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A Transfer Function Representation of Thermo-Acoustic Dynamics for Combustors

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***Abstract*—In this paper, we present a transfer function representation of a general one-dimensional combustor. The input of the transfer function is a heat rate perturbation of a burner and the output is a flow velocity perturbation at the burner. This paper considers a general combustor model composed of multiple cans with different cross sectional areas, along with a non-zero flow rate.**

***Keywords*—Thermoacoustics, dynamics, combustor, transfer function.**

I. Introduction

An appropriate modeling of thermo-acoustic behaviors of a combustor is a critical for a prediction and prevention of the combustion instability. The combustion instability is a self-excited thermal and/or mechanical oscillation of a combustor system, which is caused by a dynamic interplay between a heat rate perturbation and velocity perturbation ; (i) a heat rate perturbation of a burner can cause an acoustic velocity perturbation and (ii) conversely, a velocity perturbation in return makes a heat rate perturbation of a burner. A positive feedback among those two dynamics is a source of a combustion instability.

The second *velocity-to-heat* dynamics commonly described as a flame transfer function, is usually obtained from experiments. The first *heat-to-velocity* dynamics, we call it an *acoustic transfer function*, is very hard to be precisely found. This is because the combustor is a distributed parameter system and thus its thermal and acoustic behaviors are characterized by a set of coupled nonlinear partial differential equations (PDE's) in the fields of fluid dynamics and acoustics. In order to circumvent those complications, a one-dimensional acoustic model of a combustor is widely adopted in literature. A good reference for this topic can be found in [1].

This paper presents a step-by-step procedure for a derivation of an acoustic transfer function. In principle we adopt existing approaches in literature such as [2]-[5] but we also make some generalizations for combustors composed of multiple cans with different areas.

II. Acoustic Model

A. Wave Model

In this section, we consider a one-dimensional combustor model composed of two cans (different cross sectional areas) as shown in Fig. 1. The pressure and velocity perturbations in two sections have the following representations ($k = i, i+1$) ;

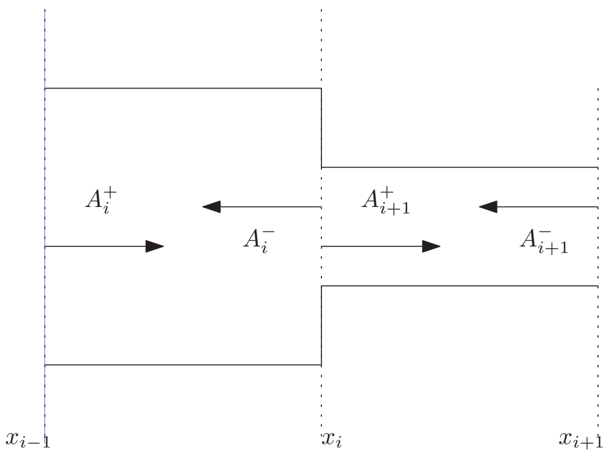

Fig. 1 Combustor with two cans

$$\begin{aligned}
&p_k(x,t) - \overline{p}_k := p'_k(x,t) \\
&\quad = A_k^+\left(t - \frac{x - x_{k-1}}{\overline{c}_k + \overline{u}_k}\right) + A_k^-\left(t - \frac{x_k - x}{\overline{c}_k - \overline{u}_k}\right) \\
&u_k(x,t) - \overline{u}_k := u'_k(x,t) \\
&\quad = \frac{1}{\overline{\rho}_k \overline{c}_k}\left[A_k^+\left(t - \frac{x - x_{k-1}}{\overline{c}_k + \overline{u}_k}\right) + A_k^-\left(t - \frac{x_k - x}{\overline{c}_k - \overline{u}_k}\right)\right] \\
&\rho_i(x,t) - \overline{\rho}_i := \rho'_k(x,t) \\
&\quad = \frac{1}{\overline{c}_i^2}\left[A_i^+\left(t - \frac{x - x_{i-1}}{\overline{c}_i + \overline{u}_i}\right) + A_i^-\left(t - \frac{x_i - x}{\overline{c}_i - \overline{u}_i}\right)\right]
\end{aligned} \tag{1}$$

where p_k, u_k, ρ_k, c_k denote the pressure, velocity, density, sound speed of the interval $x \in (x_{k-1}, x_k)$. In addition the overbar symbol denotes mean value and $A_k^{\pm}(x,t)$ are unknown functions.

Choosing $x = x_i$ and applying the Laplace transformation to (1), we have

$$\begin{aligned}
\widetilde{p'_k}(s) &= \widetilde{A}_k^+ e^{-\tau_k^+ s} + \widetilde{A}_k^- \\
\overline{\rho}_k \overline{c}_k \widetilde{u'_k}(s) &= \widetilde{A}_k^+ e^{-\tau_k^+ s} - \widetilde{A}_k^- \\
\overline{c}_i^2 \widetilde{\rho'_i}(s) &= \widetilde{A}_i^+ e^{-\tau_i^+ s} + \widetilde{A}_i^- \\
\tau_k^{\pm} :&= \frac{x_k - x_{k-1}}{\overline{c}_k \pm \overline{u}_k} \quad (k = i, i+1)
\end{aligned} \tag{2}$$

where the symbol ˜ represents the Laplace transform and $s \in \mathbb{C}$ is a complex Laplace variable.

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B. Governing Equations

We wish to find relations between four wave functions $A_k^{\pm}(x_i,t)$ $(k = i, i+1)$ across at $x = x_i$ in Fig. 1. The relations come from the following three (mass, momentum and energy) conservations laws

$$\begin{aligned} \left[\rho u \mathcal{A}\right]_1^2 &= 0, \\ \left[(p+\rho u^2)\mathcal{A}\right]_1^2 &= 0, \\ \left[(\eta p u + \rho u^3/2)\mathcal{A}\right]_1^2 &= \dot{q}_i, \quad \eta := \frac{\gamma}{\gamma-1} \end{aligned} \tag{3}$$

where the subscript/superscript $\{1,2\}$ denote $\{x_i-\epsilon, x_i+\epsilon\}$ for small $\epsilon > 0$. $\mathcal{A}_i$ denotes the cross-sectional areaes of the interval $x \in (x_{i-1}, x_i)$ and $\dot{q}_i'$ denotes a heat rate perturbation at the point $x = x_i$ in Fig. 1.

Explicitly, (3) can be written as

$$\begin{aligned} \alpha_i\rho_2 u_2 &= \rho_1 u_1, \\ \alpha_i(p_2+\rho_2 u_2^2) &= p_1 + \rho_1 u_1^2, \\ \alpha_i(\eta_2 p_2 u_2 + \rho_2 u_2^3/2) &= \eta_1 p_1 u_1 + \rho_1 u_1^3/2 + \dot{q}_i/\mathcal{A}_1, \\ \alpha_i &:= \mathcal{A}_{i+1}/\mathcal{A}_i \end{aligned} \tag{4}$$

where, for notational simplicity, we mixed subscripts $\{1,2\}$ with $\{i, i+1\}$.

We will use the mass and energy conservation laws as given in (4) but for the momentum conservation, a modification is to be made as will be explained in the following sections.

1) Mass Conservation: The mass conservation law in (4) gives a conservation condition at an equilibrium state

$$\alpha_i \bar{\rho}_2 \bar{u}_2 = \bar{\rho}_1 \bar{u}_1 \tag{5}$$

and its perturbed form

$$\alpha_i \rho_2' \bar{u}_2 = \rho_1' \bar{u}_1 + \bar{\rho}_1 u_1' - \alpha_i \bar{\rho}_2 u_2' \tag{6}$$

which can be rewritten as

$$\bar{u}_2^2 \rho_2' = \frac{\bar{u}_1\bar{u}_2}{\alpha_i}\rho_1' + \frac{\bar{\rho}_1\bar{u}_2}{\alpha_i}u_1' - \bar{\rho}_2\bar{u}_2 u_2' \tag{7}$$

The equillibrium and perforbation forms (5) and (6) will be merged into the momentum and energy conservation laws below.

2) Momentum Equation: Note that, under the next conditions

$$\mathcal{A}_1 \neq \mathcal{A}_2 \text{ (or } \alpha_i \neq 1), \quad u_1 \approx 0, \quad u_2 \approx 0, \tag{8}$$

the momentum conservation law (4) gives rise to a discontinuity $p_1'(x_1,t) \neq p_2'(x_1,t)$ at the flame, which is not physically intuitive. For a resolution of this, while keeping the momentum conservation law alive as much as possible, one may consider a simple modification

$$p_2 + \rho_2 u_2^2 = p_1 + \rho_1 u_1^2/\alpha_i \tag{9}$$

instead of the previous form in (4). This new law says that the momentum is not conserved but either increased if $\alpha_i > 1$ or decreased if $\alpha_i < 1$. A physical justification of the new momentum law (9) is accredited to axial forces at $x = x_i$ in [5]. A perturbed of (9) is given as

$$p_2' + \rho_2'\bar{u}_2^2 + 2\bar{\rho}_2\bar{u}_2 u_2' = p_1' + \frac{\bar{u}_1^2\rho_1' + 2\bar{\rho}_1\bar{u}_1 u_1'}{\alpha_i} \tag{10}$$

By combining this result with the mass equation (7), we can obtain

$$\begin{aligned} 0 = p_2' - p_1' + \bar{\rho}_2\bar{c}_2 u_2' M_2 &+ \frac{\bar{u}_1}{\bar{c}_1^2\alpha_i}\left(\bar{u}_2 - \bar{u}_1\right)\bar{c}_1^2\rho_1' \\ &+ \frac{\bar{\rho}_1}{\bar{\rho}_1\bar{c}_1\alpha_i}\left(\bar{u}_2 - 2\bar{u}_1\right)\bar{\rho}_1\bar{c}_1 u_1' \end{aligned} \tag{11}$$

From the representation (10), perturbation form (2) and next two identities ;

$$\begin{aligned} &\text{(i)} \quad \frac{\bar{u}_1(\bar{u}_2-\bar{u}_1)}{\bar{c}_1^2\alpha_i} = \frac{\bar{u}_1^2(\bar{u}_2/\bar{u}_1 - 1)}{\bar{c}_1^2\alpha_i} = \frac{M_1^2}{\alpha_i}\left(\frac{\bar{u}_2}{\bar{u}_1} - 1\right), \\ &\text{(ii)} \quad \frac{(\bar{u}_2 - 2\bar{u}_1)}{\bar{c}_1\alpha_i} = \frac{\bar{u}_1}{\bar{c}_1\alpha_i}\left(\frac{\bar{u}_2}{\bar{u}_1} - 2\right) = \frac{M_1}{\alpha_i}\left(\frac{\bar{u}_2}{\bar{u}_1} - 2\right) \\ &\text{(iii)} \quad -\alpha_i + M_1^2\left(\frac{\bar{u}_2}{\bar{u}_1} - 1\right) \pm M_1\left(\frac{\bar{u}_2}{\bar{u}_1} - 2\right) \\ &\qquad = -\alpha_i \mp M_1 + M_1(M_1 \pm 1)\left(\frac{\bar{u}_2}{\bar{u}_1} - 1\right) \end{aligned}$$

where $M_k := \bar{u}_k/\bar{c}_k$, one can obtain that

$$\begin{aligned} &\left[-\alpha_i - M_1 + M_1(M_1+1)\left(\frac{\bar{u}_2}{\bar{u}_1} - 1\right)\right]\widetilde{A}_i^+ e^{-\tau_i^+ s} \\ &\quad + \left[-\alpha_i + M_1 + M_1(M_1-1)\left(\frac{\bar{u}_2}{\bar{u}_1} - 1\right)\right]\widetilde{A}_i^- \\ &\quad + \alpha_i\left(1 + M_2\right)\widetilde{A}_{i+1}^+ \\ &\quad + \alpha_i\left(1 - M_2\right)\widetilde{A}_{i+1}^- e^{-\tau_{i+1}^- s} = 0 \end{aligned} \tag{12}$$

3) Energy Conservation: A perturbation form of the energy conservation law (4) is given as

$$\begin{aligned} &\alpha_i\eta_2\bar{u}_2 p_2' + \alpha_i\eta_2\bar{p}_2 u_2' + \frac{\bar{u}_2^3}{2}\alpha_i\rho_2' + \frac{3}{2}\alpha_i\bar{\rho}_2\bar{u}_2^2 u_2' \\ &- \eta_1\bar{p}_1 u_1' - \eta_1\bar{u}_1 p_1' - \frac{\bar{u}_1^3}{2}\rho_1' - \frac{3\bar{\rho}_1\bar{u}_1^2}{2}u_1' = \widetilde{q}'_i(s)/\mathcal{A}_i \end{aligned} \tag{13}$$

where $\widetilde{q}'_i(s)$ denotes the Laplace transform of the heat rate perturbation $\dot{q}'(x_i,t)$.

From (7), one can rewrite

$$\begin{aligned} \widetilde{q}'_i(s)/\mathcal{A}_i &= \alpha_i\eta_2\bar{u}_2 p_2' - \eta_1\bar{u}_1 p_1' \\ &+ \frac{\alpha_i}{\bar{\rho}_2\bar{c}_2}\left(\eta_2\bar{p}_2 - \frac{\bar{u}_2^2}{2}\bar{\rho}_2 + \frac{3\bar{\rho}_2\bar{u}_2^2}{2}\right)\bar{\rho}_2\bar{c}_2 u_2' \\ &+ \frac{\bar{u}_1}{2\bar{c}_1^2}\left(\bar{u}_2^2 - \bar{u}_1^2\right)\bar{c}_1^2\rho_1' \\ &- \frac{1}{\bar{\rho}_1\bar{c}_1}\left(\eta_1\bar{p}_1 - \frac{\bar{u}_1^2}{2}\bar{\rho}_1 + \frac{3\bar{\rho}_1\bar{u}_1^2}{2}\right)\bar{\rho}_1\bar{c}_1 u_1' \end{aligned} \tag{14}$$

Now, making uses of the next facts [6], (p.35).

$$\bar{p}_1 = \frac{1}{\gamma_1}\bar{\rho}_1\bar{c}_1^2, \quad \bar{p}_2 = \frac{1}{\gamma_2}\bar{\rho}_2\bar{c}_2^2, \tag{15}$$

one can easily derive the following identities

(i) $\alpha_i \eta_2 \overline{u}_2 = \alpha_i \overline{c}_2 \dfrac{\gamma_2 M_2}{\gamma_2 - 1}$

(ii) $\eta_1 \overline{u}_1 = \overline{c}_1 \dfrac{\gamma_1 M_1}{\gamma_1 - 1}$

(iii) $\dfrac{\alpha_i}{\overline{\rho}_2 \overline{c}_2} \left(\eta_2 \overline{p}_2 + \overline{\rho}_2 \overline{u}_2^2\right) = \alpha_i \overline{c}_2 \left(\dfrac{1}{\gamma_2 - 1} + M_2^2\right)$

(vi) $\dfrac{\overline{u}_1}{2\overline{c}_1^2} \left(\overline{u}_2^2 - \overline{u}_1^2\right) = \dfrac{\overline{c}_1 M_1^3}{2} \left(\dfrac{\overline{u}_2^2}{\overline{u}_1^2} - 1\right)$

(v) $\dfrac{1}{\overline{\rho}_1 \overline{c}_1} \left(-\eta_1 \overline{p}_1 + \dfrac{\overline{u}_2^2}{2} \overline{\rho}_1 - \dfrac{3\overline{\rho}_1 \overline{u}_1^2}{2}\right)$
$= \overline{c}_1 \left[-\dfrac{1}{\gamma_1 - 1} + \dfrac{M_1^2}{2} \left(\dfrac{\overline{u}_2^2}{\overline{u}_1^2} - 3\right)\right]$

Making use of these identities and (14), we can obtain

$$
\begin{aligned}
&\widetilde{q}'_i(s)/\mathcal{A}_i = \\
&\overline{c}_1 \left[-\frac{\gamma_1 M_1 + 1}{\gamma_1 - 1} + \frac{M_1^2}{2}(M_1 + 1)\left(\frac{\overline{u}_2^2}{\overline{u}_1^2} - 1\right) - M_1^2\right] \widetilde{A}_i^+ e^{-\tau_i^+ s} \\
&+ \overline{c}_1 \left[-\frac{\gamma_1 M_1 - 1}{\gamma_1 - 1} + \frac{M_1^2}{2}(M_1 - 1)\left(\frac{\overline{u}_2^2}{\overline{u}_1^2} - 1\right) + M_1^2\right] \widetilde{A}_i^- \\
&\qquad + \alpha_i \overline{c}_2 \left[\frac{\gamma_2 M_2 + 1}{\gamma_2 - 1} + M_2^2\right] \widetilde{A}_{i+1}^+ \\
&\qquad + \alpha_i \overline{c}_2 \left[\frac{\gamma_2 M_2 - 1}{\gamma_2 - 1} - M_2^2\right] \widetilde{A}_{i+1}^- e^{-\tau_{i+1}^- s}
\end{aligned} \tag{16}
$$

C. Relations between Wave Functions

From now on we recover the subscript $\{i, i+1\}$ instead of $\{1, 2\}$ for notational consistency. Then, in a matrix form, the momentum and energy conditions can be written as

$$
Q_i \begin{bmatrix} \widetilde{A}_i^+ \\ \widetilde{A}_i^- \end{bmatrix} + D_i \begin{bmatrix} \widetilde{A}_{i+1}^+ \\ \widetilde{A}_{i+1}^- \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\widetilde{q}'_i(s)}{\mathcal{A}_i} \tag{17}
$$

where

$$
Q_i := \begin{bmatrix} q_i^{(1,1)} & q_i^{(1,2)} \\ q_i^{(2,1)} & q_i^{(2,2)} \end{bmatrix} \begin{bmatrix} e^{-\tau_i^+ s} & 0 \\ 0 & 1 \end{bmatrix} \tag{18}
$$

$$
D_i := \begin{bmatrix} d_i^{(1,1)} & d_i^{(1,2)} \\ d_i^{(2,1)} & d_i^{(2,2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\tau_{i+1}^- s} \end{bmatrix} \tag{19}
$$

$$
\begin{cases}
q_i^{(1,1)} &= -\alpha_i - M_i + M_i(1 + M_i)\left(\overline{u}_{i+1}/\overline{u}_i - 1\right) \\
q_i^{(1,2)} &= -\alpha_i + M_i - M_i(1 - M_i)\left(\overline{u}_{i+1}/\overline{u}_i - 1\right) \\
q_i^{(2,1)} &= \overline{c}_i \Big[-\frac{\gamma_i M_i + 1}{\gamma_i - 1} - M_i^2 \\
&\qquad + \frac{1}{2} M_i^2 (1 + M_i)(\overline{u}_{i+1}^2/\overline{u}_i^2 - 1)\Big] \\
q_i^{(2,2)} &= \overline{c}_i \Big[-\frac{\gamma_i M_i - 1}{\gamma_i - 1} + M_i^2 \\
&\qquad - \frac{1}{2} M_i^2 (1 - M_i)(\overline{u}_{i+1}^2/\overline{u}_i^2 - 1)\Big] \\
d_i^{(1,1)} &= \alpha_i (1 + M_{i+1}) \\
d_i^{(1,2)} &= \alpha_i (1 - M_{i+1}) \\
d_i^{(2,1)} &= \alpha_i \overline{c}_{i+1} \left[\frac{\gamma_{i+1} M_{i+1} + 1}{\gamma_{i+1} - 1} + M_{i+1}^2)\right] \\
d_i^{(2,2)} &= \alpha_i \overline{c}_{i+1} \left[\frac{\gamma_{i+1} M_{i+1} - 1}{\gamma_{i+1} - 1} + M_{i+1}^2)\right]
\end{cases} \tag{20}
$$

We note that if the heat perturbation at $x = x_i$ satisfies $\dot{q}_i' = 0$ then (17) can be written as

$$
\begin{bmatrix} \widetilde{A}_i^+ \\ \widetilde{A}_i^- \end{bmatrix} = -Q_i^{-1} D_i \begin{bmatrix} \widetilde{A}_{i+1}^+ \\ \widetilde{A}_{i+1}^- \end{bmatrix} \tag{21}
$$

D. General One-Dimensional Model

Consider a general combustor composed of multiple area places as illustrated in Fig. 2. We assume that this combustor has only one hear source at $x = x_{n-1}$, that is,

$$
\dot{q}_k' = 0 \quad (k = 1, \cdots, n-2), \quad \dot{q}_{n-1}' \neq 0 \tag{22}
$$

This assumption is not essential but can be easily removed with slight modifications of the following results.

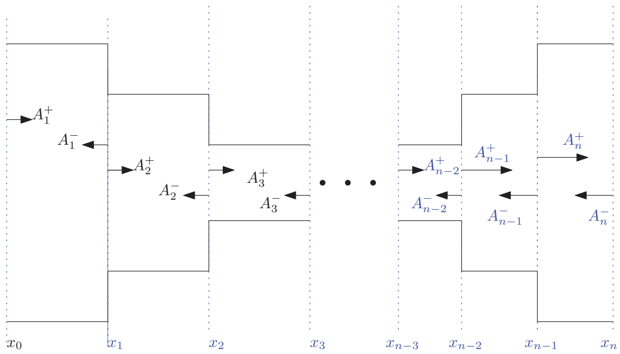

Fig. 2 Combustor with n-cans

It should be noted that we made no assumptions on the area ratios $\{\alpha_i\,;\, i = 1, \cdots, n\}$. The particular shape of the combustor in Fig. 2 whose cross sectional areas decrease firstly and then increases as n increases, is only an illustration and any general shape can be considered in our model to be developed below.

An application of the wave function relations (17) to $x = x_k$ for every $k = 1, \cdots, n-1$, gives $(k = 1, \cdots, n-2)$

$$
\begin{aligned}
Q_k \begin{bmatrix} \widetilde{A}_k^+ \\ \widetilde{A}_k^- \end{bmatrix} + D_k \begin{bmatrix} \widetilde{A}_{k+1}^+ \\ \widetilde{A}_{k+1}^- \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
Q_{n-1} \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} + D_{n-1} \begin{bmatrix} \widetilde{A}_n^+ \\ \widetilde{A}_n^- \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\widetilde{q}'_{n-1}(s)}{\mathcal{A}_{n-1}}
\end{aligned} \tag{23}
$$

Now, from (21), we can eliminate $\widetilde{A}_k^{\pm}$ for $k = 2, \cdots, n-2$ in the recursive equation (23) to have

$$
\begin{aligned}
Q_1 \begin{bmatrix} \widetilde{A}_1^+ \\ \widetilde{A}_1^- \end{bmatrix} + V \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
Q_{n-1} \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} + D_{n-1} \begin{bmatrix} \widetilde{A}_n^+ \\ \widetilde{A}_n^- \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\widetilde{q}'_{n-1}(s)}{\mathcal{A}_{n-1}}
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
V &= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \\
&:= D_1(-Q_2^{-1} D_2)(-Q_3^{-1} D_3) \cdots (-Q_{n-2}^{-1} D_{n-2})
\end{aligned} \tag{25}
$$

Notice that (24) has six unknowns and four equalities. Two additional equalities come from the boundary conditions at $x \in \{x_0, x_n\}$. The boundary condition are generally characterized by the *reflection coefficients*

$$R_i(s) := \frac{\widetilde{A}_1^+}{\widetilde{A}_1^- e^{-\tau_1^- s}}, \quad R_o(s) := \frac{\widetilde{A}_n^-}{\widetilde{A}_n^+ e^{-\tau_n^- s}} \tag{26}$$

In general, the reflection coefficients $R_i(s), R_o(s)$ can be functions of the Laplace variable $s \in \mathbb{C}$ but we suppress their dependency on s for notational simplicity.

By substituting $\widetilde{A}_1^+ = R_i e^{-\tau_1^- s}\widetilde{A}_1^-$, $\widetilde{A}_n^- = R_o e^{-\tau_n^+ s}\widetilde{A}_n^+$ into (24), we obtain four equalities with four unknowns;

$$\begin{gathered}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \widetilde{A}_1^- + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} q_{n-1}^{(1,1)} e^{-\tau_{n-1}^+ s} & q_{n-1}^{(1,2)} \\ q_{n-1}^{(2,1)} e^{-\tau_{n-1}^+ s} & q_{n-1}^{(2,2)} \end{bmatrix} \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \widetilde{A}_n^+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\widetilde{q}'_{n-1}(s)}{\mathcal{A}_{n-1}}\end{gathered} \tag{27}$$

where

$$\begin{cases} k_1 := q_1^{(1,1)} R_i e^{-(\tau_1^+ + \tau_1^-)s} + q_1^{(1,2)} \\ k_2 := q_1^{(2,1)} R_i e^{-(\tau_1^+ + \tau_1^-)s} + q_1^{(2,2)} \\ h_1 := d_{n-1}^{(1,1)} + d_{n-1}^{(1,2)} R_o e^{-(\tau_n^+ + \tau_n^-)s} \\ h_2 := d_{n-1}^{(2,1)} + d_{n-1}^{(2,2)} R_o e^{-(\tau_n^+ + \tau_n^-)s} \end{cases} \tag{28}$$

In addition, an elimination of two unknowns $\widetilde{A}_1^-, \widetilde{A}_n^+$ in (27) gives

$$\mathcal{F}(s) \begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\widetilde{q}'_{n-1}(s)}{\mathcal{A}_{n-1}} \tag{29}$$

where

$$\mathcal{F}(s) := \begin{bmatrix} k_2 v_{11} - k_1 v_{21} & k_2 v_{12} - k_1 v_{22} \\ \left(h_2 q_{n-1}^{(1,1)} - h_1 q_{n-1}^{(2,1)}\right) e^{-\tau_{n-1}^+ s} & h_2 q_{n-1}^{(1,2)} - h_1 q_{n-1}^{(2,2)} \end{bmatrix} \tag{30}$$

Define a matrix determinant $\Delta(s) := |\mathcal{F}(s)|$. Then (29) gives

$$\begin{bmatrix} \widetilde{A}_{n-1}^+ \\ \widetilde{A}_{n-1}^- \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} k_2 v_{12} - k_1 v_{22} \\ -k_2 v_{11} + k_1 v_{21} \end{bmatrix} \frac{\widetilde{q}'_{n-1}(s)}{\mathcal{A}_{n-1}} \tag{31}$$

Note that, similar to (2), the velocity perturbation at $x = x_{n-1}$ is given

$$\bar{\rho}_{n-1}\bar{c}_{n-1}\widetilde{u}'_{n-1}(s) = \widetilde{A}_{n-1}^+ e^{-\tau_{n-1}^+ s} - \widetilde{A}_{n-1}^- \tag{32}$$

As a final step, from (31) and (32), we obtain a transfer function from the hear rate perturbation to the velocity perturbation given

$$\frac{\widetilde{u}'_{n-1}(s)}{\widetilde{q}'_{n-1}(s)} = \left(\frac{1}{\bar{\rho}_{n-1}\bar{c}_{n-1}\mathcal{A}_{n-1}}\right) \times \frac{(k_2 v_{12} - k_1 v_{22}) e^{-\tau_{n-1}^+ s} + (k_2 v_{11} - k_1 v_{21})}{\Delta(s)} \tag{33}$$

III. Conclusion

We have derived a thermo-acoustic transfer function of a general one-dimensional combustor mode. Our transfer function representation can handle a general one-dimensional combustor model with multiple cans with different cross sectional areas, without assuming zero flow rate.

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