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Almost periodic sequence solutions of a discrete cooperation system with feedback controls

Ziping Li and Yongkun Li

Abstract—In this paper, we consider the almost periodic solutions of a discrete cooperation system with feedback controls. Assuming that the coefficients in the system are almost periodic sequences, we obtain the existence and uniqueness of the almost periodic solution which is uniformly asymptotically stable.

Keywords—Discrete cooperation model; Almost periodic solution; Feedback control; Lyapunov function.

I. Introduction

IN Ref. [1], Cui and Chen studied the following continuous cooperation model:

$$\begin{cases} \dot{u} = r_1(t)u\left[1 - \dfrac{u}{a_1(t)+b_1(t)v} - c_1(t)u\right], \\ \dot{v} = r_2(t)v\left[1 - \dfrac{v}{a_2(t)+b_2(t)u} - c_2(t)v\right], \end{cases} \tag{1}$$

where $r_i(t), a_i(t), b_i(t), c_i(t) (i = 1, 2)$ are continuous functions bounded above and below by positive constants. They investigated the asymptotic behavior of system (1). Also, under the assumption that $r_i(t), a_i(t), b_i(t), c_i(t)(i = 1, 2)$ are all continuous T-periodic functions, they obtained sufficient conditions which guarantee the existence of a unique globally asymptotically stable strictly positive periodic solution of system (1).

Since many authors (see for e.g. [2], [3]) have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, then, discrete-timemodels can provide efficient computational types of continuous models for numerical simulations. It is reasonable to study discrete-time population models governed by difference equations.

In Ref. [4], Bai, Fan and Wang studied the existence of periodic solutions of the following discrete cooperation system:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\left\{r_1(k)\left[1 - \dfrac{x_1(k)}{a_1(k)+b_1(k)x_2(k)} \right.\right.\\ \qquad\qquad \left.\left. -c_1(k)x_1(k)\right]\right\}, \\ x_2(k+1) = x_2(k)\exp\left\{r_2(k)\left[1 - \dfrac{x_2(k)}{a_2(k)+b_2(k)x_1(k)} \right.\right.\\ \qquad\qquad \left.\left. -c_2(k)x_2(k)\right]\right\}. \end{cases}$$

Feedback control is the basic mechanism by which systems, whether mechanical, electrical, or biological, maintain their equilibrium or homeostasis. During the last decade, a series of mathematical models have been established to describe the dynamics of feedback control systems ([5], [6], [7], [8]).

In this paper, we are concerned with the following discrete cooperation system with feedback controls:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\big\{r_1(k)\big[1 - \dfrac{x_1(k)}{a_1(k)+b_1(k)x_2(k)} \\ \qquad\qquad -c_1(k)x_1(k)\big] - d_1(k)u_1(k)\big\}, \\ x_2(k+1) = x_2(k)\exp\big\{r_2(k)\big[1 - \dfrac{x_2(k)}{a_2(k)+b_2(k)x_1(k)} \\ \qquad\qquad -c_2(k)x_2(k)\big] - d_2(k)u_2(k)\big\}, \\ \Delta u_1(k) = -f_1(k)u_1(k) + g_1(k)x_1(k), \\ \Delta u_2(k) = -f_2(k)u_2(k) + g_2(k)x_2(k), \end{cases} \tag{2}$$

where $x_i(k)$ $(i = 1, 2)$ is the density of cooperation species i at the nth generation, $r_i(k)$ denotes the intrinsic growth rate of species i and $u_i(k)(i = 1, 2)$ is the control variables (see [1,2] and the references cited therein). Under the assumptions of almost periodicity of coefficients of system (2), we will discuss the existence and uniqueness of almost periodic solutions for system (2).

For any bounded sequence $\{r(k)\}$, we denote

$$r^u = \sup_{k\in N}\{r(k)\}, \quad r^l = \inf_{k\in N}\{r(k)\}.$$

Throughout this paper, we assume that

(H) $\{r_i(k)\}, \{a_i(k)\}, \{b_i(k)\}, \{c_i(k)\}, \{d_i(k)\}, \{f_i(n)\}$ and $\{g_i(n)\}(i = 1, 2)$ are bounded non-negative almost periodic sequences such that

$$\begin{cases} 0 < r_i^l \le r_i(k) \le r_i^u, \\ 0 < a_i^l \le a_i(k) \le a_i^u, \\ 0 < b_i^l \le b_i(k) \le b_i^u, \\ 0 < c_i^l \le c_i(k) \le c_i^u, \\ 0 < d_i^l \le d_i(k) \le d_i^u, \\ 0 < f_i^l \le f_i(k) \le f_i^u < 1, \\ 0 < g_i^l \le g_i(k) \le g_i^u. \end{cases} \tag{3}$$

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By the biological meaning, we focus our discussion on the positive solutions of the system (2). Hence, it is assumed that the initial conditions of (2) are of the form

$$x_i(0) > 0,\ u_i(0) > 0,\ i = 1, 2. \tag{4}$$

One can easily show that the solutions of (2) with the initial condition (4) are defined and remain positive for all $k \in Z^+ = \{0, 1, 2, \ldots\}$.

II. Preliminaries

In this section, we will introduce two definitions and some useful lemmas.

Definition 1. *[9] A sequence $x : Z \longrightarrow R^k$ is called an almost periodic sequence if the ε-translation set of x*

$$E\{\varepsilon, x\} := \{\tau \in Z : |x(k+\tau) - x(k)| < \varepsilon\}$$

for all $k \in Z$ is a relatively dense set in Z for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each discrete interval of length $l(\varepsilon)$ contains an integer $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(k+\tau) - x(k)| < \varepsilon$$

for all $k \in Z$, τ is called the ε-translation number of $x(k)$.

Definition 2. *[9] Let $f : Z \times D \longrightarrow R^k$, where D is an open set in R^k, $f(k, x)$ is said to be almost periodic in k uniformly for $x \in D$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and any compact set S in D, there exists a positive integer $l(\varepsilon, S)$ such that any interval of length $l(\varepsilon, S)$ contains an integer τ for which*

$$|f(k+\tau, x) - f(k, x)| < \varepsilon$$

for all $k \in Z$ and $x \in S$. τ is called the ε-translation number of $f(k, x)$.

Lemma 1. *[9] If $\{x(k)\}$ is an almost periodic sequence, then $\{x(k)\}$ is bounded.*

Lemma 2. *[9] $\{x(k)\}$ is an almost periodic sequence if and only if for sequence $\{h'_k\} \subset Z$, there exists a subsequence $\{h_k\} \subset \{h'_k\}$,such that $\{x(k+h_k)\}$ converges uniformly on $x \in S$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Lemma 3. *[9] Let $k \in N^+_{k_0} = \{k_0, k_0+1, \ldots, k_0+r, \ldots\}$, $r \geq 0$. For any fixed k, $g(k, r)$ is a non-decreasing function with respect to r, and for $k \geq k_0$, the following inequalities hold: $y(k+1) \leq g(k, y(k))$, $u(k+1) \leq g(k, u(k))$. If $y(k_0) \leq u(k_0)$, then $y(k) \leq u(k)$ for all $k \geq k_0$.*

Now let us consider the following single species discrete model:

$$N(k+1) = N(k)\exp\{a(k) - b(k)N(k)\},$$

where $\{a(k)\}$ and $\{b(k)\}$ are strictly positive sequences of real numbers defined for $k \in N = \{0, 1, 2, \ldots\}$ and $0 < a^l < a^u$, $0 < b^l < b^u$.

Lemma 4. *[10] If $\{x(k)\}$ with initial condition $x(k) > 0$ and for all $k \in N$ satisfies*

$$x(k+1) \leq x(k)\exp\{a(k) - b(k)x(k)\},$$

then

$$\lim_{k\to+\infty}\sup x(k) \leq M,$$

where $a(k)$ and $b(k)$ are nonnegative sequences with positive bounded below, $M = \frac{1}{b^l}\exp\{a^u - 1\}$.

Lemma 5. *[10] If $\{x(k)\}$ satisfies*

$$x(k+1) \geq x(k)\exp\{a(k) - b(k)x(k)\}, k \geq N_0$$

and

$$\lim_{k\to+\infty}\sup x(k) \leq M,\ \frac{b^u}{a^l}M > 1,\ x(N_0) > 0,$$

then

$$\lim_{k\to+\infty}\inf x(k) \geq m,$$

where $a(k)$ and $b(k)$ are nonnegative sequences with positive bounded below, $m = \frac{a^l}{b^u}\exp\{a^l - b^u M\}$.

III. Persistence

In this section, we establish a persistence result for model (2).

Theorem 1. *Assume that* (3) *and* (4) *hold, furthermore,*
(H_1) $r_i^l - d_i^u u_i^* > 0, i = 1, 2$
is satisfied. Then for any positive solution $(x_1(k), x_2(k), u_1(k), u_2(k))$ of (2)*, we have*

$$\begin{aligned} x_{i*} &\leq \lim_{k\to+\infty}\inf x(k) \leq \lim_{k\to+\infty}\sup x(k) \leq x_i^*, \\ u_{i*} &\leq \lim_{k\to+\infty}\inf u(k) \leq \lim_{k\to+\infty}\sup u(k) \leq u_i^*, \end{aligned} \tag{5}$$

where

$$x_i^* = \frac{1}{r_i^l c_i^l}\exp\{r_i^u - 1\},$$
$$x_{i*} = \frac{r_i^l - d_i^u u_i^*}{\left(\frac{r_i^l}{a_i^l} + r_i^l c_i^u\right)}\exp\left\{r_i^l - d_i^u u_i^* - \left(\frac{r_i^l}{a_i^l} + r_i^l c_i^u\right)x_i^*\right\}.$$
$$u_i^* = \frac{g_i^u x_i^*}{f_i^l},\ u_{i*} = \frac{g_i^l x_{i*}}{f_i^u}.$$

Proof: We first prove that

$$\lim_{k\to+\infty}\sup x_i(k) \leq x_i^*,\ \ i = 1, 2.$$

By the first equation of system (2), we have

$$\begin{aligned} &x_1(k+1) \leq x_1(k)\exp\{r_1(k)[1 - c_1(k)x_1(k)]\} \\ &= x_1(k)\exp\{r_1(k) - r_1(k)c_1(k)x_1(k)\}. \end{aligned}$$

By applying Lemma 4, we have

$$\lim_{k\to+\infty}\sup x_1(k) \leq \frac{1}{r_1^l c_1^l}\exp\{r_1^u - 1\} \triangleq x_1^*. \tag{6}$$

By using the second equation of system (2), similar to the above analysis, we can obtain

$$\lim_{k\to+\infty}\sup x_2(k) \le \frac{1}{r_2^l c_2^l}\exp\{r_2^u - 1\} \triangleq x_2^*. \quad (7)$$

Therefore, for each $\varepsilon > 0$, there exists a large enough integer k_0 such that

$$x_i(k) \le x_i^* + \varepsilon,\ i = 1, 2, \text{ whenever } k \ge k_0.$$

Now we prove that

$$\lim_{k\to+\infty}\sup u_i(k) \le u_i^*,\ i = 1, 2.$$

By the third equation of system (2), we can get that

$$\begin{aligned} u_1(k) &= \prod_{i=0}^{k-1}(1 - f_1(i))\Big[u_1(0) + \sum_{i=0}^{k-1}\frac{g_1(i)x_1(i)}{\prod_{j=0}^{i}(1 - f_1(j))}\Big] \\ &\le (1 - f_1^l)^k(u_1(0) + v_1) \\ &\quad + g_1^u(x_1^* + \varepsilon)\sum_{i=k_0}^{k-1}\prod_{j=i+1}^{k-1}(1 - f_1(j)) \\ &\le (1 - f_1^l)^k(u_1(0) + v_1) \\ &\quad + g_1^u(x_1^* + \varepsilon)\sum_{i=k_0}^{k-1}(1 - f_1^l)^{k-i-1}, \end{aligned}$$

where $v_1 = \sum_{i=0}^{k-1}\frac{g_1(i)x_1(i)}{\prod_{j=0}^{i}(1-f_1(j))}$. Since $0 < f_1^l < 1$, we can find a positive number d such that $1 - f_1^l = e^{-d}$, then, by Stolz's theorem, we have

$$\begin{aligned} &\sum_{i=k_0}^{k-1}(1 - f_1^l)^{k-i-1} \\ &= \frac{\sum_{i=k_0}^{k-1} e^{d(i+1)}}{e^{dk}} \to \frac{1}{1 - e^{-d}} = \frac{1}{f_1^l}, (k \to \infty). \end{aligned}$$

Thus

$$\lim_{k\to+\infty}\sup u_1(k) \le \frac{g_1^u x_1^*}{f_1^l} \triangleq u_1^*. \quad (8)$$

In the similar way, we can prove that

$$\lim_{k\to+\infty}\sup u_2(k) \le \frac{g_2^u x_2^*}{f_2^l} \triangleq u_2^*. \quad (9)$$

Therefore, for each $\varepsilon > 0$, there exists $k_0 \in N$ such that

$$u_i(k) \le u_i^* + \varepsilon,\ i = 1, 2.$$

Next, we prove that

$$\lim_{k\to+\infty}\inf x_i(k) \ge x_{i*},\ i = 1, 2.$$

By the first equation of system (2), we have

$$\begin{aligned} x_1(k+1) &\ge x_1(k)\exp\Big\{r_1(k)\Big[1 - \frac{x_1(k)}{a_1^l} - c_1^u x_1(k)\Big] \\ &\quad - d_1^u(u_1^* + \varepsilon)\Big\} \\ &\ge x_1(k)\exp\Big\{[r_1^l - d_1^u(u_1^* + \varepsilon)] \\ &\quad - \Big(\frac{r_1^l}{a_1^l} + r_1^l c_1^u\Big)x_1(k)\Big\}. \end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} \lim_{k\to+\infty}\inf x_1(k) &\ge \frac{r_1^l - d_1^u(u_1^* + \varepsilon)}{\Big(\frac{r_1^l}{a_1^l} + r_1^l c_1^u\Big)}\exp\Big\{r_1^l - d_1^u(u_1^* + \varepsilon) \\ &\quad - \Big(\frac{r_1^l}{a_1^l} + r_1^l c_1^u\Big)x_1^*\Big\}. \end{aligned}$$

Let $\varepsilon \to 0$, it follows that

$$\begin{aligned} \lim_{k\to+\infty}\inf x_1(k) &\ge \frac{r_1^l - d_1^u u_1^*}{\Big(\frac{r_1^l}{a_1^l} + r_1^l c_1^u\Big)}\exp\Big\{r_1^l - d_1^u u_1^* \\ &\quad - \Big(\frac{r_1^l}{a_1^l} + r_1^l c_1^u\Big)x_1^*\Big\} \triangleq x_{1*}. \quad (10) \end{aligned}$$

In the similar way, we can prove that

$$\begin{aligned} \lim_{k\to+\infty}\inf x_2(k) &\ge \frac{r_2^l - d_2^u u_2^*}{\Big(\frac{r_2^l}{a_2^l} + r_2^l c_2^u\Big)}\exp\Big\{r_2^l - d_2^u u_2^* \\ &\quad - \Big(\frac{r_2^l}{a_2^l} + r_2^l c_2^u\Big)x_2^*\Big\} \triangleq x_{2*}. \quad (11) \end{aligned}$$

Therefore, for each $\varepsilon > 0$, there exists a large enough integer k_0 such that

$$x_i(k) > x_{i*} - \varepsilon,\ i = 1, 2.$$

Finally, we prove that

$$\lim_{k\to+\infty}\inf u_i(k) \ge u_{i*},\ i = 1, 2.$$

By the third equation of system (2), we can get that

$$\begin{aligned} u_1(k) &= \prod_{i=0}^{k-1}(1 - f_1(i))\Big[u_1(0) + \sum_{i=0}^{k-1}\frac{g_1(i)x_1(i)}{\prod_{j=0}^{i}(1 - f_1(j))}\Big] \\ &\ge (1 - f_1^u)^k(u_1(0) + v_1) \\ &\quad + g_1^l(x_{1*} - \varepsilon)\sum_{i=k_0}^{k-1}\prod_{j=i+1}^{k-1}(1 - f_1(j)) \\ &\ge (1 - f_1^u)^k(u_1(0) + v_1) \\ &\quad + g_1^l(x_{1*} - \varepsilon)\sum_{i=k_0}^{k-1}(1 - f_1^u)^{k-i-1}, \end{aligned}$$

where $v_1 = \sum_{i=0}^{k-1}\frac{g_1(i)x_1(i)}{\prod_{j=0}^{i}(1-f_1(j))}$. Since $0 < f_1^u < 1$, we can find a positive number d such that $1 - f_1^u = e^{-d}$, then, by Stolz's theorem, we have

$$\begin{aligned} &\sum_{i=k_0}^{k-1}(1 - f_1^u)^{k-i-1} \\ &= \frac{\sum_{i=k_0}^{k-1} e^{d(i+1)}}{e^{dk}} \to \frac{1}{1 - e^{-d}} = \frac{1}{f_1^u}, (k \to \infty). \end{aligned}$$

Thus

$$\lim_{k\to+\infty}\inf u_1(k) \ge \frac{g_1^l x_{1*}}{f_1^u} \triangleq u_{1*}. \tag{12}$$

In the similar way, we can prove that

$$\lim_{k\to+\infty}\inf u_2(k) \ge \frac{g_2^l x_{2*}}{f_2^u} \triangleq u_{2*}. \tag{13}$$

Hence, from (5)-(13), we can get that

$$x_{i*} \le \lim_{k\to+\infty}\inf x(k) \le \lim_{k\to+\infty}\sup x(k) \le x_i^*,$$
$$u_{i*} \le \lim_{k\to+\infty}\inf u(k) \le \lim_{k\to+\infty}\sup u(k) \le u_i^*,$$

where $i = 1, 2$. This completes the proof of Theorem 1. ∎

IV. Main results

Consider the following almost periodic difference system

$$x(n+1) = f(n, x(n)), n \in Z^+, \tag{14}$$

where $f : Z \times S_B \to R^k, S_B = \{x \in R^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x. The product system of (14) is as follows:

$$x(n+1) = f(n, x(n)), y(n+1) = f(n, y(n)). \tag{15}$$

Lemma 6. *[11] Suppose that there exists a Lyapunov functional $V(n, x, y)$ defined for $n \in N$, $\|x\| < B$, $\|y\| < B$ satisfying the following conditions:*

(i) $a(\|x-y\|) \le V(n,x,y) \le b(\|x-y\|)$, where $a, b \in K$, with $K = \{a \in C(R^+, R^+) : a(0) = 0\}$ and a is increasing;

(ii) $|V(n,x_1,y_1)-V(n,x_2,y_2)| \le L(\|x_1-x_2\|+\|y_1-y_2\|)$, where $L > 0$ is a constant;

(iii) $\Delta V_{(15)}(n,x,y) \le -aV(n,x,y)$, where $0 < a < 1$ is a constant and

$$\Delta V_{(15)}(n,x,y) = V(n+1, f(n,x), f(n,y)) - V(n,x,y).$$

Moreover, if there exists a solution $\varphi(n)$ of (14) such that $\|\varphi(n)\| \le B^* < B$, for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of system (14) which is bounded by B^*. In particular, if $f(n,x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of (14) of period ω.

According to Lemma 6, we first prove that there exists a bounded solution of (2), then construct an adaptive Lyapunov functional for (2).

We denote by Ω the set of all solutions $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))$ of system (2) satisfying $x_{i*} \le x_i(k) \le x_i^*$, $u_{i*} \le u_i(k) \le u_i^*$, $i = 1, 2$ for all $k \in Z^+$.

Lemma 7. *Assume that (H) and the conditions of Theorem 1 hold, then $\Omega \ne \emptyset$.*

Proof: It is now possible to show by an inductive argument that the system (2) leads to

$$\begin{cases} x_1(k) = x_1(0)\exp\sum_{l=0}^{k-1}\Big\{r_1(l)\Big[1-\frac{x_1(l)}{a_1(l)+b_1(l)x_2(l)} \\ \qquad -c_1(l)x_1(l)\Big]-d_1(l)u_1(l)\Big\}, \\ x_2(k) = x_2(0)\exp\sum_{l=0}^{k-1}\Big\{r_2(l)\Big[1-\frac{x_2(l)}{a_2(l)+b_2(l)x_1(l)} \\ \qquad -c_2(l)x_2(l)\Big]-d_2(l)u_2(l)\Big\}, \\ u_1(k) = u_1(0)-\sum_{l=0}^{k-1}\Big\{f_1(l)u_1(l)-g_1(l)x_1(l)\Big\}, \\ u_2(k) = u_2(0)-\sum_{l=0}^{k-1}\Big\{f_2(l)u_2(l)-g_2(l)x_2(l)\Big\}. \end{cases}$$

From Theorem 1, for any solution $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))$ of system (2) with initial condition (4) satisfy (5). Hence, for any $\varepsilon > 0$, there exist k_0, if k_0 is sufficiently large, we have

$$x_{i*}-\varepsilon \le x_i(k) \le x_i^*+\varepsilon, u_{i*}-\varepsilon \le u_i(k) \le u_i^*+\varepsilon, i = 1, 2.$$

Let $\{\tau_\alpha\}$ be any integer valued sequence such that $\tau_\alpha \to \infty$ as $\alpha \to \infty$, we claim that there exists a subsequence of $\{\tau_\alpha\}$, we still denote it by $\{\tau_\alpha\}$, such that

$$x_i(k+\tau_\alpha) \to x_i^*(k)$$

uniformly in n on any finite subset B of Z as $\alpha \to \infty$, where $B = \{a_1, a_2, \ldots, a_m\}$, $a_h \in Z$, $h = (1, 2, \ldots, m)$ and m is a finite number.

In fact, for any finite subset $B \subset Z$, when α is large enough, $\tau_\alpha + a_h > k_0$, $h = 1, 2, \ldots, m$. So

$$x_{i*}-\varepsilon \le x_i(k+\tau_\alpha) \le x_i^*+\varepsilon,\ i = 1, 2,$$
$$u_{i*}-\varepsilon \le u_i(k+\tau_\alpha) \le u_i^*+\varepsilon,\ i = 1, 2.$$

That is, $\{x_i(k+\tau_\alpha), u_i(k+\tau_\alpha)\}$ are uniformly bounded for large enough α.

Now, for $a_1 \in B$, we can choose a subsequence $\{\tau_\alpha^{(1)}\}$ of $\{\tau_\alpha\}$ such that $\{x_i(a_1+\tau_\alpha^{(1)}), u_i(a_1+\tau_\alpha^{(1)})\}$ uniformly converges on Z^+ for α large enough.

Similarly, for $a_2 \in B$, we can choose a subsequence $\{\tau_\alpha^{(2)}\}$ of $\{\tau_\alpha^{(1)}\}$ such that $\{x_i(a_2+\tau_\alpha^{(2)}), u_i(a_2+\tau_\alpha^{(2)})\}$ uniformly converges on Z^+ for α large enough.

Repeating this procedure, for $a_m \in B$, we can choose a subsequence $\{\tau_\alpha^{(m-1)}\}$ of $\{\tau_\alpha^{(m)}\}$ such that $\{x_i(a_m+\tau_\alpha^{(m)}), u_i(a_m+\tau_\alpha^{(m)})\}$ uniformly converges on Z^+ for α large enough.

Now pick the sequence $\{\tau_\alpha^{(m)}\}$ which is a subsequence of $\{\tau_\alpha\}$, we still denote it as $\{\tau_\alpha\}$ then for all $k \in B$, we have

$$\{x_i(k+\tau_\alpha) \to x_i^*,\ u_i(k+\tau_\alpha) \to u_i^*\}$$

uniformly in $k \in B$ as $\alpha \to \infty$.

By the arbitrary of B, the conclusion is valid.

Since $\{r_i(k)\}$, $\{a_i(k)\}$, $\{b_i(k)\}$, $\{c_i(k)\}$, $\{d_i(k)\}$, $\{f_i(k)\}$ and $\{g_i(k)\}$ are almost periodic sequence, for above sequence

$\{\tau_\alpha\}, \tau_\alpha \to \infty$ as $\alpha \to \infty$, there exists a subsequence still denote it by $\{\tau_\alpha\}$ (if necessary, we take subsequence), such that

$$\begin{aligned}
&r_i(k+\tau_\alpha) \to r_i(k),\ a_i(k+\tau_\alpha) \to a_i(k),\\
&b_i(k+\tau_\alpha) \to b_i(k),\ c_i(k+\tau_\alpha) \to c_i(k),\\
&d_i(k+\tau_\alpha) \to d_i(k),\ f_i(k+\tau_\alpha) \to f_i(k),\\
&g_i(k+\tau_\alpha) \to g_i(k),
\end{aligned}$$

as $\alpha \to \infty$ uniformly on Z^+.

For any $\delta \in Z$, we can assume that $\tau_\alpha + \delta \geq k_0$ for δ large enough. Let $k \geq 0$ and $k \in Z$, by an inductive argument of (2) from $\tau_\alpha + \delta$ to $k + \tau_\alpha + \delta$ leads to

$$\begin{aligned}
&x_i(k+\tau_\alpha+\delta)\\
&= x_i(\tau_\alpha+\delta)\exp\sum_{l=\tau_\alpha+\delta}^{k+\tau_\alpha+\delta-1}\bigg\{r_i(l)\bigg[1-\frac{x_i(l)}{a_i(l)+b_i(l)x_j(l)}\\
&-c_i(l)x_i(l)\bigg]-d_i(l)u_i(l)\bigg\},\\
&u_i(k+\tau_\alpha+\delta)\\
&= u_i(\tau_\alpha+\delta)-\sum_{l=\tau_\alpha+\delta}^{k+\tau_\alpha+\delta-1}\bigg\{f_i(l)u_i(l)-g_i(l)x_i(l)\bigg\}.
\end{aligned}$$

Then, for $i, j = 1, 2,\ i \neq j$, we have

$$\begin{aligned}
&x_i(k+\tau_\alpha+\delta)\\
&= x_i(\tau_\alpha+\delta)\exp\sum_{l=\delta}^{k+\delta-1}\bigg\{r_i(l+\tau_\alpha)\bigg[1\\
&-\frac{x_i(l+\tau_\alpha)}{a_i(l+\tau_\alpha)+b_i(l+\tau_\alpha)x_j(l+\tau_\alpha)}\\
&-c_i(l+\tau_\alpha)x_i(l+\tau_\alpha)\bigg]-d_i(l+\tau_\alpha)u_i(l+\tau_\alpha)\bigg\},\\
&u_i(k+\tau_\alpha+\delta)\\
&= u_i(\tau_\alpha+\delta)-\sum_{l=\delta}^{k+\delta-1}\{f_i(l+\tau_\alpha)u_i(l+\tau_\alpha)\\
&-g_i(l+\tau_\alpha)x_i(l+\tau_\alpha)\}.
\end{aligned}$$

Let $\alpha \to \infty$, for any $k \geq 0$,

$$\begin{aligned}
x_i^*(k+\delta) &= x_i^*(\delta)\exp\sum_{l=\delta}^{k+\delta-1}\bigg\{r_i(l)\bigg[1-\frac{x_i^*(l)}{a_i(l)+b_i(l)x_j^*(l)}\\
&\quad -c_i(l)x_i^*(l)\bigg]-d_i(l)u_i^*(l)\bigg\},\\
u_i^*(k+\delta) &= u_i^*(\delta)-\sum_{l=\delta}^{k+\delta-1}\bigg\{f_i(l)u_i^*(l)-g_i(l)x_i^*(l)\bigg\}.
\end{aligned}$$

By the arbitrariness of δ, $X^*(k) = (x_1^*(k), x_2^*(k), u_1^*(k), u_2^*(k))$ is a solution of system (2) on Z^+. It is clear that

$$0 < x_{i*} \leq x_i^*(k) \leq x_i^*, 0 < u_{i*} \leq u_i^*(k) \leq u_i^*,\ k \in Z^+.$$

So $\Omega \neq \Phi$. Lemma 7 is valid. ∎

Theorem 2. *Suppose the conditions of Lemma 7 are satisfied, moreover,* $0 < \beta < 1$*, where*

$$\beta = \min\{r_{ij}, r_{ij}^*\},$$

$$\begin{aligned}
r_{ij} &= \frac{2r_i^l x_{i*}}{a_i^u + b_i^u x_j^*} + 2c_i^l r_i^l x_{i*} - \frac{r_i^{u^2}\xi_i^{u^2}}{(a_i^l + b_i^l x_{j*})^2} - r_i^{u^2} c_i^{u^2} x_i^{*2}\\
&\quad -2r_i^{u^2} c_i^u x_i^{*^2} - g_i^{u^2} x_i^{*^2} - \frac{r_i^{u^2} b_i^u x_i^{*^2} x_j^*}{(a_i^l + b_i^l x_j*)^3}\\
&\quad -\frac{r_i^{*^2} c_i^u b_i^u x_i^* x_j^*}{(a_i^l + b_i^l x_j*)^2} - \frac{r_i^u b_i^u x_i^{*^2} x_j^{*^2}}{(a_i^l + b_i^l x_{j*})^2} - \frac{r_i^u d_i^u x_i^*}{a_i^l + b_i^l x_{j*}}\\
&\quad -r_i^u c_i^u d_i^u x_i^* - (1 - f_i^l)x_i^* + d_i^l - \frac{b_j^{u^2} x_j^{*^2} x_i^{*^2}}{(a_j^l + b_j^l x_{i*})^4}\\
&\quad -\frac{r_j^{u^2} b_j^u x_j^{*^2} x_i^*}{(a_j^l + b_j^l x_{i*})^3} - \frac{r_j^{*^2} c_j^u b_j^u x_j^* x_i^*}{(a_j^l + b_j^l x_{i*})^2} - \frac{r_j^u b_j^u x_j^{*^2} x_i^{*^2}}{(a_j^l + b_j^l x_{i*})^2}\\
&\quad -\frac{r_j^u b_j^u d_j^u x_j^* x_i^*}{(a_j^l + b_j^l x_{i*})^2},\\
r_{ij}^* &= 2f_i^l - \frac{r_i^u d_i^u x_i^*}{a_i^l + b_i^l x_{j*}} - r_i^u c_i^u d_i^u x_i^* - (1 - f_i^l)x_i^* + d_i^l\\
&\quad -\frac{r_i^u b_i^u d_i^u x_i^* x_j^*}{(a_i^l + b_i^l x_{j*})^2} - d_i^{u^2} - f_i^{u^2},
\end{aligned}$$

for $i, j = 1, 2,\ i \neq j$. *Then there exists a unique uniformly asymptotically stable almost periodic solution* $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))$ *of* (2) *which is bounded by* Ω *for all* $k \in Z^+$.

Proof: Let $p_i(k) = \ln x_i(k)$, from system (2), we have

$$\begin{cases}
p_i(k+1) = p_i(k) + r_i(k)\bigg[1 - \dfrac{\exp\{p_i(k)\}}{a_i(k) + b_i(k)\exp\{p_j(k)\}}\\
\qquad\qquad -c_i(k)\exp\{p_i(k)\}\bigg] - d_i(k)u_i(k),\\
\Delta u_i(k) = -f_i(k)u_i(k) + g_i(k)\exp\{p_i(k)\},
\end{cases} \tag{16}$$

where $i, j = 1, 2, i \neq j$. From Lemma 7, we know that system (16) has a bounded solution $Y(k) = (p_1(k), p_2(k), u_1(k), u_2(k))$ satisfying

$$\ln x_{i*} \leq p_i(k) \leq \ln x_i^*, u_{i*} \leq u_i(k) \leq u_i^*,\ i = 1, 2,\ \ k \in Z^+.$$

Hence, $|p_i(k)| \leq A_i, |u_i(k)| \leq B_i, A_i = \max\{|\ln x_{i*}|, |\ln x_i^*|\}, B_i = \max\{u_{i*} u_i^*\}, i = 1, 2.$

For $(X, U) \in R^{2+2}$, we define the norm $\|(X, U)\| = \sum_{i=1}^2 |x_i| + \sum_{i=1}^2 |u_i|$.

Consider the product system of system (16)

$$\begin{cases}
p_i(k+1) = p_i(k) + r_i(k)\Big[1 - \frac{\exp\{p_i(k)\}}{a_i(k)+b_i(k)\exp\{p_j(k)\}}\\
\qquad\qquad -c_i(k)\exp\{p_i(k)\}\Big] - d_i(k)u_i(k),\\
\Delta u_i(k) = -f_i(k)u_i(k) + g_i(k)\exp\{p_i(k)\},\\
q_i(k+1) = q_i(k) + r_i(k)\Big[1 - \frac{\exp\{q_i(k)\}}{a_i(k)+b_i(k)\exp\{q_j(k)\}}\\
\qquad\qquad -c_i(k)\exp\{q_i(k)\}\Big] - d_i(k)\omega_i(k),\\
\Delta\omega_i(k) = -f_i(k)\omega_i(k) + g_i(k)\exp\{q_i(k)\}.
\end{cases} \tag{17}$$

Suppose that $Z = (p_1(k), p_2(k), u_1(k), u_2(k)), W = (q_1(k), q_2(k), \omega_1(k), \omega_2(k))$ are any two solutions of system (16) defined on $Z^+ \times S^* \times S^*$, then $\|Z\| \leq B\|$, $\|W\| \leq B$, where $B = \sum_{i=1}^{2}\{A_i + B_i\}$, $S^* =$

$\{(p_1(k), p_2(k), u_1(k), u_2(k)) | \ln x_{i*} \le p_i(k) \le \ln x_i^*, u_{i*} \le u_i(k) \le u_i^*, i = 1, 2, k \in Z^+\}$.

Consider a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows

$$V(k, Z, W) = \sum_{i=1}^{2} \{(p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2\}.$$

It is easy to see that the norm

$$\|Z - W\| = \sum_{i=1}^{2} \{| p_i(k) - q_i(k) | + | u_i(k) - \omega_i(k) |\}$$

and the norm

$$\|Z - W\|_* = \left\{ \sum_{i=1}^{2} \{(p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2\} \right\}^{\frac{1}{2}}$$

are equivalent. That is, there exist two constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1 \|Z - W\| \le \|Z - W\|_* \le C_2 \|Z - W\|,$$

then

$$(C_1 \|Z - W\|)^2 \le V(k, Z, W) \le (C_2 \|Z - W\|)^2.$$

Let $a \in C(R^+, R^+)$, $a(x) = C_1^2 x^2$, $b \in C(R^+, R^+)$, $b(x) = C_2^2 x^2$, thus the condition (i) in Lemma 6 is satisfied.

In addition,

$$\begin{aligned}
& |V(n, Z, W) - V(n, \tilde{Z}, \tilde{W})| \\
= & \Big| \sum_{i=1}^{2} \Big\{ (p_i(n) - q_i(n))^2 + (u_i(n) - w_i(n))^2 \Big\} \\
& - \sum_{i=1}^{2} \Big\{ (\tilde{p}_i(n) - \tilde{q}_i(n))^2 + (\tilde{u}_i(n) - \tilde{w}_i(n))^2 \Big\} \Big| \\
\le & \sum_{i=1}^{2} \Big| (p_i(n) - q_i(n))^2 - (\tilde{p}_i(n) - \tilde{q}_i(n))^2 \Big| \\
& + \sum_{i=1}^{2} \Big| (u_i(n) - w_i(n))^2 - (\tilde{u}_i(n) - \tilde{w}_i(n))^2 \Big| \\
= & \sum_{i=1}^{2} \Big\{ |(p_i(n) - q_i(n)) + (\tilde{p}_i(n) - \tilde{q}_i(n))| \\
& \times |(p_i(n) - q_i(n)) - (\tilde{p}_i(n) - \tilde{q}_i(n))| \Big\} \\
& + \sum_{i=1}^{2} \Big\{ |(u_i(n) - w_i(n)) + (\tilde{u}_i(n) - \tilde{w}_i(n))| \\
& \times |(u_i(n) - w_i(n)) - (\tilde{u}_i(n) - \tilde{w}_i(n))| \Big\} \\
\le & \sum_{i=1}^{2} \Big\{ (|p_i(n)| + |q_i(n)| + |\tilde{p}_i(n)| + |\tilde{q}_i(n)|) \\
& \times (|(p_i(n)| + |q_i(n)| + |\tilde{p}_i(n)| + |\tilde{q}_i(n)|) \Big\} \\
& + \sum_{i=1}^{2} \Big\{ (|u_i(n)| + |w_i(n)| + |\tilde{u}_i(n)| + |\tilde{w}_i(n)|) \\
& \times (|(u_i(n)| + |w_i(n)| + |\tilde{u}_i(n)| + |\tilde{w}_i(n)|) \Big\} \\
\le & L\{ \sum_{i=1}^{2} \{ |p_i(n) - \tilde{p}_i(n)| + |u_i(n) - \tilde{u}_i(n)| \} \\
& + \sum_{i=1}^{2} \{ |q_i(n) - \tilde{q}_i(n)| + |w_i(n) - \tilde{w}_i(n)| \} \} \\
= & L\{ \|Z - \tilde{Z}\| + \| \|W - \tilde{W}\| \},
\end{aligned}$$

where $L = 4 \max\{A_i, B_i\} (i = 1, 2)$. Hence, the condition (ii) of Lemma 6 is satisfied.

Finally, calculate the ΔV of $V(k)$ along the solutions of (17), we can obtain

$$\begin{aligned}
& \Delta V_{(17)}(k) \\
= & V(k+1) - V(k) \\
= & \sum_{i=1}^{2} \{ (p_i(k+1) - q_i(k+1))^2 \\
& + (u_i(k+1) - \omega_i(k+1))^2 \} \\
& - \sum_{i=1}^{2} \{ (p_i(k) - q_i(k))^2 + (u_i(k) - \omega_i(k))^2 \} \\
= & \sum_{i=1}^{2} \Big\{ \Big(p_i(k) + r_i(k) \Big[1 - \frac{\exp\{p_i(k)\}}{a_i(k) + b_i(k) \exp\{p_j(k)\}} \\
& - c_i(k) \exp\{p_i(k)\} \Big] - d_i(k) u_i(k) - q_i(k) \\
& - r_i(k) \Big[1 - \frac{\exp\{q_i(k)\}}{a_i(k) + b_i(k) \exp\{q_j(k)\}} \\
& - c_i(k) \exp\{q_i(k)\} \Big] - d_i(k) \omega_i(k) \Big)^2 \\
& + [((1 - f_i(k)) u_i(k) + g_i(k) \exp\{p_i(k)\} \\
& - (1 - f_i(k)) \omega_i(k) - g_i(k) \exp\{q_i(k)\}]^2 \\
& - (p_i(k) - q_i(k))^2 - (u_i(k) - \omega_i(k))^2 \Big\} \\
= & \sum_{i=1}^{2} \Big\{ \Big[\frac{r_i^2(k)}{(a_i(k) + b_i(k) e^{q_j(k)})^2} + r_i^2(k) c_i^2(k) \\
& + 2 r_i^2(k) c_i(k) + g_i^2(k) \Big] (e^{p_i(k)} - e^{q_i(k)})^2 \\
& + \frac{b_i^2(k) (e^{p_i(k)})^2 (e^{p_j(k)} - e^{q_j(k)})^2}{(a_i(k) + b_i(k) e^{p_j(k)})^2 (a_i(k) + b_i(k) e^{q_j(k)})^2} \\
& - \Big[\frac{2 r_i^2(k) b_i(k) e^{p_i(k)} (e^{p_j(k)} - e^{q_j(k)})^2}{(a_i(k) + b_i(k) e^{p_j(k)})(a_i(k) + b_i(k) e^{q_j(k)})^2} \\
& + \frac{2 r_i^2(k) b_i(k) c_i(k)}{(a_i(k) + b_i(k) e^{p_j(k)})(a_i(k) + b_i(k) e^{q_j(k)})} \Big] \\
& \times (e^{p_i(k)} - e^{q_i(k)})(e^{p_j(k)} - e^{q_j(k)}) \\
& - \Big[\frac{2 r_i(k)}{a_i(k) + b_i(k) e^{q_j(k)}} + 2 c_i(k) r_i(k) \Big] \\
& \times (p_i(k) - q_i(k))(e^{p_i(k)} - e^{q_i(k)}) \\
& + \frac{2 r_i(k) b_i(k) e^{p_i(k)}}{(a_i(k) + b_i(k) e^{p_j(k)})(a_i(k) + b_i(k) e^{q_j(k)})} \\
& \times (p_i(k) - q_i(k))(e^{p_j(k)} - e^{q_j(k)})
\end{aligned}$$

$$
\begin{aligned}
&-2d_i(k)(p_i(k)-q_i(k))(u_i(k)-\omega_i(k))\\
&+\Big[\frac{2r_i(k)d_i(k)}{a_i(k)+b_i(k)e^{p_j(k)}}+2c_i(k)r_i(k)d_i(k)\\
&+2(1-f_i(k))\Big](u_i(k)-\omega_i(k))(e^{p_i(k)}-e^{q_i(k)})\\
&-\frac{2r_i(k)b_i(k)d_i(k)e^{p_i(k)}}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&\times(u_i(k)-\omega_i(k))(e^{p_j(k)}-e^{q_j(k)})+(d_i^2(k)\\
&-2f_i(k)+f_i^2(k))(u_i(k)-\omega_i(k))^2\Big\}. \qquad (18)
\end{aligned}
$$

Using the mean value theorem we get

$$e^{p_i(k)}-e^{q_i(k)}=\xi_i(k)(p_i(k)-q_i(k)),\ \ i=1,2, \qquad (19)$$

where $\xi_i(k)$ lies between $e^{p_i(k)}$ and $e^{q_i(k)}, i=1,2$. From (18) and (19), we have

$$
\begin{aligned}
&\Delta V_{(17)}(k)\\
&=\sum_{i=1}^{2}\Big\{\Big[\frac{r_i^2(k)}{(a_i(k)+b_i(k)e^{q_j(k)})^2}+r_i^2(k)c_i^2(k)\\
&+2r_i^2(k)c_i(k)+g_i^2(k)\Big]\xi_i^2(k)(p_i(k)-q_i(k))^2\\
&+\frac{b_i^2(k)(e^{p_i(k)})^2\xi_j^2(k)(p_j(k)-q_j(k))^2}{(a_i(k)+b_i(k)e^{p_j(k)})^2(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&-\Big[\frac{2r_i^2(k)b_i(k)e^{p_i(k)}}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&+\frac{2r_i^2(k)b_i(k)c_i(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\Big]\\
&\times\xi_i(k)(p_i(k)-q_i(k))\xi_j(k)(p_j(k)-q_j(k)))\\
&-\Big[\frac{2r_i(k)}{a_i(k)+b_i(k)e^{q_j(k)}}+2c_i(k)r_i(k)\Big]\xi_i(k)(p_i(k)\\
&-q_i(k))^2+\frac{2r_i(k)b_i(k)e^{p_i(k)}}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&\times\xi_j(k)(p_j(k)-q_j(k))(p_i(k)-q_i(k))-2d_i(k)(p_i(k)\\
&-q_i(k))(u_i(k)-\omega_i(k))+\Big[\frac{2r_i(k)d_i(k)}{a_i(k)+b_i(k)e^{q_j(k)}}\\
&+2c_i(k)r_i(k)d_i(k)+2(1-f_i(k))\Big](u_i(k)\\
&-\omega_i(k))\xi_i(k)(p_i(k)-q_i(k))\\
&-\frac{2r_i(k)b_i(k)d_i(k)e^{p_i(k)}}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&\times(u_i(k)-\omega_i(k))\xi_j(k)(p_j(k)-q_j(k))\\
&+(d_i^2(k)-2f_i(k)+f_i^2(k))(u_i(k)-\omega_i(k))^2\Big\}\\
&\le\sum_{i=1}^{2}\Big\{\Big[\frac{r_i^2(k)\xi_i^2(k)}{(a_i(k)+b_i(k)e^{q_j(k)})^2}+r_i^2(k)c_i^2(k)\xi_i^2(k)\\
&+2r_i^2(k)c_i(k)\xi_i^2(k)+g_i^2(k)\xi_i^2(k)\\
&-\frac{2r_i(k)\xi_i(k)}{a_i(k)+b_i(k)e^{q_j(k)}}\\
&-2r_i(k)c_i(k)\xi_i(k)\Big](p_i(k)-q_i(k))^2\\
&+\frac{b_i^2(k)(e^{p_i(k)})^2\xi_j^2(k)}{(a_i(k)+b_i(k)e^{p_j(k)})^2(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&\times(p_j(k)-q_j(k))^2\\
&+2\Big|\Big(-\frac{r_i^2(k)b_i(k)e^{p_i(k)}\xi_i(k)\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&-\frac{r_i^2(k)b_i(k)c_i(k)\xi_i(k)\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&+\frac{r_i(k)b_i(k)e^{p_i(k)}\xi_i(k)\xi_j^2(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\Big)\\
&\times(p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big|\\
&+2\Big|\Big[\frac{r_i(k)d_i(k)\xi_i(k)}{a_i(k)+b_i(k)e^{p_j(k)}}+c_i(k)r_i(k)d_i(k)\xi_i(k)\\
&+(1-f_i(k))\xi_i(k)-d_i(k)\Big](p_i(k)-q_i(k))\\
&\times(u_i(k)-\omega_i(k))\Big|\\
&+\Big|\frac{2r_i(k)b_i(k)d_i(k)e^{p_i(k)}\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&\times(p_j(k)-q_j(k))(u_i(k)-\omega_i(k))\Big|\\
&+[d_i^2(k)-2f_i(k)+f_i^2(k)](u_i(k)-\omega_i(k))^2\Big\}.
\end{aligned}
$$

We get for $j=1,2$,

$$\Delta V_{(17)}=\sum_{i=1}^{2}\{V_{1ij}+V_{2ij}+V_{3ij}+V_{4ij}+V_{5ij}+V_{6ij}\},$$

where

$$
\begin{aligned}
V_{1ij}\ =\ &\Big[\frac{r_i^2(k)\xi_i^2(k)}{(a_i(k)+b_i(k)e^{q_j(k)})^2}+r_i^2(k)c_i^2(k)\xi_i^2(k)\\
&+2r_i^2(k)c_i(k)\xi_i^2(k)+g_i^2(k)\xi_i^2(k)\\
&-\frac{2r_i(k)\xi_i(k)}{a_i(k)+b_i(k)e^{q_j(k)}}-2r_i(k)c_i(k)\xi_i(k)\Big]\\
&\times(p_i(k)-q_i(k))^2\\
\le\ &\Big[\frac{r_i^{u^2}\xi_i^{u^2}}{(a_i^l+b_i^lx_{j*})^2}+r_i^{u^2}c_i^{u^2}x_i^{*^2}+2r_i^{u^2}c_i^ux_i^{*^2}\\
&+g_i^{u^2}x_i^{*^2}-\frac{2r_i^lx_{i*}}{a_i^u+b_i^ux_j^*}-2r_i^lc_i^lx_{i*}\Big]\\
&\times(p_i(k)-q_i(k))^2,\\
V_{2ij}\ =\ &\frac{b_i^2(k)(e^{p_i(k)})^2\xi_j^2(k)}{(a_i(k)+b_i(k)e^{p_j(k)})^2(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&\times(p_j(k)-q_j(k))^2\\
\le\ &\frac{b_i^{u^2}x_i^{*^2}x_j^{*^2}}{(a_i^l+b_i^lx_{j*})^4}(p_j(k)-q_j(k))^2,\\
V_{3ij}\ =\ &2\Big|\Big(-\frac{r_i^2(k)b_i(k)e^{p_i(k)}\xi_i(k)\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})^2}\\
&-\frac{r_i^2(k)b_i(k)c_i(k)\xi_i(k)\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\\
&+\frac{r_i(k)b_i(k)e^{p_i(k)}\xi_i(k)\xi_j^2(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times (p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big| \\
&\le \left[\frac{r_i^{u^2} b_i^u x_i^{*^2} x_j^*}{(a_i^l+b_i^l x_{j*})^3}+\frac{r_i^{u^2} b_i^u c_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}+\frac{r_i^u b_i^u x_i^{*^2} x_j^{*^2}}{(a_i^l+b_i^l x_{j*})^2}\right] \\
&\quad \times ((p_i(k)-q_i(k))^2+(p_j(k)-q_j(k))^2), \\
V_{4ij} &= 2\Bigg|\left[\frac{r_i(k)d_i(k)\xi_i(k)}{a_i(k)+b_i(k)e^{p_j(k)}}+c_i(k)r_i(k)d_i(k)\xi_i(k)\right. \\
&\quad \left. +(1-f_i(k))\xi_i(k)-d_i(k)\right] \\
&\quad (p_i(k)-q_i(k))(u_i(k)-\omega_i(k))\Bigg| \\
&\le \left[\frac{r_i^u d_i^u x_i^*}{a_i^l+b_i^l x_{j*}}+r_i^u c_i^u d_i^u x_i^*+(1-f_i^l)x_i^*-d_i^l\right] \\
&\quad \times ((p_i(k)-q_i(k))^2+(u_i(k)-\omega_i(k))^2), \\
V_{5ij} &= \Bigg|\frac{2r_i(k)b_i(k)d_i(k)e^{p_i(k)}\xi_j(k)}{(a_i(k)+b_i(k)e^{p_j(k)})(a_i(k)+b_i(k)e^{q_j(k)})} \\
&\quad \times (p_j(k)-q_j(k))(u_i(k)-\omega_i(k))\Bigg| \\
&\le \frac{r_i^u b_i^u d_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}((p_j(k)-q_j(k))^2+(u_i(k)-\omega_i(k))^2), \\
V_{6ij} &= [d_i^2(k)-2f_i(k)+f_i^2(k)](u_i(k)-\omega_i(k))^2 \\
&\le (d_i^{u^2}-2f_i^l+f_i^{u^2})(u_i(k)-\omega_i(k))^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Delta V_{(17)} &\le \sum_{i=1}^{2}\left[\frac{r_i^{u^2}\xi_i^{u^2}}{(a_i^l+b_i^l x_{j*})^2}+r_i^{u^2} c_i^{u^2} x_i^{*^2}\right. \\
&\quad +2r_i^{u^2} c_i^u x_i^{*^2}+g_i^{u^2} x_i^{*^2}-\frac{2r_i^l x_{i*}}{a_i^u+b_i^u x_j^*}-2r_i^l c_i^l x_{i*} \\
&\quad +\frac{r_i^{u^2} b_i^u x_i^{*^2} x_j^*}{(a_i^l+b_i^l x_{j*})^3}+\frac{r_i^{u^2} b_i^u c_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}+\frac{r_i^u b_i^u x_i^{*^2} x_j^{*^2}}{(a_i^l+b_i^l x_{j*})^2} \\
&\quad \left.+\frac{r_i^u d_i^u x_i^*}{a_i^l+b_i^l x_{j*}}+r_i^u c_i^u d_i^u x_i^*+(1-f_i^l)x_i^*-d_i^l\right] \\
&\quad \times (p_i(k)-q_i(k))^2+\left[\frac{b_i^{u^2} x_i^{*^2} x_j^{*^2}}{(a_i^l+b_i^l x_{j*})^4}\right. \\
&\quad +\frac{r_i^{u^2} b_i^u x_i^{*^2} x_j^*}{(a_i^l+b_i^l x_{j*})^3}+\frac{r_i^{u^2} b_i^u c_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2} \\
&\quad \left.+\frac{r_i^u b_i^u x_i^{*^2} x_j^{*^2}}{(a_i^l+b_i^l x_{j*})^2}+\frac{r_i^u b_i^u d_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}\right] \\
&\quad \times (p_j(k)-q_j(k))^2+\left[\frac{r_i^u d_i^u x_i^*}{a_i^l+b_i^l x_{j*}}\right. \\
&\quad +r_i^u c_i^u d_i^u x_i^*+(1-f_i^l)x_i^*-d_i^l \\
&\quad \left.+\frac{r_i^u b_i^u d_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}+d_i^{u^2}-2f_i^l+f_i^{u^2}\right] \\
&\quad \times (u_i(k)-\omega_i(k))^2 \\
&= -\sum_{i=1}^{2}\left[\frac{2r_i^l x_{i*}}{a_i^u+b_i^u x_j^*}+2r_i^l c_i^l x_{i*}-\frac{r_i^{u^2}\xi_i^{u^2}}{(a_i^l+b_i^l x_{j*})^2}\right. \\
&\quad -r_i^{u^2} c_i^{u^2} x_i^{*^2}-2r_i^{u^2} c_i^u x_i^{*^2}-g_i^{u^2} x_i^{*^2} \\
&\quad -\frac{r_i^{u^2} b_i^u x_i^{*^2} x_j^*}{(a_i^l+b_i^l x_{j*})^3}-\frac{r_i^{u^2} b_i^u c_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2}-\frac{r_i^u b_i^u x_i^{*^2} x_j^{*^2}}{(a_i^l+b_i^l x_{j*})^2} \\
&\quad -\frac{r_i^u d_i^u x_i^*}{a_i^l+b_i^l x_{j*}}-r_i^u c_i^u d_i^u x_i^*-(1-f_i^l)x_i^* \\
&\quad +d_i^l-\frac{b_j^{u^2} x_j^{*^2} x_i^{*^2}}{(a_j^l+b_j^l x_{i*})^4}-\frac{r_j^{u^2} b_j^u x_j^{*^2} x_i^*}{(a_j^l+b_j^l x_{i*})^3} \\
&\quad \left.-\frac{r_j^{u^2} b_j^u c_j^u x_j^* x_i^*}{(a_j^l+b_j^l x_{i*})^2}-\frac{r_j^u b_j^u x_j^{*^2} x_i^{*^2}}{(a_j^l+b_j^l x_{i*})^2}-\frac{r_j^u b_j^u d_j^u x_j^* x_i^*}{(a_j^l+b_j^l x_{i*})^2}\right] \\
&\quad \times (p_i(k)-q_i(k))^2+\left[2f_i^l-\frac{r_i^u d_i^u x_i^*}{a_i^l+b_i^l x_{j*}}\right. \\
&\quad -r_i^u c_i^u d_i^u x_i^*-(1-f_i^l)x_i^*+d_i^l-\frac{r_i^u b_i^u d_i^u x_i^* x_j^*}{(a_i^l+b_i^l x_{j*})^2} \\
&\quad \left.-d_i^{u^2}-f_i^{u^2}\right](u_i(k)-\omega_i(k))^2 \\
&\le -\sum_{i=1}^{2}\left\{r_{ij}(p_i(k)-q_i(k))^2+r_{ij}^*(u_i(k)-\omega_i(k))^2\right\} \\
&\le -\beta\sum_{i=1}^{2}\left\{(p_i(k)-q_i(k))^2+(u_i(k)-\omega_i(k))^2\right\} \\
&\le -\beta V(k),
\end{aligned}
$$

where $\beta=\min\{r_{ij},r_{ij}^*\}, i,j=1,2, i\neq j$. That is, there exists a positive constant $0<\beta<1$ such that

$$\Delta V_{(17)}(k)\le -\beta V(k).$$

From $0<\beta<1$, the condition (iii) of Lemma 6 is satisfied. So, from Lemma 6, there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(k)=(p_1(k),p_2(k),u_1(k),u_2(k))$ of (16) which is bounded by $s*$ for all $k\in Z^+$, which means that there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(k)=(x_1(k),x_2(k),u_1(k),u_2(k))$ of (2) which is bounded by Ω for all $k\in Z^+$. This completes the proof. ■

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