e^4}}\frac{1-e^{4-4e^4}}{30e^8}>0,$$

$$K=\frac{0.5e^4}{e^4-4}+\frac{1}{30e^4},\ r=\frac{2}{e^4-4}<1,$$

then $\frac{K}{1-r}<1$ and $r<\alpha_{ij}, i,j=1,2.$

Now, we can see that all conditions are hold, according to Theorem 3.1 and Theorem 3.2, system (22) has one unique nonzero almost periodic solution which is exponentially stable.

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