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A Study on Stochastic Integral Associated with Catastrophes

M. Reni Sagayaraj, S. Anand Gnana Selvam, R. Reynald Susainathan

Abstract**—**We analyze stochastic integrals associated with a mutation process. To be specific, we describe the cell population process and derive the differential equations for the joint generating functions for the number of mutants and their integrals in generating functions and their applications. We obtain first-order moments of the processes of the two-way mutation process in first-order moment structure of X (t) and Y (t) and the second-order moments of a one-way mutation process. In this paper, we obtain the limiting behaviour of the integrals in limiting distributions of X (t) and Y (t).

Keywords**—**Stochastic integrals, single–server queue model, catastrophes, busy period.

I. Introduction

The queueing theory has played an important role in the theory of probability and related concepts. Its applications have been utilized varies fields like communication system, industrial sector and so on. Human beings, telephone calls flow of finished products, failed machines and so on may be considered as queueing units. In modern days, the queueing models have been analyzed by assuming the telephone calls as the units for demanding service.

In the analysis of some queueing system, we come across situations where the annihilation of all the system and paralysation of the service facility may take place upon the arrival of some kind of special events. These special events are called catastrophical events and they themselves form a point process which may be independent of the arrival and service pattern of the queueing system. Such events occur quite commonly in computer networks. For example, when an infected job or file arrive at a service station, the job or the file acts as a catastrophic event destroying all the files in the processor and paralysing momentarily the processor.

II. The Basic Stochastic Model

Consider a cell population such as bacteria consisting of two types of cells called Red and White. Each cell type may undergo mutation or the divisional process or encounter death in case of action of bactericidal drugs. The following assumptions are made with the occurrence of these events.

- The probability that any white cell at time becomes a red cell at time$t+\delta t$ is $\alpha\delta t+0(\delta t)$.
- The probability that any red cell at time becomes a white cell at time$t+\delta t$ is$\beta\delta t+0(\delta t)$
- The probability that any cell type at time t encounters death at time $t+\delta t$ is$\mu\delta t+0(\delta t)$
- The probability that any white cell at time t splits into White cell at time$t+\delta t$ is$\lambda_2\delta t+0(\delta t)$
- The probability that any white cell at time t splits into red cells at time t splits into Red cells at time$t+\delta t$ is$\lambda_2\delta t+0(\delta t)$
- The parametersα , β and $\mu$are non-negative constants λ_1 and $\lambda_2$are positive constants.
- The mutation process, the splitting process and death process are independent of one another.
- The cells behave independently of one another.

when time $t=0$ with one white particle. Let$R(t)$ and$W(t)$ denote respectively the numbers of Red and White particle at time t. Then the total number$N(t)$ of individuals at time t is given by:

$$N(t) = R(t) + W(t) \tag{1}$$

Along with the stochastic processes $R(t), W(t)$and $N(t)$ consider the following the stochastic integrals:

$$\int_0^1 R(T)dT , \int_0^1 W(T)dT , \int_0^1 N(T)dt \tag{2}$$

The main impetus to study these integrals arises since they are associated with some cumulative response of the mutation process. For example, In the case of βgalactosidase gene, the stochastic integrals $\int_0^1 R(T)dT$ and $\int_0^1 W(T)dT$ represent respectively the durations of enzyme activities of the red particles up to time t. We denote:

$$X(t) = \int_0^1 W(T)dT , Y(t) = \int_0^1 R(T)dT$$

We derive a system of integral equations for the generating functions associated with the population process.

III. The Generating Functions and Their Applications

We represent the generating functions with the following notations

M. Reni Sagayaraj, Associate Professor and Head, and S. Anand Gnana Selvam, Research Scholar, are with the Department of Mathematics, Sacred Heart College, Accredited by NAAC (3rd Cycle) with ‘A’ Grade [3.43], Tirupattur-635601, Vellore District. Tamil Nadu, India (e-mail: reni.sagaya@gmail.com, anandjuslin@gmail.com).

R. Reynald Susainathan is with the Reckitt Benckiser, India Limited, Gurgaon, New Delhi. India (e-mail: reynaldsusain@gmail.com).

$$\emptyset_n(s_1, s_2, s_3, s_4; t) = E\left\{s_1^{W(t)} s_2^{R(t)} e^{-s_3 X(t)} e^{-s_4 X(t)} \mid \eta(0)\right\}$$

where $\eta = W$ with $\eta(0) = \{W(0) = 1, B(0) = 0\}$ with $\eta(0) = \{W(0) = 1, B(0) = 1\}$. Conditioning on the time of occurrence of the first splitting from timet $= 0$and using probabilistic arguments, we have the following system of integro-differential equations for the joint-probability moment generating functions $\emptyset(W)$and$\emptyset(R)$. (3) and (4) are not solvable easily:

$$\begin{aligned}\phi_W(s_1,s_2,s_3,s_4;t) &= s_1 e^{-(\alpha+\lambda+\mu+s_3)t} \\ &+ \alpha\int_0^t e^{-(\alpha+\lambda+\mu+s_3)T}\phi_R(s_1,s_2,s_3,s_4;t-T)\,dT \\ &+ \lambda_1\int_0^t e^{-(\alpha+\lambda+\mu+s_3)T}(\phi_W(s_1,s_2,s_3,s_4;t-T))^2\,dT \\ &+ \mu\int_0^t e^{-(\alpha+\lambda+\mu+s_3)T}\,dT\end{aligned} \quad (3)$$

$$\begin{aligned}\phi_R(s_1,s_2,s_3,s_4;t) &= s_2 e^{-(\beta+\lambda+\mu+s_4)t} \\ &+ \beta\int_0^t e^{-(\beta+\lambda+\mu+s_4)T}\phi_W(s_1,s_2,s_3,s_4;t-T)\,dT \\ &+ \lambda_2\int_0^t e^{-(\beta+\lambda+\mu+s_4)T}(\phi_R(s_1,s_2,s_3,s_4;t-T))^2\,dT \\ &+ \mu\int_0^t e^{-(\beta+\lambda+\mu+s_4)T}\,dT\end{aligned} \quad (4)$$

IV. A Single Queue Model

Consider a single–server queue model with infinite services and catastrophes. Customer departure at the queue according to the Poisson process with rate λ. We assume that the service –time has exponential distribution with parameter μ. Let the service discipline be FIFO. We assume that the system capacity is infinite. Let the catastrophic events departure independently at the service facility according to a Poisson process with rate γ [1]. The nature of a catastrophic event is that upon its departure at the service station it destroys all the customers there waiting. The catastrophe departure at queue j from the outside of the network according to the Poisson process with rate $\delta_j, j = 1,2,..j$. Whenever a catastrophe departure at a queue, either from the outside or from another queue, all the customers in the queue are destroyed immediately and the server is ready to serve new customers [3]. Let $P_n(t)$ be the probability that there are n customers in the system at time t , by routine procedure, we have:

$$\begin{aligned}P_n(t) &= \mu P_{n+1}(t) - (\lambda+\mu+\gamma)P_n(t) + \lambda P_{n-1}(t), n = 1,2\ldots \\ P_0'(t) &= \mu P_1(t) - \lambda P_0(t) + \gamma[1 - P_0(t)]\end{aligned} \quad (5)$$

where λ and μ have the usual meanings. We assume that a customer departures to an empty at the service facility at time t=0 so that the busy period starts at time t=0.Then $P_n(0) = \delta_{n,1}$, n=0 ,1, 2, ... where:

$$\delta_{i,j} = \begin{cases}1\ wheni = j \\ 0 wheni \neq j\end{cases}$$

To solve (7), we proceed as follows. Defining:

$$P(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$$

We obtain from (4):

$$\frac{\partial P(s,t)}{\partial t} = \left[\lambda s + \frac{\mu}{s} - (\lambda+\mu+\gamma)\right]P(s,t) + \mu\left(1-\frac{1}{s}\right)P_0(t) + \gamma \quad (6)$$

Subject to condition P(s,0) = s. (4) can be solved and we obtain:

$$\begin{aligned}P(s,t) &= se^{-(\lambda+\mu+\gamma)t}e^{(\lambda s+\frac{\mu}{s})t} \\ &+ \mu\left(1-\frac{1}{s}\right)\int_0^t p_0(u)e^{-(\lambda+\mu+\gamma)(t-u)}e^{(\lambda s+\frac{\mu}{s})(t-u)}\,du \\ &+ \gamma\int_0^t e^{(\lambda s+\frac{\mu}{s})(t-u)}\,du\end{aligned} \quad (7)$$

In (5), we use the generating function:

$$exp\left\{\left(\frac{\mu}{s}+\lambda s\right)t\right\} = \exp\left\{\frac{1}{2}\left((\beta s)+\frac{1}{(\beta s)}\right)\alpha t\right\} = \sum_{-\infty}^{\infty}\beta s^n I_n(\alpha t) \quad (8)$$

where we have set $\lambda = \frac{\alpha\beta}{2}$ and $\mu = \frac{\alpha}{2\beta}$we have $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \frac{\lambda}{\mu}$. In the above, $I_n(\alpha t), n = 0, \pm1, \pm2, \ldots$are modified Bessel functions of the first kind given by:

$$I_n(u) = \sum_{k=0}^{\infty}\frac{u^{n+2k}}{2^{n+2k}k!(n+k)!}, n > -1; I_{-n}(u) = I_n(u)$$

Then equating the power of ++ on both idles, we obtain:

$$\begin{aligned}p_0(t) &= \frac{1}{\mu}\sum_{n=1}^{\infty}\frac{(n+1)}{\beta^{n+1}}\frac{I_{n+1}(\alpha t)}{t}e^{-(\lambda+\mu+\gamma)t} \\ &+ \frac{\gamma}{\mu}\int_0^t\frac{n}{\beta^n}\frac{I_n(\alpha u)}{u}e^{-(\lambda+\mu+\gamma)t}\,du\end{aligned}$$

$$\begin{aligned}p_n(t) &= \frac{2\gamma\beta^{n+1}}{\alpha}\int_0^t\sum_{k=0}^{\infty}\frac{(n+k+1)I_{n+k+1}(\alpha u)}{\beta^{k+1}u}e^{-(\lambda+\mu+\gamma)t}\,du \\ &+ \sum_{k=0}^{\infty}e^{-(\lambda+\mu+\gamma)t}\left[\frac{I_{m+n+2}(\alpha t)}{\beta^{b-n+1}}\frac{I_{m+n+3}(\alpha t)}{\beta^{m-n+1}}\right] \\ &+ \beta^{n-1}I_{n-1}(\alpha t)e^{-(\lambda+\mu+\gamma)t}, n = 0,1,2,\ldots,\end{aligned}$$

The above probabilities are completely describing the queueing process.

V. The Busy Period

We have already mentioned that the busy and the idle periods develop a random evolution in problems related to the queueing systems. We proceed to obtain the probability law of the busy period [7]. To do this, we impose further that there is an absorbing barrier at zero system size so that $P_0'(t)$gives the probability density function of the busy period[8], where$P_n(t)$represents the probability that the system size at time **t** is **n .** We assume that the server enters into the busy period at time t = 0. Then$P_1(0) = 1, P_n(0) = 0$ for n≠1 With absorption at the state 0, $P_n(t)$are:

$$P_0'(t) = \mu P_1(t) + \gamma[1 - P_0(t)] \quad (9)$$

$$P_n(t) = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), n \geq 1 \quad (10)$$

Equations (9) and (10) are subject to the condition $P_n(0) = \delta_{1,n}$ $n = 0,1,2..$ it is clear that$P_0'(t)$ is the probability density function of the busy period. To find it, we proceed as follows: Define:

$$K(s,t) = \sum_{n=0}^{\infty} P_n(t) s^n$$

Then, K(S,0) = S and:

$$\frac{\partial K(s,t)}{\partial t} = -\left(\lambda + \mu + \gamma - \lambda s - \frac{\mu}{s}\right) K(s,t) + \left(\lambda + \mu - \frac{\mu}{s}\right) p_0(t) + \gamma . \quad (11)$$

Integrating (11) we get:

$$K(s,t) = s e^{-(\lambda+\mu+\gamma)t} e^{\left(\lambda s + \frac{\mu}{s}\right)t} + \left(\lambda + \mu - \frac{\mu}{s}\right) \int_0^t e^{-(\lambda+\mu+\gamma)(t-u)} e^{\left(\lambda s + \frac{\mu}{s}\right)(t-u)} p_0(u) du + \lambda \int_0^t e^{-(\lambda+\mu+\gamma)(t-u)} e^{\left(\lambda s + \frac{\mu}{s}\right)(t-u)} du. \quad (12)$$

Substituting the series expressions for $K(s,t)$and$e^{\left(\lambda s + \frac{\mu}{s}\right)t}$ into (12) and equating the coefficients of s^0 on both sides, we get:

$$p_0(t) = \frac{I_1(\alpha t)}{\beta} e^{-(\lambda+\mu+\gamma)t} + (\lambda + \mu) \int_0^t e^{-(\lambda+\mu+\gamma)(t-u)} I_0(\alpha(t-u)) p_0(u) du - \mu\beta \int_0^t e^{-(\lambda+\mu+\gamma)(t-u)} I_1(\alpha(t-u)) p_0(u) du + \gamma \int_0^t e^{-(\lambda+\mu+\gamma)(t-u)} I_0(\alpha(t-u)) du. \quad (13)$$

The integral (12) admits a Laplace transform solution for $P_0(t)$. If$P_0'(t)$ is the Laplace transform of$P_0(t)$ and$I_n^*(\theta)$ is the Laplace transform of$I_n(\theta)$then we obtain:

$$\theta p_0^*(\theta) = \frac{1}{\beta} \left[\frac{\beta\gamma I_0^*(\frac{\theta+\lambda+\mu+\gamma}{\alpha}) + I_1^*((\frac{\theta+\lambda+\mu+\gamma}{\alpha})}{\alpha - (\lambda+\mu) I_0^*(\frac{\theta+\lambda+\mu+\gamma}{\alpha}) + \beta\mu I_1^*((\frac{\theta+\lambda+\mu+\gamma}{\alpha})} \right] \quad (14)$$

Inverting (14), we obtain the derivative $P_0'(t)$and this gives the probability density function of the busy period.

VI. The Random Evolution of a Stochastical Integral

In this proposed project the idle period occurs whenever a catastrophic event occurs when server is busy [2]. Let there be a positive income when the server is busy, and a cost to pay when it is idle [6]. To study the net gain, we define the following costs. Let C_1 be the profit per unit time of the busy period, C_2 be the cost per unit time of the idle period not initiated by the departure of a catastrophic event and C_3 be the cost per unit time of the idle period initiated by the departure of a catastrophic event[9]. Then the time – course of the net profit can be described by the random motion of a stochastic integral. To achieve this, we define stochastic process Z(t) as:

$$Z(t) = \begin{cases} 1\ iftheserverisbusy \\ 2\ iftheserverisidlenotinitiatedbycatastrophicevents \\ 3\ iftheserverisidlenotinitiatedbycatastrophicevents \end{cases}$$

The stochastic process Z(t) is a market- point process and its probability law can be obtained in terms of the distributions of the busy and idle periods [4]. The idle periods are of two types and are characterized by the point process of catastrophic events [5], [10]. We not that:

$$P_r\{Z(t) = 1\} = \sum_{n=1}^{\infty} \int_0^t P_n(u)\, e^{-(\alpha+\lambda+\mu)(t-u)} du$$

$$P_r\{Z(t) = 2\} = \int_0^t P_1(u)\, e^{-\lambda(t-u)} du$$

$$P_r\{Z(t) = 3\} = \sum_{n=1}^{\infty} \int_0^t P_1(u)\, \gamma\, e^{-\lambda(t-u)} du$$

If C(t) is the instantaneous cost at time t then:

$$C(t) = \begin{cases} C_1, if\ z(t) = 1 \\ C_2, if\ z(t) = 2 \\ C_3, if\ z(t) = 3 \end{cases}$$

Then the net gain X(t) is given by$X(t) = \int_0^t C(u)du$is identified as the position of the particle and the probability law of X(t) can be obtained by considering the random motion of the particle.

VII. Conclusion

In this paper, we analyzed the single server queueing model and we obtain the derivative$P_0'(t)$and the probability density function of the busy period. Then the time-course of the net profit is described by the random motion of a stochastic integral. To achieved stochastic process Z(t).

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Dr. Major. M. Reni Sagayaraj is an Associate Professor and Head, Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, Vellore (District), Tamil Nadu, India. He has 34 years of teaching experience and 20 years of research experience. He has received Ph. D in "Studies on the random evolution of certain Queueing system" from the University of Madras. He has published 68 research papers in the International Journals.

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