

# A Utilitarian Approach to Modeling Information Flows in Social Networks

Usha Sridhar and Sridhar Mandyam

**Abstract**—We propose a multi-agent based utilitarian approach to model and understand information flows in social networks that lead to Pareto optimal informational exchanges. We model the individual expected utility function of the agents to reflect the net value of information received. We show how this model, adapted from a theorem by Karl Borch dealing with an actuarial Risk Exchange concept in the Insurance industry, can be used for social network analysis. We develop a utilitarian framework that allows us to interpret Pareto optimal exchanges of value as potential information flows, while achieving a maximization of a sum of expected utilities of information of the group of agents. We examine some interesting conditions on the utility function under which the flows are optimal. We illustrate the promise of this new approach to attach economic value to information in networks with a synthetic example.

**Keywords**—Borch's Theorem, Economic value of information, Information Exchange, Pareto Optimal Solution, Social Networks, Utility Functions

## I. INTRODUCTION

THE study of social and economic behavior of groups of individuals interconnected through social networks has emerged as an important area of interdisciplinary research in the recent years.

This has often been attributed to the dramatic growth of social networking platforms, which have brought millions on to the internet, sharing their experiences, beliefs, social and personal preferences, choices, and economic decisions; and perhaps in turn shaping the beliefs and choices of millions of others through social influences.

Central to many of the diverse research approaches is the issue of how individuals perceive, evaluate, and learn from the information that reaches them from the different sources that they communicate with, particularly neighbors who are close to them on the network, such as family, friends and co-workers. This class of social learning is thought to occur through the aggregation of information dispersed in the network.

In much of the recent research on learning in social networks, information that is received, processed and shared by connected individuals through network communication has been associated with measures of beliefs and opinions, or the probability of occurrences of events which are observed by agents. Such information measures play a central role in building models to understand the dynamics and evolution of opinions and decisions of agents in a social network.

A utilitarian framework that endows the agents with the ability to consider decision making options on the basis of how much utility they can extract by valuing the information that reaches them in some economic sense has been extensively researched. It has been shown, for instance, that the microeconomic decision theoretic framework accommodates incorporation of information structures and the association of values to them under the rubric of individual expected utility maximization criteria. It is also established that incorporation of information and its values cannot leave a utility maximizer worse off. [Lawrence, D.L. (1999), Chade and Schlee(2002)]

Even though there is no fundamental hurdle in conceptualizing utility functions built on the idea of valuation of information, there are difficulties in extending the notion to a *simultaneous maximization* of the individual *expected* utilities of the interconnected group of agents – an action that would be naturally expected of them as economic agents. The difficulty is largely due to three issues: i) Solutions to the simultaneous *expected* utility maximization problem in a traditional optimization framework may require explicit forms of the underlying probability distributions of information flows, and this assumption of knowledge of such distributions may be unrealistic; ii) Detailed modeling of network effects within such optimization frameworks can add very significantly to overall complexity; and iii) Implementations may fail to handle the large scale of practical networks.

In this paper, we take a fresh approach to modeling information driven utilitarian agents which value incident (i.e. received) information using private valuation measures in an exponential utility function. We show that the group expected utility maximization problem can be elegantly and Pareto optimally solved, even allowing for balancing information exchanges among agents using Borch's theorem on Risk Exchanges, a result adopted from the actuarial world of Reinsurance. We describe a new kind of framework that accommodates utilitarian agents in a network, where they can simultaneously maximize their expected utilities, with the theory not only yielding an idealized 'distribution' of information among the agents to achieve that maximum, but also the means to understand alternative flows that mimic other information exchange scenarios, for instance, for the achievement of a consensus.

The scope of this paper covers the essential theoretical structure for obtaining Pareto optimal information flows under the assumptions of full connectivity, even though the framework can accommodate other kinds of connectivity conditions.

The paper is organized as follows: in the next section we discuss how individual's perceive information, and a mechanism to arrive at a Pareto exchange of information that

Usha Sridhar is with Ecometrix Research.(e-mail: ushasridhar1158@gmail.com).

Sridhar Mandyam is with Ecometrix Research. (e-mail: sridhar.mandyam@ecometrix.in).

individually and simultaneously maximizes agents' utility functions using Borch's theorem. We then go on to validate the theory with an empirical example on a small social network. We conclude in the last section.

## II. UTILITARIAN FRAMEWORK AND MODEL

### A. Information and its Utility

We consider a somewhat stylized setting in which a group of  $N$  individuals are inter-connected as a social network. By this, we mean that each agent has the means to communicate with all the other agents. The central theme of activity we consider is essentially the same as that in nearly all analytical approaches to studying social networks – that agents observe some aspect of the state of the world, form an opinion about it, and communicate information about it to all other agents on the network – setting in motion a dynamical process of assimilating and updating a notional 'value' associated with the received information through more flows of information.

Generally the information that is assimilated and shared by agents is associated with some metric of belief and opinion, or a measure of probability of occurrence of one or more events. For instance Bayesian social learning models [1],[2],[11] consider 'signals' received by agents conveying a probability that the state of the world is  $\theta \in \Theta$ , a set of possible states. This class of models would then view aggregation of information by an agent within the network as a process of updating prior probabilities on agent's belief measures through exchange of new information among agents – essentially a process of recalculating likelihoods on the beliefs given new signals – as agents interact among themselves. In non-Bayesian settings where researchers have studied information diffusion and aggregation [7], [8], [12], beliefs and opinions are often thought of in terms of a measure of conviction or agreement of agents on some issue, and represented by a real number in some suitable range. It is thus possible for an agent to aggregate information on beliefs communicated by different agents by adding or averaging numbers in the positive real space.

In empirical social network analysis [3], the dynamics of beliefs and opinions in networks are often associated with ordinal or interval-scaled measures of approval, or agreement. Ratings on products or services (such movies on Netflix), and brand positioning statements derived as 'sentiments' from text strings (such as "tweets" on Twitter) have been considered the classical carriers of opinions.

In this paper, we approach the issue of characterizing information as measure of content of belief and opinions, which can be communicated without noise over the network, received without loss by agents, and most importantly, *valued* by agents as an economic entity, and hence associate with it a utility. In order to achieve this Utilitarian view, we shall assume that there exists a universal mapping - from beliefs and opinion measures in any suitable space, or from some event probability space that captures their occurrence probabilities as information shared among agents – to the real space. In other words information, to our agent, is a number that tells her the quantity of a transferable economic good, represented by a variable  $v \in \mathcal{R}_+$ .

The assignment of economic value to information has been an important area of research attention in economics, and has been long debated in the literature [14]. Many authors have established that under conditions where the marginal utility of a small amount of information is small, and the continuity of the function with the value of information is guaranteed, the utility can be considered concave [9],[13],[15]. Equivalently, the decision maker is assumed to have a normal prior and that she observes a signal that is normally distributed with mean  $s$  and variance  $1/\theta$ , i.e.  $N(s, 1/\theta)$ , where  $\theta > 0$  [6]. Varian [16] shows how the utility of information value approximates a concave function in the context of document search application.

At an axiomatic level, we shall consider that i) a value of  $v = 0$  implies no information; ii) a higher of value of  $v$  implies more information; and iii) two amounts of information may be added to yield the equivalent numerical sum of information. We see that these assumptions are, in principle, not at odds with any of the theoretical or empirical social learning approaches.

Let  $v_j; j = 1, \dots, N$  represent the sum of all the information sent by others in the network and is thus *incident* to agent  $j$ . We use the phrase 'incident on agent' in the sense 'reaching the agent'. We shall let  $v_j \in \mathcal{R}$  in the range  $[0, V]$ , where  $V$  is some notional upper bound. As mentioned above, higher values of  $v_j$ , indicate higher information content. They are considered as random variables as they are the result of subjective observations of the state of the world by the agents. Though their distributional properties are not required in our calculations we shall assume that we have an estimate of their sum:

$$\sum_{j=1}^N v_j = W \quad (1)$$

In concept,  $W$  is the total amount of information over all agents. We shall later see that  $W$  turns out to be a simple scale factor that may be set aside. Let  $\rho_j; j = 1, \dots, N$  denote the value that agent  $j$  associates with the incident information. We shall let  $\rho_j \in \mathcal{R}$  and place the bound  $[0, 1]$  on its value. Higher values of  $\rho_j$ , indicates that an agent gives higher value to that information.

We now model the utility of information to agent  $j$  as :

$$u_j(v_j) = \frac{1}{\rho_j} (1 - e^{-\rho_j v_j}) \quad (2)$$

which represents a standard, robust, utility function that captures utility as a concave, non-decreasing function of information<sup>1</sup>. The utilities are normalized such that zero information will offer zero utility, while the maximum utility for any large information is unity.

The valuation of information by agents via  $\rho_j$  characterizes the economic behaviour of agents. Clearly, the higher the valuation measure, the sharper the utility, implying that the same level of incident information can lead to different utility achievements. Figure 1 illustrates this feature, where agent 2

<sup>1</sup> When an agent in a social network receives the same information from multiple sources, it need not be 'worthless'. In fact, such multiple affirmations have value (albeit diminishing, perhaps) in enhancing the agent's convictions in the beliefs she might hold.

with a higher  $\rho_j = 0.7$  has utility (A) than agent 1 with a  $\rho_j = 0.2$ . Equally, more information is required by agent 1 ( $v_1 = 4$ ) than agent 2 ( $v_2 \cong 1.2$ ) to achieve the same level of utility ( $D=C=B$ ).

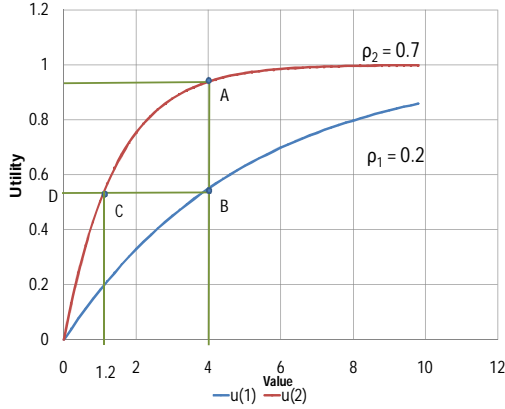


Fig. 1 Utility of Information

#### B. Maximization of Group Expected Utilities

We now posit that the agents, assumed rational decision makers, attempt to maximize their expected utilities – and in particular, a linear sum of their expected utilities:

$$S = \max \sum_{j=1}^N k_j E[u_j(v_j)] \quad (3)$$

subject to (1).

We therefore seek a Pareto optimal solution  $v_j^*; j = 1, \dots, N$  to (1) – (3) such that  $S$  maximized in such a manner that

$$k_j E[u_j(v_j)] \leq k_j E[u_j(v_j^*)] \quad \forall j = 1, \dots, N \quad (4)$$

implying that it would not be possible to improve the utility of any one agent without lowering the utility of some other agent. The solution to (3)-(4) in a general manner via optimization may require explicit definitions of underlying probability distribution of the information values.

However, a Theorem by Borch (Borch [4], Buhlmann [5], Gerber [10]), often used in the context of evaluating Risk Exchanges in the actuarial Re-insurance calculations offers an elegant method to obtain a solution utilizing the concavity properties of the utility functions and the notional sum  $W$  in the constraint

The theorem established that, for  $k_j > 0; j = 1, \dots, N$ , there exists a Pareto optimal solution  $v_j^*; j = 1, \dots, N$  that maximizes the weighted sum of  $N$  expected utilities in (3) under the constraint  $\sum_{j=1}^N v_j = W$ , which yields (4), exactly as required.

The classical Risk Exchange problem was posed as a method to exchange risk among a group of  $N$  entities at the end of a time interval. The  $N$  entities are thought to be risk-averse entities having concave utilities on ‘wealth’. Given that

$N$  random variables represented the ‘wealth’ of the entities at the start of the period and their sum,  $W$ , the problem was to determine another allocation of  $W$  at the end of the interval such that the allocation Pareto optimally maximized a sum of the form (3).

**Borch’s Theorem:** Borch’s Theorem established that the linear sum of expected values in (3) is maximized at values  $v_j^*; j = 1, \dots, N$ , such that

$$k_j u_j'(v_j^*) = \Lambda \quad \forall j = 1, \dots, N \quad (5)$$

The theorem implies that the slopes of the utility function evaluated at the optimal solution, taken in product with the positive multipliers,  $k_j; j = 1, \dots, N$ , are the same,  $\Lambda$ , for all the agents. The standard proof is sketched in the Appendix for the sake of completeness.

Following up on the logic of risk exchanges, where we seek a new optimal combination of  $N$  random variables whose sum remains the same, yet maximizes (3), we find that we can draw an analog in the world of information utilities. Equation (5), for the specific case of the exponential utility function of (2) yields the basic framework the mechanism we seek:

$$v_j^* = \frac{\rho}{\rho_j} W + \frac{\ln k_j}{\rho_j} - \frac{\rho}{\rho_j} \sum_{i=1}^N \frac{\ln k_i}{\rho_i} \quad \forall j = 1, \dots, N \quad (6)$$

where  $\frac{1}{\rho} = \sum_{j=1}^N \frac{1}{\rho_j}$ . Equation (6) says that the Pareto optimal solution we sought is a sum of three items: the first is an ‘distribution’ of total information  $W$  in (inverse) proportion to the value perceived by the agents, the second and third terms express what are termed as side-payments in the Risk Exchange world. We shall have vital use for these last two terms, which exhibit an information exchange property. If we denote:

$$d_j = \frac{\ln k_j}{\rho_j} - \frac{\rho}{\rho_j} \sum_{i=1}^N \frac{\ln k_i}{\rho_i} \quad \forall j = 1, \dots, N \quad (7)$$

we find that  $\sum_{j=1}^N d_j = 0$ . The first term on the rhs of (6) represents a Pareto optimal ‘distribution’ of total information,  $W$ , which would maximize the weighted sum of expected utilities of individual agents, implying that if each agent received information equal to  $v_j^*; j = 1, \dots, N$ , then any other level of available information would be sub-optimal in the sense that one would have to trade-off the achievement of a lower expected utility for some agent, in order to achieve a higher utility for another agent.

The potential power that this formulation offers to understand information exchange in a network of agents comes from (6), which says that for **any** choice of  $k_j > 0 \quad \forall j = 1, \dots, N$ , the ‘distribution’ of  $W$  *continues* to be Pareto optimal. We actually have a  $(n-1)$  family of Pareto optimal solutions, all of which share the same optimal distribution ratio  $\frac{\rho}{\rho_j} W$ . Note that in order to arrive at the Pareto optimal solution (6) of the maximization problem (4), it wasn’t necessary to make any assumptions on the information probability distributions at all. We only made the assumption

of exponential utility functions in order to reuse a convenient form of the result of Borch's Theorem.

### III. PARETO OPTIMAL INFORMATION EXCHANGES

What the application of Borch's Theorem delivered to our basic utility maximization problem with incident information values is precisely the following: *In order for all agents to extract maximum utility from available information  $v_j$ 's under given private valuations  $\rho_j$ 's, a total available information value of  $W$  would best (in a Pareto optimal sense) be 'distributed' among the agents in the ratio  $\frac{\rho}{\rho_j} W, \forall j = 1, \dots, N$ . Should the 'distribution' deviate from this optimal ratio, then there needs to be information exchanges between agents in such a manner that  $d_j$ 's in (7) define the exchanges on a per agent basis, but ensure that the sum total of all exchanges in the network is zero.*

If we multiply out  $\rho_j$  on both sides of (6), and taking the optimal ratio  $\frac{\rho}{\rho_j} W$  along with  $v_j^*$  to the left yields the vector

$$\mathbf{b}^* = [\rho_1(v_1^* - \frac{\rho}{\rho_1} W) \quad \rho_2(v_2^* - \frac{\rho}{\rho_2} W) \dots \rho_N(v_N^* - \frac{\rho}{\rho_N} W)]^T \quad (8)$$

Equation (8) in matrix form is thus simply

$$\mathbf{b}^* = \mathbf{R} \cdot \mathbf{x} \quad (9)$$

where  $\mathbf{x} = [\ln k_1 \quad \ln k_2 \quad \dots \quad \ln k_N]^T$  and  $\mathbf{R}$  is the matrix of coefficients, which represent a set valuation multipliers that describe how much information exchange between agents (Pareto) optimally will render the utilities maximized:

$$\mathbf{R} = \begin{bmatrix} 1 - \frac{\rho}{\rho_1} & -\frac{\rho}{\rho_2} & \dots & -\frac{\rho}{\rho_N} \\ -\frac{\rho}{\rho_1} & 1 - \frac{\rho}{\rho_2} & \dots & -\frac{\rho}{\rho_N} \\ \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\rho}{\rho_1} & -\frac{\rho}{\rho_2} & \dots & 1 - \frac{\rho}{\rho_N} \end{bmatrix} \quad (10)$$

For example, the element  $R(1,2)$  says that  $-\frac{\rho}{\rho_2} \ln k_2$  is the amount of information that needs to flow from agent 1 to 2. By its very construction,  $\mathbf{R}$  is a rank deficient matrix ( $\text{rank}(\mathbf{R}) = n-1$ ) and its rows sum to 0. However, we can calculate its pseudo-inverse  $\mathbf{R}^+$ , and obtain Pareto optimal exchanges. With this framework we therefore can study the following interesting propositions:

**Proposition 1:** *Under the assumption of the framework, the Pareto optimal incident information values which maximize the linear sum of expected utilities of all the  $N$  connected agents is  $v_j^* = \frac{\rho}{\rho_j} W; j = 1, \dots, N$*

We see that since the exchange matrix  $\mathbf{R}$  has a non-zero vector in its null space, the corresponding eigenvector, which has all its elements identical will drive  $\mathbf{R}\mathbf{x}$  to zero, implying

that  $v_j^* - \frac{\rho}{\rho_j} W = 0 \forall j = 1, \dots, N$ . Hence the elements of matrix  $\mathbf{R}$  themselves represent a Pareto optimal information exchange to achieve the maximization of the sum of expected utilities.

**Proposition 2:** *Under the assumptions of the framework, it is possible to obtain a Pareto optimal information exchange solution that requires the optimal information value incident on each agent to be identical, i.e.,  $\forall j : v_j^* = z$ .*

Since we have  $W = \sum_{j=1}^N v_j^* = Nz$ , the elements of the vector  $\mathbf{b}$  are  $b_j = (\frac{\rho_j}{N} - \rho) W$ . We can then find  $\mathbf{x} = \mathbf{R}^+ \mathbf{b}$ , where  $\mathbf{R}^+$  is a pseudo-inverse of  $\mathbf{R}$ . The optimal exchange levels are then obtained as the matrix  $\mathbf{E}$ :

$$\mathbf{E} = \begin{bmatrix} (1 - \frac{\rho}{\rho_1}) \ln k_1 & -\frac{\rho}{\rho_2} \ln k_2 & \dots & -\frac{\rho}{\rho_N} \ln k_N \\ -\frac{\rho}{\rho_1} \ln k_1 & 1 - \frac{\rho}{\rho_2} \ln k_2 & \dots & -\frac{\rho}{\rho_N} \ln k_N \\ \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\rho}{\rho_1} \ln k_1 & -\frac{\rho}{\rho_2} \ln k_2 & \dots & 1 - \frac{\rho}{\rho_N} \ln k_N \end{bmatrix}$$

### IV. ILLUSTRATIVE EXAMPLE

Consider a group of  $N=5$  utilitarian agents with their exponential utility functions as defined in (2), and private valuation measures  $\rho_j; j = 1, 2, \dots, N$  as in the vector  $[0.2 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7]^T$ . We obtain from  $\frac{1}{\rho} = \sum_{j=1}^N \frac{1}{\rho_j}$ , the value of  $\rho = 0.0794$ . Forming the matrix  $\mathbf{R}$  as in (10) we obtain

$$\mathbf{R} = \begin{bmatrix} 0.6030 & -0.1985 & -0.1588 & -0.1323 & -0.1134 \\ -0.3970 & 0.8015 & -0.1588 & -0.1323 & -0.1134 \\ -0.3970 & -0.1985 & 0.8412 & -0.1323 & -0.1134 \\ -0.3970 & -0.1985 & -0.1588 & 0.8677 & -0.1134 \\ -0.3970 & -0.1985 & -0.1588 & -0.1323 & 0.8866 \end{bmatrix}$$

and the Pareto optimal 'distribution' solution vector of elements  $\frac{\rho}{\rho_j} W, j = 1, \dots, N$  is as in the vector  $\mathbf{pos} = W[0.3970 \quad 0.1985 \quad 0.1588 \quad 0.1323 \quad 0.1134]^T$ . Note that since we have set the valuation measure in an increasing order, the optimal 'distributions' reduce in that order, agreeing with the notion that the higher the  $\rho_j$  value for an agent, the sharper the utility – implying also that a higher utility may then be achievable for a lower level for incident information.  $\mathbf{R}$  has a rank of 4, and its rows sum to zero. In order to maximize the linear sum of expected utilities as in (3)-(4), and the Pareto optimal vector  $\mathbf{pos}$  above, the value of  $\mathbf{x} = [1 \ 1 \ 1 \ 1 \ 1]^T$  will satisfy  $\mathbf{R}\mathbf{x} = \mathbf{b}$  as in (9). Hence  $\mathbf{R}$  directly represents the scaled 'information exchanges' that are Pareto optimal.

The numbers in the first row of the  $R$  matrix in this case tell us that if agents 2,3,4 and 5, transfer information amounts - 0.1985, -0.1588, -0.1323, -0.1134, respectively, (with negative sign showing direction of transfer), and agent 1 itself assimilates a value of 0.603, then these transfers will optimally balance out the information flows to deliver the an optimal amount of information as in the vector, pos. This in turn guarantees the best balance between availability of information and the value the agents associate with the information, as expressed by their respective utility functions. Let us now examine the scenario where we seek an optimal information transfer among the agents in such a manner that the amount of information reaching each agent is identical. We set  $b_j = \left(\frac{\rho_j}{N} - \rho\right) W$  for  $W=1$  ( $W$  is merely a scale factor here since all elements of the exchange matrix are scaled by  $W$ ). We solve  $Rx = b$  by computing the pseudo-inverse  $R^+$  using the SVD of  $R$ , and obtain the solution  $x = R^+b$  to obtain  $x = [-0.0560 \ -0.0160 \ 0.0040 \ 0.0240 \ 0.0440]^T$ . Using these values we obtain the information exchange matrix  $E$  as:

$$E = \begin{pmatrix} -0.0338 & 0.0032 & -0.0006 & -0.0032 & -0.0050 \\ 0.0222 & -0.0128 & -0.0006 & -0.0032 & -0.0050 \\ 0.0222 & 0.0032 & 0.0034 & -0.0032 & -0.0050 \\ 0.0222 & 0.0032 & -0.0006 & 0.0208 & -0.0050 \\ 0.0222 & 0.0032 & -0.0006 & -0.0032 & 0.0390 \end{pmatrix}$$

The rows of  $E$  naturally sum to the values of  $b$ . We interpret the required information exchanges in the following manner: should the net information incident on each agent be the same, then the Pareto optimal amounts of information that have to be exchanged among the agents are captured in  $E$ , so as ensure that the agents maximize the linear sum of their utilities. Of course, these amounts are all for  $W=1$ , and all amounts will scale linearly with  $W$ .

We note at this stage that our model assumed a fully connected network interconnecting the agents, as result of which we broke down information exchanges between all agents. We observe that the assumption of full connectivity can be clearly relaxed so long as those agents who are connected are able to exchange the sum required in any other optimal combination to balance out the total. For instance, if agents 2 and 3 only were connected to 1, they could compensate for the lack of connectivity between 1 and 4 and 5, so long as the total incidence requirement for 1 is met.

## V. CONCLUDING REMARKS

The exchanges of information among agents in social networks are thought to have a critical bearing on the way agents shape their beliefs and make choices. In this paper we have proposed a Utilitarian approach to model the economic value of these information exchanges. We posed the overall information use problem as a group expected utility maximization problem and proposed the use of Borch's result on Risk Exchange as a means to obtain Pareto optimal distributions of information that achieves simultaneous maximization of a linear sum of their expected utilities. Further work is underway in several directions. Key among them is the exploration of means to fully incorporate an

information theoretic interpretation to exchanges, and placement of these ideas on a firm decision theoretic foundation that also accommodates notions of social influence.

## APPENDIX

*Borch's Theorem and the Risk Exchange (Gerber and Pafumi(2007))*

Let us suppose that there are  $n$  agents, each with a wealth  $W_i$  at the end of a predefined time period.  $W_1, W_2, \dots, W_n$ , are considered as random variables with a some joint distribution only so that total wealth at the end of year is

$$W = \sum_{i=1}^n W_i.$$

A risk exchange facilitates a redistribution of wealth and each agent gets,  $X_1, X_2, \dots, X_n$ , which are also considered as random variables such that

$$W = X_1 + X_2 + \dots + X_n.$$

Assume the utility of the agents are non-decreasing and concave. Let utility functions be denoted as  $U_i(X_i)$  and the expected utility as  $E[U_i(X_i)]$ .

A risk exchange  $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n)$  is said to be Pareto optimal, if it is not possible to improve the situation of one agent without worsening the situation of at least one other agent. In other words, there is no other exchange  $(X_1, X_2, \dots, X_n)$  with

$$E[U_i(X_i)] \geq E[U_i(\tilde{X}_i)] \quad \forall i = 1, \dots, n.$$

To obtain a family of Pareto optimal risk exchanges with  $n-1$  parameters we obtain for  $k_i > 0, i = 1, \dots, n$  the maximum of the linear sum of the expected utilities

$$\text{Max } \sum_{i=1}^n k_i E[U_i(X_i)] \quad (1)$$

where the maximum is taken over all risk possible exchanges  $(X_1, X_2, \dots, X_n)$ . Borch's Theorem below offers an explicit solution to this maximization problem.

*Borch's Theorem:* A risk exchange  $(\tilde{X}_1, \dots, \tilde{X}_n)$  maximizes (1) if and if only the random variables  $k_i u'_i(\tilde{X}_i)$  are the same for  $i = 1, \dots, n$ .

*Proof*

a) Suppose that  $(\tilde{X}_1, \dots, \tilde{X}_n)$  maximizes (A1). Let  $j \neq h$  and let  $V$  be an arbitrary random variable.

We define

$$\begin{aligned} X_i &= \tilde{X}_i, \text{ for } i \neq j, h; \\ X_j &= \tilde{X}_j + tV \end{aligned}$$

and

$$X_h = \tilde{X}_h - tV,$$

where  $t$  is a parameter. Let

$$f(t) = \sum_{i=1}^n k_i E[u_i(X_i)].$$

According to initial assumption, the function  $f(t)$  has a maximum at  $t=0$ . Hence  $f'_t = 0$ , or

$$k_j E[Vu'_j(\tilde{X}_j)] - k_h E[Vu'_h(\tilde{X}_h)] = 0 \quad (2)$$

Utilizing the associativity of the expectation operator, (2) can be rewritten as

$$E[V(k_j u'_j(\tilde{X}_j) - k_h u'_h(\tilde{X}_h))] = 0.$$

Since this holds for an arbitrary  $V$ , we must have

$$k_j u'_j(\tilde{X}_j) - k_h u'_h(\tilde{X}_h) = 0$$

showing that  $k_i u'_i(\tilde{X}_i)$  is independent of  $i$ .

(b) Conversely, if we let  $(\tilde{X}_1, \dots, \tilde{X}_n)$  be a risk exchange such that

$$k_i u'_i(\tilde{X}_i) = \Lambda \quad (3)$$

where  $\Lambda$  is the same random variable for all  $i$ .

Let  $(X_1, \dots, X_n)$  be any other risk exchange. If we use the concavity condition for the utility function, we have

$$u_i(X_i) \leq u_i(\tilde{X}_i) + u'_i(\tilde{X}_i)(X_i - \tilde{X}_i).$$

Multiplying this inequality by  $k_i$ , summing over  $i$  and using (A3), we get

$$\begin{aligned} \sum_{i=1}^n k_i u_i(X_i) &\leq \sum_{i=1}^n k_i u_i(\tilde{X}_i) + \Lambda (X_i - \tilde{X}_i) \\ &= \sum_{i=1}^n k_i u_i(\tilde{X}_i) \end{aligned}$$

Hence

$$\sum_{i=1}^n k_i E[u_i(X_i)] \leq \sum_{i=1}^n k_i E[u_i(\tilde{X}_i)]$$

showing that expression (3) is indeed maximal for  $(\tilde{X}_1, \dots, \tilde{X}_n)$ .

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