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A Study on Intuitionistic Fuzzy h-ideal in Γ-Hemirings

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Abstract—The notions of intuitionistic fuzzy h-ideal and normal intuitionistic fuzzy h-ideal in Γ-hemiring are introduced and some of the basic properties of these ideals are investigated. Cartesian product of intuitionistic fuzzy h-ideals is also defined. Finally a characterization of intuitionistic fuzzy h-ideals in terms of fuzzy relations is obtained.

Keywords—Γ-hemiring, fuzzy h-ideal, normal, cartesian product.

Mathematics Subject Classification[2000]:08A72,16Y99

I. Introduction

THE concept of fuzzy set was introduced by Zadeh[14]. Jun and Lee[9] applied the concept of fuzzy sets to the theory of Γ-rings. The notion of Γ-semiring was introduced by Rao[12] as a generalization of Γ-ring as well as of semiring. Ideals of semirings (hemirings) play a central role in the structure theory and are important in many other purposes. However they do not in general coincide with the usual ring ideals. So, many results of rings apparently have no analogues in hemirings using only ideals. To solve this problem, Henriksen[7] defined a more restricted class of ideals in semirings, named k-ideals. Another more restricted, but very important class of ideals, called as h-ideals, has been defined and investigated by Izuka[8] and La Torre[11]. These concepts are extended to Γ-semiring by Rao[12], Dutta and Sardar[5]. The basic concepts of fuzzification of h-ideals in hemirings and Γ-hemiring were discussed in [10] and [13], respectively. In ([2],[3]), Dudek discussed about the intuitionistic fuzzy h-ideals and their properties in hemirings.

As a continuation of this, we introduce here the intuitionistic fuzzy h-ideals in Γ-hemirings and investigate some of their properties. In section II, we recall some definitions. We investigate some basic properties of intuitionistic fuzzy h-ideals such as characteristic and level subset criterion, their behavior under intersection, fuzzy translation etc. in section 3. Then we study and characterize normal intuitionistic fuzzy h-ideal and cartesian product of intuitionistic fuzzy h-ideals in section 4 and 5 respectively.

II. Preliminaries

As an important generalization of the notion of fuzzy sets, Atanassov[1] introduced the concept of an Intuitionistic fuzzy set defined on a non-empty set R, denoted by IFS(R). An IFS(R) is an object having the form

$$A = (\mu_A, \lambda_A) = \{x, \mu_A(x), \lambda_A(x) : x \in R\}$$

where the fuzzy sets μ_A and λ_A denote the degree of membership(namely $\mu_A(x)$) and the degree of non-membership(namely $\lambda_A(x)$) of each element $x \in R$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in R$. According to [1], for every two intuitionistic fuzzy sets $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ in R,we define $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in R$. Obviously $A = B$ means that $A \subseteq B$ and $B \subseteq A$.

***Definition** 2.1:* [6] A hemiring(respectively semiring) is a nonempty set R on which operations addition and multiplication have been defined such that (R,+) is a commutative monoid with identity 0_R, (R,.) is a semigroup(respectively monoid with identity 1_R), multiplication distributes over addition from either side, $1_R \neq 0$ and $0_R.r = 0 = r.0_R$ for all $r \in R$.

***Definition** 2.2:* Let R and Γ be two additive commutative semigroups with zero. Then R is called a Γ-hemiring if there exists a mapping $R \times \Gamma \times R \to R((a, \alpha, b) \to a\alpha b)$ satisfying the follwing conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$
(v) $0_S\alpha a = 0_S = a\alpha 0_S$
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$ for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

***Definition** 2.3:* A left ideal A of a Γ-hemiring S is called a left h-ideal if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z \Rightarrow x \in A$.

A right h-ideal is defined analogously.

***Definition** 2.4:* Let μ be a nonempty fuzzy subset of a Γ-hemiring R(i.e. $\mu(x) \neq 0$ for some $x \in R$). Then μ is called a fuzzy left h-ideal(fuzzy right h-ideal) of R if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and
(ii) $\mu(x\gamma y) \geq \mu(y)$(respectively $\mu(x\gamma y) \geq \mu(x)$) for all $x, y \in$R$, \gamma \in \Gamma$.
(iii) For all $x, a, b, z \in S$, $x + a + z = b + z$ implies $\mu(x) \geq \min\{\mu(a), \mu(b)\}$

In a similar manner we can define fuzzy right ideal of a Γ hemiring R.

In the coming sections we will give definitions and results for intuitionistic fuzzy left h-ideals. Since the proofs of those results for intuitionistic fuzzy right h-ideals follow similarly, we have omitted those.

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III. Intuitionistic fuzzy left h-ideals

Definition *3.1:* An intuitionistic fuzzy subset $A = (\mu_A, \lambda_A)$ in a Γ-hemiring R is called an intuitionistic fuzzy left h-ideal if

(1) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in R$.
(2) $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$.
(3) $\mu_A(x\gamma y) \geq \mu_A(y)$ for all $x, y \in R$ and for all $\gamma \in \Gamma$.
(4) $\lambda_A(x\gamma y) \leq \lambda_A(y)$ for all $x, y \in R$ and for all $\gamma \in \Gamma$.
(5) $x+a+z = b+z \Rightarrow \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$ where $x, a, b, z \in R$.
(6) $x+a+z = b+z \Rightarrow \lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$ where $x, a, b, z \in R$.

An intuitionistic fuzzy subset (μ_A, λ_A) satisfying the first four conditions is called an intuitionistic fuzzy left ideal in the Γ-hemiring R.

Clearly, $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$ for all $x \in R$.

Example *3.2:* Let us consider $R = \mathbf{N}, \Gamma = \mathbf{N}$ with usual addition(+) and multiplication (.) defined on natural numbers. Then (R,+,.) is a Γ-hemiring.
Consider $A = (\mu_A, \lambda_A)$ where

$$\mu_A(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0.8 & \text{for } x \in p\mathbf{N} \sim \{0\} \\ 0.6 & \text{for } x \notin p\mathbf{N} \text{ and } x \neq 0 \end{cases}$$

$$\lambda_A(x) = \begin{cases} 0 & \text{for } x = 0 \\ 0.2 & \text{for } x \in p\mathbf{N} \sim \{0\} \\ 0.4 & \text{for } x \notin p\mathbf{N} \text{ and } x \neq 0 \end{cases}$$

where p is a prime number.Then it is easy to check that A is an Intuitionistic fuzzy left h-ideal of R.

Theorem *3.3:* An intuitionistic fuzzy subset A = (μ_A, λ_A) of a Γ-hemiring R is an intuitionistic fuzzy left h-ideal of R if and only if any level subset

$$R_A^{(\alpha,\beta)} = \{x \in R : \mu_A(x) \geq \alpha \text{ and } \lambda_A(x) \leq \beta; \alpha, \beta \in [0,1] \text{ such that } \alpha + \beta \leq 1\}$$

is a left h-ideal of R provided it is nonempty.

Proof: Let A = (μ_A, λ_A) be an intuitionistic fuzzy left h-ideal of R and let $R_A^{(\alpha,\beta)} \neq \phi$, where $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$.
Let $x, y \in R_A^{(\alpha,\beta)}$. Then $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq \alpha$ and $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\} \leq \beta$. So, $x+y \in R_A^{(\alpha,\beta)}$.
Now let $x \in R_A^{(\alpha,\beta)}$, $y \in R$ and $\gamma \in \Gamma$.
Then $\mu_A(y\gamma x) \geq \mu_A(x) \geq \alpha$ and $\lambda_A(y\gamma x) \leq \lambda_A(x) \leq \beta$. which implies $y\gamma x \in R_A^{(\alpha,\beta)}$.
Hence $R_A^{(\alpha,\beta)}$ is a left ideal.
Again let $x, z \in R$ and $a, b \in R_A^{(\alpha,\beta)}$ be such that $x+a+z = b+z$.
Then $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\} \geq \alpha$ and $\lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\} \leq \beta$. Hence $x \in R_A^{(\alpha,\beta)}$. Therefore $R_A^{(\alpha,\beta)}$ is a left h-ideal of R.
Conversely, let $R_A^{(\alpha,\beta)}$ be a left h-ideal of R , for any $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$.
Let $x, y \in R$ be such that $\mu_A(x) = \alpha_1, \mu_A(y) = \alpha_2$ and $\lambda_A(x) = \beta_1, \lambda_A(y) = \beta_2$. Then $\alpha_1 + \beta_1 \leq 1, \alpha_2 + \beta_2 \leq 1$.
So, $x, y \in R_A^{(\tilde{\alpha},\tilde{\beta})}$, a left h-ideal, where $\tilde{\alpha} = \alpha_1 \wedge \alpha_2$, $\tilde{\beta} = \beta_1 \vee \beta_2$ which implies $x+y \in R_A^{(\tilde{\alpha},\tilde{\beta})}$ i.e. $\mu_A(x+y) \geq \tilde{\alpha} = \alpha_1 \wedge \alpha_2 = \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(x+y) \leq \tilde{\beta} = \beta_1 \vee \beta_2 = \max\{\lambda_A(x), \lambda_A(y)\}$.
Again, let $x, y \in R$ and $\gamma \in \Gamma$ be arbitrary.
Let $\mu_A(y) = c_1$ and $\lambda_A(y) = c_2$ with $c_1 + c_2 \leq 1$
Then $y \in R_A^{(c_1,c_2)}$ which implies $x\gamma y \in R_A^{(c_1,c_2)}$ i.e. $\mu_A(x\gamma y) \geq c_1 = \mu_A(y)$ and $\lambda_A(x\gamma y) \leq c_2 = \lambda_A(y)$.
Therefore A is an intuitionistic fuzzy left ideal.
Again, suppose $A = (\mu_A, \lambda_A)$ is not an intuitionistic fuzzy left h-ideal.
So $\exists\ x_0, z_0, a_0, b_0 \in R$ such that $x_0 + a_0 + z_0 = b_0 + z_0$ and $\mu_A(x_0) < \min\{\mu_A(a_0), \mu_A(b_0)\}$
Taking $\alpha_0 = \frac{1}{2}\{\mu_A(x_0) + \min\{\mu_A(a_0), \mu_A(b_0)\}\}$, we have $\mu_A(x_0) < \alpha_0 < \min\{\mu_A(a_0), \mu_A(b_0)\}$.
Clearly $\mu_A(a_0), \mu_A(b_0) \geq \alpha_0$.
As $\mu_A(a_0) + \lambda_A(a_0) \leq 1$, then $\lambda_A(a_0) \leq 1 - \mu_A(a_0)$.
So $\lambda_A(a_0) \leq 1 - \alpha_0$ and similarly $\lambda_A(b_0) \leq 1 - \alpha_0$.
Consider $R_A^{(\alpha_0, 1-\alpha_0)}$, which is clearly nonempty, is a left h-ideal of R and $a_0, b_0 \in R_A^{(\alpha_0,1-\alpha_0)}$. Therefore $x_0 \in R_A^{(\alpha_0,1-\alpha_0)}$. So, $\mu_A(x_0) \geq \alpha_0$, which is a contradiction. ■

Theorem *3.4:* Let A be a non-empty subset of a Γ-hemiring R. Then an intuitionistic fuzzy subset (μ_A, λ_A) defined by

$$\mu_A(x) = \begin{cases} \alpha_2 & \text{for } x \in A \\ \alpha_1 & \text{for } x \notin A \end{cases}$$

$$\lambda_A(x) = \begin{cases} \beta_2 & \text{for } x \in A \\ \beta_1 & \text{for } x \notin A \end{cases}$$

where $0 \leq \alpha_1 < \alpha_2 \leq 1$, $0 \leq \beta_2 < \beta_1 \leq 1$ and $\alpha_i + \beta_i \leq 1$ for each i=1,2 is an intuitionistic fuzzy left h-ideal if and only if A is a left h-ideal of R.

Proof: Let A be a left h-ideal of R. Let $x, y \in R$. If $x, y \in$A, then $x+y, x\gamma$y$\in$A for all $\gamma \in \Gamma$. Then $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, $\mu_A(x\gamma y) \geq \mu_A(y)$ and $\lambda_A(x\gamma y) \leq \lambda_A(y)$. If either x or $y \notin A$, then also $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$, $\mu_A(x\gamma y) \geq \mu_A(y)$ and $\lambda_A(x\gamma y) \leq \lambda_A(y)$.
Let $x, z, a, b \in R$ be such that $x+a+z = b+z$,then if $a, b\ \in$A, we obtain $\mu_A(a) = \mu_A(b) = \alpha_2, \lambda_A(a) = \lambda_A(b) = \beta_2$ and hence $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$, $\lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$. If either a or $b \notin A$ then $\min\{\mu_A(a), \mu_A(b)\} \leq \alpha_1 \leq \mu_A(x)$, $\max\{\lambda_A(a), \lambda_A(b)\} \geq \beta_2 \geq \lambda_A(x)$. Hence (μ_A, λ_A) is an intuitionistic fuzzy left h-ideal of R.
Conversely, let (μ_A, λ_A) is an intuitionistic fuzzy left h-ideal of R. Then $R_{(\mu_A,\lambda_A)}^{(\alpha_1,\beta_1)} = A$. So, by previous theorem[cf.Th.3.2] A must be a left h-ideal of R. ■

Corollary *3.5:* Let A be a non-empty subset of a Γ-hemiring R. Then A is a left h-ideal of A if and only if the intuitionistic fuzzy subset (μ_A, λ_A) defined by

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

$$\lambda_A(x) = \begin{cases} 0 & \text{for } x \in A \\ 1 & \text{for } x \notin A \end{cases}$$

is an intuitionistic fuzzy left h-ideal of R.

Definition *3.6:* Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two intuitionistic fuzzy subsets of a Γ-hemiring R. Then $A \bigcap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} : x \in R\}$.

Proposition *3.7:* Intersection of a non-empty collection of intuitionistic fuzzy left h-ideals of a Γ-hemiring R is also an intuitionistic fuzzy left h-ideal of R.

Definition *3.8:* [4] Let R, S be Γ-hemirings and $f : R \to S$ be a function. Then f is said to be a Γ-homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(a\alpha b) = f(a)\alpha f(b)$ for $a, b \in$R and $\alpha \in \Gamma$.
(iii) $f(0_R) = 0_S$ where 0_R and 0_S are the zeroes of R and S respectively.

Proposition *3.9:* Let $f : R \to S$ be a homomorphism of Γ-hemirings. If $A = (\sigma_A, \eta_A)$ is an intuitionistic fuzzy left h-ideal of S, then $f^{-1}(A)$ defined as

$$f^{-1}(A) = (f^{-1}(\sigma_A), f^{-1}(\eta_A))$$

where

$$(f^{-1}(\sigma_A))(x) = \sigma_A(f(x)) \text{ and } (f^{-1}(\eta_A))(x) = \eta_A(f(x))$$

for all x in S, is an intuitionistic fuzzy left h-ideal of R,

Definition *3.10:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of X. Let $\alpha \in [0, \inf\{\lambda_A(x) : x \in X\}]$, $\beta \in [0, 1 - \sup\{\mu_A(x) : x \in X\}]$.
Then $A^T_{\beta,\alpha} = ((\mu_A)^T_\beta, (\lambda_A)^T_\alpha)$ is called an intuitionistic fuzzy translation of A if
$(\mu_A)^T_\beta(x) = \mu_A(x)+\beta, (\lambda_A)^T_\alpha(x) = \lambda_A(x)-\alpha$ for all $x \in X$.

Definition *3.11:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of X. Let $\gamma \in [0, 1]$.
Then $A^M_\gamma = ((\mu_A)^M_\gamma, (\lambda_A)^M_\gamma)$ is called an intuitionistic fuzzy multiplication of A if
$(\mu_A)^M_\gamma(x) = \gamma.\mu_A(x), (\lambda_A)^M_\gamma(x) = \gamma.\lambda_A(x)$ for all $x \in X$.

Definition *3.12:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of X. Let $\alpha \in [0, \inf\{\gamma.\lambda_A(x) : x \in X\}]$, $\beta \in [0, 1 - \sup\{\gamma.\mu_A(x) : x \in X\}]$ where $\gamma \in [0, 1]$.
Then $A^{MT}_{(\beta,\alpha),\gamma} = ((\mu_A)^{MT}_{\beta,\gamma}, (\lambda_A)^{MT}_{\alpha,\gamma})$ is called an intuitionistic fuzzy magnified translation of A if $(\mu_A)^{MT}_{\beta,\gamma}(x) = \gamma.\mu_A(x) + \beta, (\lambda_A)^{MT}_{\alpha,\gamma}(x) = \gamma.\lambda_A(x) - \alpha$ for all $x \in X$.

Theorem *3.13:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of a Γ-hemiring R. A is an intuitionistic fuzzy left h-ideal of R if and only if $A^{MT}_{(\beta,\alpha),\gamma}$ is an intuitionistic fuzzy left h-ideal of R,where $\gamma \in [0, 1], \alpha \in [0, \inf\{\gamma.\lambda_A(x) : x \in X\}]$, $\beta \in [0, 1 - \sup\{\gamma.\mu_A(x) : x \in X\}]$.

Proof: Suppose A is an intuitionistic fuzzy left h-ideal of R. Then for any $x, y \in R$ and $\eta \in \Gamma$,

$$\begin{aligned} (\mu_A)^{MT}_{\beta,\gamma}(x + y) &= \gamma.\mu_A(x + y) + \beta \\ &\geq \gamma.\min\{\mu_A(x), \mu_A(y)\} + \beta \\ &= \min\{(\gamma.\mu_A(x) + \beta), (\gamma.\mu_A(y) + \beta)\} \\ &= \min\{(\mu_A)^{MT}_{\beta,\gamma}(x), (\mu_A)^{MT}_{\beta,\gamma}(y)\}, \end{aligned}$$

and

$$\begin{aligned} (\mu_A)^{MT}_{\beta,\gamma}(x\eta y) &= \gamma.\mu_A(x\eta y) + \beta \\ &\geq \gamma.\mu_A(y) + \beta \\ &= (\mu_A)^{MT}_{\beta,\gamma}(y) \end{aligned}$$

Similarly,

$$\begin{aligned} (\lambda_A)^{MT}_{\alpha,\gamma}(x + y) &= \gamma.\lambda_A(x + y) - \alpha \\ &\leq \gamma.\max\{\lambda_A(x), \lambda_A(y)\} - \alpha \\ &= \max\{(\gamma.\lambda_A(x) - \alpha), (\gamma.\lambda_A(y) - \alpha)\} \\ &= \max\{(\lambda_A)^{MT}_{\alpha,\gamma}(x), (\lambda_A)^{MT}_{\alpha,\gamma}(y)\} \end{aligned}$$

and

$$\begin{aligned} (\lambda_A)^{MT}_{\alpha,\gamma}(x\eta y) &= \gamma.\lambda_A(x\eta y) - \alpha \\ &\leq \gamma.\lambda_A(y) - \alpha \\ &= (\lambda_A)^{MT}_{\alpha,\gamma}(y) \end{aligned}$$

Let $x, a, b, z \in$R be such that $x + a + z = b + z$. Then

$$\begin{aligned} (\mu_A)^{MT}_{\beta,\gamma}(x) &= \gamma.\mu_A(x) + \beta \\ &\geq \gamma.\min\{\mu_A(a), \mu_A(b)\} + \beta \\ &= \min\{(\gamma.\mu_A(a) + \beta), (\gamma.\mu_A(b) + \beta)\} \\ &= \min\{(\mu_A)^{MT}_{\beta,\gamma}(a), (\mu_A)^{MT}_{\beta,\gamma}(b)\} \end{aligned}$$

and

$$\begin{aligned} (\lambda_A)^{MT}_{\alpha,\gamma}(x) &= \gamma.\lambda_A(x) - \alpha \\ &\leq \gamma.\max\{\lambda_A(a), \lambda_A(b)\} - \alpha \\ &= \max\{(\gamma.\lambda_A(a) - \alpha), (\gamma.\lambda_A(b) - \alpha)\} \\ &= \max\{(\lambda_A)^{MT}_{\alpha,\gamma}(a), (\lambda_A)^{MT}_{\alpha,\gamma}(b)\} \end{aligned}$$

Hence $A^{MT}_{(\beta,\alpha),\gamma}$ is an intuitionistic fuzzy left h-ideal of R.
Conversely, let $A^{MT}_{(\beta,\alpha),\gamma}$ be an intuitionistic fuzzy left h-ideal of R.
Then, for any $x, y \in$R and any $\eta \in \Gamma$,
$(\mu_A)^{MT}_{\beta,\gamma}(x + y) \geq \min\{(\mu_A)^{MT}_{\beta,\gamma}(x), (\mu_A)^{MT}_{\beta,\gamma}(y)\}$
$\Rightarrow \gamma.\mu_A(x + y) + \beta \geq \min\{(\gamma.\mu_A(x) + \beta), (\gamma.\mu_A(y) + \beta)\}$
$= \gamma.\min\{\mu_A(x), \mu_A(y)\} + \beta$
$\Rightarrow \mu_A(x + y) \geq \min\{\mu_A(x), \mu_A(y)\}$

$(\mu_A)^{MT}_{\beta,\gamma}(x\eta y) \geq (\mu_A)^{MT}_{\beta,\gamma}(y)$
$\Rightarrow \gamma.\mu_A(x\eta y) + \beta \geq \gamma.\mu_A(y) + \beta$
$\Rightarrow \mu_A(x\eta y) \geq \mu_A(y)$

$(\lambda_A)^{MT}_{\alpha,\gamma}(x + y) \leq \max\{(\lambda_A)^{MT}_{\alpha,\gamma}(x), (\lambda_A)^{MT}_{\alpha,\gamma}(y)\}$
$\Rightarrow \gamma.\lambda_A(x + y) - \alpha \leq \max\{(\gamma.\lambda_A(x) - \alpha), (\gamma.\lambda_A(y) - \alpha)\}$
$= \gamma.\max\{\lambda_A(x), \lambda_A(y)\} - \alpha$
$\Rightarrow \lambda_A(x + y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$

$(\lambda_A)^{MT}_{\alpha,\gamma}(x\eta y) \leq (\lambda_A)^{MT}_{\alpha,\gamma}(y)$
$\Rightarrow \gamma.\lambda_A(x\eta y) - \alpha \leq \gamma.\lambda_A(y) - \alpha$
$\Rightarrow \lambda_A(x\eta y) \leq \lambda_A(y)$

Let $x, a, b, z \in$R such that $x + a + z = b + z$.
Then, $(\mu_A)^{MT}_{\beta,\gamma}(x) \geq \min\{(\mu_A)^{MT}_{\beta,\gamma}(a), (\mu_A)^{MT}_{\beta,\gamma}(b)\}$
$\Rightarrow \gamma.\mu_A(x) + \beta \geq \min\{(\gamma.\mu_A(a) + \beta), (\gamma.\mu_A(b) + \beta)\}$
$= \gamma.\min\{\mu_A(a), \mu_A(b)\} + \beta$
$\Rightarrow \mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$

$(\lambda_A)^{MT}_{\alpha,\gamma}(x) \leq \max\{(\mu_A)^{MT}_{\alpha,\gamma}(a), (\lambda_A)^{MT}_{\alpha,\gamma}(b)\}$
$\Rightarrow \gamma.\lambda_A(x) - \alpha \leq \max\{(\gamma.\lambda_A(a) - \alpha), (\gamma.\lambda_A(b) - \alpha)\}$

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$$= \gamma.\max\{\lambda_A(a), \lambda_A(b)\} - \alpha$$
$$\Rightarrow \lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$$

Then A is an intuitionistic fuzzy left h-ideal of R. ■

***Corollary** 3.14:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of a Γ-hemiring R and $\gamma \in [0,1], \beta \in [0, 1 - \sup\{\gamma.\mu_A(x) : x \in X\}], \alpha \in [0, \inf\{\gamma.\lambda_A(x) : x \in X\}]$. Then the following statements are equivalent:

(i) A is an intuitionistic fuzzy left h-ideal of R.
(ii) $A^T_{\beta,\alpha}$,the intuitionistic fuzzy translation of A, is an intuitionistic fuzzy left h-ideal of R.
(iii) A^M_{γ},the intuitionistic fuzzy multiplication of A, is an intuitionistic fuzzy left h-ideal of R.

IV. Normal intuitionistic fuzzy left h-ideals

***Definition** 4.1:* An intuitionistic fuzzy left h-ideal A=(μ_A, λ_A) of a Γ hemiring R is said to be normal if A(0)=(1,0); i.e., $\mu_A(0) = 1, \lambda_A(0) = 0$.

***Example** 4.2:* The intuitionistic fuzzy left h-ideal $A = (\mu_A, \lambda_A)$ of Γ-hemiring (R,+,.), defined in the **Example 1.**, is a normal intuitionistic fuzzy left h-ideal of R.

***Theorem** 4.3:* Given an intuitionistic fuzzy left h-ideal A=(μ_A, λ_A) of a Γ-hemring R. Let $\mu^+_A(x) = \mu_A(x) + 1 - \mu_A(0)$ and $\lambda^+_A(x) = \lambda_A(x) - \lambda_A(0)$, for all $x \in R$. Then $A^+ = (\mu^+_A, \lambda^+_A)$ is a normal intuitionistic fuzzy left h-ideal,containing A=(μ_A, λ_A), of R.

Proof: For any $x, y \in R$ and $\gamma \in \Gamma$,

$$\begin{aligned} \mu^+_A(x+y) &= \mu_A(x+y) + 1 - \mu_A(0) \\ &\geq \min\{\mu_A(x), \mu_A(y)\} + 1 - \mu_A(0) \\ &= \min\{\mu_A(x) + 1 - \mu_A(0), \mu_A(y) + 1 - \mu_A(0)\} \\ &= \min\{\mu^+_A(x), \mu^+_A(y)\}, \end{aligned}$$

$$\begin{aligned} \mu^+_A(x\gamma y) = \mu_A(x\gamma y) + 1 - \mu_A(0) &\geq \mu_A(y) + 1 - \mu_A(0) \\ &= \mu^+_A(y), \end{aligned}$$

and

$$\begin{aligned} \lambda^+_A(x+y) &= \lambda_A(x+y) - \lambda_A(0) \\ &\leq \max\{\lambda_A(x), \lambda_A(y)\} - \lambda_A(0) \\ &= \max\{\lambda_A(x) - \lambda_A(0), \lambda_A(y) - \lambda_A(0)\} \\ &= \max\{\lambda^+_A(x), \lambda^+_A(y)\}, \end{aligned}$$

$$\begin{aligned} \lambda^+_A(x\gamma y) &= \lambda_A(x\gamma y) - \lambda_A(0) \\ &\leq \lambda_A(y) - \lambda_A(0) \\ &= \lambda^+_A(y). \end{aligned}$$

So A^+ is an intuitionistic fuzzy left ideal of R.
Let $x, a, b, z \in R$ such that $x + a + z = b + z$.
Then

$$\begin{aligned} \mu^+_A(x) &= \mu_A(x) + 1 - \mu_A(0) \\ &\geq \min\{\mu_A(a), \mu_A(b)\} + 1 - \mu_A(0) \\ &= \min\{\mu_A(a) + 1 - \mu_A(0), \mu_A(b) + 1 - \mu_A(0)\} \\ &= \min\{\mu^+_A(a), \mu^+_A(b)\} \end{aligned}$$

and

$$\begin{aligned} \lambda^+_A(x) &= \lambda_A(x) - \lambda_A(0) \\ &\leq \max\{\lambda_A(a), \lambda_A(b)\} - \lambda_A(0) \\ &= \max\{\lambda_A(a) - \lambda_A(0), \lambda_A(b) - \lambda_A(0)\} \\ &= \max\{\lambda^+_A(a), \lambda^+_A(b)\}. \end{aligned}$$

Hence A^+ is an intuitionistic fuzzy left h-ideal of R. Again we have, $\mu^+_A(0) = \mu_A(0) + 1 - \mu_A(0) = 1$ and $\lambda^+_A(0) = \lambda_A(0) - \lambda_A(0) = 0$. Hence A^+ is a normal intuitionistic fuzzy left h-ideal of R and by definition $A \subseteq A^+$. ■

***Corollary** 4.4:* Let A and A^+ be as in the very previous Proposition. Then

(i) for any $x \in R$, $A^+(x) = (0,1) \Rightarrow A(x) = (0,1)$, and
(ii) A is a normal intuitionistic fuzzy left h-ideal of R if and only if $A^+ = A$

***Remark** 4.5:* If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal of R, then $(A^+)^+$=A^+. In particular , if A is normal, then $(A^+)^+ = A^+ = A$.

***Theorem** 4.6:* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy left h-ideal of a Γ-hemiring R and let $f : [0,1] \to [0,1]$ be an increasing function. Then an intuitionistic fuzzy subset $A_f = ((\mu_A)_f, (\lambda_A)_f)$ where $(\mu_A)_f(x) = f(\mu_A(x))$ and $(\lambda_A)_f(x) = f(\lambda_A(x))$ for all x in R is an intuitionistic fuzzy left h-ideal of R. Moreover, if $f(\mu_A(0)) = 1$ and $f(\lambda_A(0)) = 0$, then A_f is normal.

Proof:
Let $x, y \in R$ and $\gamma \in R$.

$$\begin{aligned} (\mu_A)_f(x+y) &= f(\mu_A(x+y)) \\ &\geq f(\min\{\mu_A(x), \mu_A(y)\}) \\ &= \min\{f(\mu_A(x)), f(\mu_A(y))\} \\ &= \min\{(\mu_A)_f(x), (\mu_A)_f(y)\}, \end{aligned}$$

$$(\mu_A)_f(x\gamma y) = f(\mu_A(x\gamma y)) \geq f(\mu_A(y)) = (\mu_A)_f(y)$$

and

$$\begin{aligned} (\lambda_A)_f(x+y) &= f(\lambda_A(x+y)) \\ &\leq f(\max\{\lambda_A(x), \lambda_A(y)\}) \\ &= \max\{f(\lambda_A(x)), f(\lambda_A(y))\} \\ &= \max\{(\lambda_A)_f(x), (\lambda_A)_f(y)\}, \end{aligned}$$

$$(\lambda_A)_f(x\gamma y) = f(\lambda_A(x\gamma y)) \leq f(\lambda_A(y)) = (\lambda_A)_f(y).$$

Hence A_f is an intuitionistic fuzzy left ideal.
Let $x, a, b, z \in R$ such that $x + a + z = b + z$.
Then

$$\begin{aligned} (\mu_A)_f(x) = f(\mu_A)(x) &\geq f(\min\{\mu_A(a), \mu_A(b)\}) \\ &= \min\{f(\mu_A(a)), f(\mu_A(b))\} \\ &= \min\{(\mu_A)_f(a), (\mu_A)_f(b)\} \end{aligned}$$

and

$$\begin{aligned} (\lambda_A)_f(x) = f(\lambda_A)(x) &\leq f(\max\{\lambda_A(a), \lambda_A(b)\}) \\ &= \max\{f(\lambda_A(a)), f(\lambda_A(b))\} \\ &= \max\{(\lambda_A)_f(a), (\lambda_A)_f(b)\}. \end{aligned}$$

Hence A_f is an intuitionistic fuzzy left h-ideal of Γ-hemiring R.
If $f(\mu_A(0)) = 1$, $f(\lambda_A(0)) = 0$ then, $(\mu_A)_f(0) = 1$ and $(\lambda_A)_f(0) = 0$ and hence $A_f = ((\mu_A)_f, (\lambda_A)_f)$ is a normal intuitionistic fuzzy left h-ideal of Γ-hemiring R. ■

***Theorem** 4.7:* Let NI(R) denotes the collection of all normal intuitionistic fuzzy left h-ideals of R. Let $A = (\mu_A, \lambda_A) \in NI(R)$ be nonconstant such that it is a maximal element of $(NI(R), \subseteq)$. Then it takes only two values (1,0),(0,1).

Proof: Since A is normal intuitionistic fuzzy left h-ideal, so A(0)=(1,0). Let $x_0(\neq 0) \in R$ be arbitrary with $\mu_A(x_0) \neq 1$. We claim that $\mu_A(x_0) = 0$. If not then there exists an element $c \in R$ such that $0 < \mu_A(c) < 1$. Let $A_c = (\sigma_A, \eta_A)$ be an intuitionistic fuzzy subset of R defined by : $\sigma_A(x) = \frac{1}{2}[\mu_A(x) + \mu_A(c)]$, $\eta_A(x) = \frac{1}{2}[\lambda_A(x) + \lambda_A(c)]$. Clearly A_c is well-defined.
Now, $\sigma_A(0) = \frac{1}{2}[\mu_A(0) + \mu_A(c)] \geq \frac{1}{2}[\mu_A(x) + \mu_A(c)] = \sigma_A(x)$,
$\eta_A(0) = \frac{1}{2}[\lambda_A(0) + \lambda_A(c)] \leq \frac{1}{2}[\lambda_A(x), \lambda_A(c)] = \eta_A(x)$ for any x in R.
Again, for any $x, y \in R$ and for all $\gamma \in R$,

$$\begin{aligned}\sigma_A(x+y) &= \frac{1}{2}[\mu_A(x+y), \mu_A(c)]\\ &\geq \frac{1}{2}[\min\{\mu_A(x), \mu_A(y)\} + \mu_A(c)]\\ &= \min\{\frac{1}{2}(\mu_A(x) + \mu_A(c)), \frac{1}{2}(\mu_A(y) + \mu_A(c))\}\\ &= \min\{\sigma_A(x), \sigma_A(y)\},\end{aligned}$$

$$\begin{aligned}\sigma_A(x\gamma y) &= \frac{1}{2}[\mu_A(x\gamma y) + \mu_A(c)]\\ &\geq \frac{1}{2}[\mu_A(y) + \mu_A(c)] = \sigma_A(y),\end{aligned}$$

$$\begin{aligned}\eta_A(x+y) &= \frac{1}{2}[\lambda(x+y), \lambda_A(c)]\\ &\leq \frac{1}{2}[\max\{\lambda_A(x), \lambda_A(y)\} + \lambda_A(c)]\\ &= \max\{\frac{1}{2}(\lambda_A(x) + \lambda_A(c)), \frac{1}{2}(\lambda_A(y) + \lambda_A(c))\}\\ &= \max\{\eta_A(x), \eta_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\eta_A(x\gamma y) &= \frac{1}{2}[\lambda_A(x\gamma y) + \lambda_A(c)]\\ &\leq \frac{1}{2}[\lambda_A(y) + \lambda_A(c)]\\ &= \eta_A(y).\end{aligned}$$

Hence A_c is an intuitionistic fuzzy left ideal of R.
Let $x, a, b, z \in R$ such that $x + a + z = b + z$.
Then

$$\begin{aligned}\sigma_A(x) &= \frac{1}{2}[\mu_A(x) + \mu_A(c)]\\ &\geq \frac{1}{2}[\min\{\mu_A(a), \mu_A(b)\} + \mu_A(c)]\\ &= \min\{\frac{1}{2}(\mu_A(a) + \mu_A(c)), \frac{1}{2}(\mu_A(b) + \mu_A(c))\}\\ &= \min\{\sigma_A(a), \sigma_A(b)\}\end{aligned}$$

and

$$\begin{aligned}\eta_A(x) &= \frac{1}{2}[\lambda_A(x) + \lambda_A(c)]\\ &\leq \frac{1}{2}[\max\{\lambda_A(a), \lambda_A(b)\} + \lambda_A(c)]\\ &= \max\{\frac{1}{2}(\lambda_A(a) + \lambda_A(c)), \frac{1}{2}(\lambda_A(b) + \lambda_A(c))\}\\ &= \max\{\eta_A(a), \eta_A(b)\}.\end{aligned}$$

So A_c is an intuitionistic fuzzy left h-ideal of R.
Define $A_c^+ = (\sigma_A^+, \eta_A^+)$. Then by Proposition 4.2 A_c^+ is a normal intuitionistic fuzzy h-ideal of R, where

$$\begin{aligned}\sigma_A^+(x) &= \sigma_A(x) + 1 - \sigma_A(0)\\ &= \frac{1}{2}[\mu_A(x) + \mu_A(c)] + 1 - \frac{1}{2}[\mu_A(0) + \mu_A(c)]\\ &= \frac{1}{2}[1 + \mu_A(x)]\end{aligned}$$

and

$$\begin{aligned}\eta_A^+(x) &= \eta_A(x) - \eta_A(0)\\ &= \frac{1}{2}[\lambda_A(x) + \lambda_A(c)] - \frac{1}{2}[\lambda_A(0) + \lambda_A(c)]\\ &= \frac{1}{2}\lambda_A(x).\end{aligned}$$

Hence A$\subseteq A_c^+$.
Again since $\sigma_A^+(c) = \frac{1}{2}[1 + \mu_A(c)] < \sigma_A^+(0) = 1$, A_c^+ is non-constant.
Also A is a proper subset of A_c^+ as $\sigma_A^+(x) = \frac{1}{2}[1 + \mu_A(x)] > \mu_A(x)$ and $\eta_A^+(x) = \frac{1}{2}\lambda_A(x) \leq \lambda_A(x)$ for any x with $\mu_A(x) \neq 1$ in R. This violates the maximality of A in NI(R) and so we get a contradiction. This completes the proof. ■

V. Cartesian product of intuitionistic fuzzy left h-ideals

***Definition** 5.1:* Let $\{R_i\}_{i\in I}$ be a family of Γ-hemirings. Now if we define addition(+) and multiplication(.) on the cartesian product $\prod_{i\in I} R_i$ as follows:
$(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$ and
$(x_i)_{i\in I}\alpha(y_i)_{i\in I} = (x_i\alpha y_i)_{i\in I}$ for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} R_i$ and for all $\alpha \in \Gamma$.
Then $\prod_{i\in I} R_i$ becomes a Γ-hemiring.

***Definition** 5.2:* An intuitionistic fuzzy relation on any nonempty set X is an intuitionistic fuzzy set (μ_A, λ_A), where $\mu_A : X \times X \to [0, 1]$ and $\lambda_A : X \times X \to [0, 1]$.

***Definition** 5.3:* Let A=(μ_A, λ_A) be an intuitionistic fuzzy relation on a set X and $B = (\sigma_B, \eta_B)$ be an intuitionistic fuzzy subset of X. Then A is said to be an intuitionistic fuzzy relation on B if $\mu_A(x,y) \leq \min\{\sigma_B(x), \sigma_B(y)\}$ and $\lambda_A(x,y) \geq \max\{\eta_B(x), \eta_B(y)\}$ for all x, y in X.

***Definition** 5.4:* Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be two intuitionistic fuzzy subsets of R. Then cartesian product of A and B is defined as:

$A \times B = \{(x,y), \mu_A \times \mu_B, \lambda_A \times_1 \lambda_B : x, y \in R\}$,
where $(\mu_A \times \mu_B)(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ and $(\lambda_A \times_1 \lambda_B)(x,y) = \max\{\lambda_A(x), \lambda_B(y)\}$.

***Theorem** 5.5:* If A,B are intuitionistic Fuzzy left h-ideals of R then so is $A \times B$ in $R \times R$.

Proof: Let $(x_1,x_2),(y_1,y_2) \in R \times R$ and $\gamma \in \Gamma$.
Then $(A \times B)((x_1,x_2)+(y_1,y_2))$
$= (A \times B)((x_1+y_1),(x_2+y_2))$
$= (\mu_A \times \mu_B, \lambda_A \times_1 \lambda_B)((x_1+y_1),(x_2+y_2))$. Now
$(\mu_A \times \mu_B)((x_1+y_1),(x_2+y_2))$
$= \min\{\mu_A(x_1+y_1), \mu_B(x_2+y_2)\}$
$\geq \min\{\min\{\mu_A(x_1), \mu_A(y_1)\}, \min\{\mu_B(x_2), \mu_B(y_2)\}\}$
$= \min\{\min\{\mu_A(x_1), \mu_B(x_2)\}, \min\{\mu_A(y_1), \mu_B(y_2)\}\}$
$= \min\{(\mu_A \times \mu_B)(x_1,x_2), (\mu_A \times \mu_B)(y_1,y_2)\}$
$(\lambda_A \times_1 \lambda_B)((x_1+y_1),(x_2+y_2))$
$= \max\{\lambda_A(x_1+y_1), \lambda_B(x_2+y_2)\}$
$\leq \max\{\max\{\lambda_A(x_1), \lambda_A(y_1)\}, \max\{\lambda_B(x_2), \lambda_B(y_2)\}\}$
$= \max\{\max\{\lambda_A(x_1), \lambda_B(x_2)\}, \max\{\lambda_A(y_1), \lambda_B(y_2)\}\}$
$= \max\{(\lambda_A \times_1 \lambda_B)(x_1,x_2), (\lambda_A \times_1 \lambda_B)(y_1,y_2)\}$
$(\mu_A \times \mu_B)((x_1,x_2)\gamma(y_1,y_2))$
$= (\mu_A \times \mu_B)(x_1\gamma y_1, (x_2\gamma y_2))$
$= \min\{\mu_A(x_1\gamma y_1), \mu_B(x_2\gamma y_2)\}$
$\geq \min\{\mu_A(y_1), \mu_B(y_2)\}$
$= (\mu_A \times \mu_B)(y_1,y_2)$
and
$(\lambda_A \times_1 \lambda_B)((x_1,x_2)\gamma(y_1,y_2))$
$= (\lambda_A \times_1 \lambda_B)(x_1\gamma y_1, x_2\gamma y_2)$
$= \max\{\lambda_A(x_1\gamma y_1), \lambda_B(x_2\gamma y_2)\}$
$\leq \max\{\lambda_A(y_1), \lambda_B(y_2)\}$
$= (\lambda_A \times_1 \lambda_B)(y_1,y_2)$.
Hence $A \times B$ is an intuitionistic fuzzy left ideal.
Let $(x_1,x_2),(a_1,a_2),(b_1,b_2),(z_1,z_2) \in R \times R$ be such that $(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2)$. Then $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$.
Now
$(A \times B)(x_1,x_2) = ((\mu_A \times \mu_B)(x_1,x_2), (\lambda_A \times_1 \lambda_B)(x_1,x_2))$
So, $(\mu_A \times \mu_B)(x_1,x_2)$
$= \min\{\mu_A(x_1), \mu_B(x_2)\}$
$\geq \min\{\min\{\mu_A(a_1), \mu_A(b_1)\}, \min\{\mu_B(a_2), \mu_B(b_2)\}\}$
$= \min\{\min\{\mu_A(a_1), \mu_B(a_2)\}, \min\{\mu_A(b_1), \mu_B(b_2)\}\}$
$= \min\{(\mu_A \times \mu_B)(a_1,a_2), (\mu_A \times \mu_B)(b_1,b_2)\}$.
and
$(\lambda_A \times_1 \lambda_B)(x_1,x_2)$
$= \max\{\lambda_A(x_1), \lambda_B(x_2)\}$
$\leq \max\{\max\{\lambda_A(a_1), \lambda_A(b_1)\}, \max\{\lambda_B(a_2), \lambda_B(b_2)\}\}$
$= \max\{\max\{\lambda_A(a_1), \lambda_B(a_2)\}, \max\{\lambda_A(b_1), \lambda_B(b_2)\}\}$
$= \max\{(\lambda_A \times_1 \lambda_B)(a_1,a_2), (\lambda_A \times_1 \lambda_B)(b_1,b_2)\}$.
Hence $A \times B$ is an intuitionistic fuzzy left h-ideal of $R \times R$. ■

***Definition** 5.6:* Let X be a non-empty set and A=(μ_A, λ_A) be an intuitionistic fuzzy subset of X. Then the strongest intuitionistic fuzzy relation on X determined by A is defined as the cartesian product of A with itself and denoted by S_A.

***Theorem** 5.7:* Let A=(μ_A, λ_A) be an intuitionistic fuzzy subset of a Γ-hemiring R and S_A be the strongest intuitionistic fuzzy relation on R determined by A. Then A is an intuitionistic fuzzy left h-ideal of R if and only if S_A is an intuitionistic fuzzy left h-ideal of $R \times R$.

Proof: Let A=(μ_A, λ_A) be an intuitionistic fuzzy left h-ideal of the Γ-hemiring R. Then from the previous theorem $S_A = A \times A$ is an intuitionistic fuzzy left h-ideal of $R \times R$. Conversely, suppose S_A be an intuitionistic fuzzy left h-ideal of $R \times R$.
Let $x_1, x_2, y_1, y_2 \in R$ and $\gamma \in \Gamma$. Then
$\min\{\mu_A(x_1+y_1), \mu_A(x_2+y_2)\}$
$= (\mu_A \times \mu_A)(x_1+y_1, x_2+y_2)$
$= (\mu_A \times \mu_A)((x_1,x_2)+(y_1,y_2))$
$\geq \min\{(\mu_A \times \mu_A)(x_1,x_2), (\mu_A \times \mu_A)(y_1,y_2)\}$
$= \min\{\min\{\mu_A(x_1), \mu_A(x_2)\}, \min\{\mu_A(y_1), \mu_A(y_2)\}\}$

Let x, y be arbitrarily chosen from R. Now putting $x_1 = x, x_2 = 0, y_1 = y, y_2 = 0$ and noting that $\mu_A(0) \geq \mu_A(r)$ for all $r \in R$ we get $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}$
Again,
$\max\{\lambda_A(x_1+y_1), \lambda_A(x_2+y_2)\}$
$= (\lambda_A \times_1 \lambda_A)(x_1+y_1, x_2+y_2)$
$= (\lambda_A \times_1 \lambda_A)((x_1,x_2)+(y_1,y_2))$
$\leq \max\{(\lambda_A \times_1 \lambda_A)(x_1,x_2), (\lambda_A \times_1 \lambda_A)(y_1,y_2)\}$
$= \max\{\max\{\lambda_A(x_1), \lambda_A(x_2)\}, \max\{\lambda_A(y_1), \lambda_A(y_2)\}\}$

Let x, y be arbitrarily chosen from R. Putting $x_1 = x, x_2 = 0, y_1 = y, y_2 = 0$ and noting that $\lambda_A(0) \leq \lambda_A(r)$ for all $r \in R$ we get $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$.
Now,
$\min\{\mu_A(x_1\gamma y_1), \mu_A(x_2\gamma y_2)\}$
$= (\mu_A \times \mu_A)((x_1,x_2)\gamma(y_1,y_2))$
$\geq (\mu_A \times \mu_A)(y_1,y_2)$
$= \min\{\mu_A(y_1), \mu_A(y_2)\}$

Let x, y be arbitrarily chosen from R. Putting $x_1 = x, x_2 = 0, y_1 = y, y_2 = 0$ and noting that $\mu_A(0) \geq \mu_A(r)$ for all $r \in R$ we get $\mu_A(x\gamma y) \geq \mu_A(y)$.
Again
$\max\{\lambda_A(x_1\gamma y_1), \lambda_A(x_2\gamma y_2)\}$
$= (\lambda_A \times_1 \lambda_A)((x_1,x_2)\gamma(y_1,y_2))$
$\leq (\lambda_A \times_1 \lambda_A)(y_1,y_2)$
$= \max\{\lambda_A(y_1), \lambda_A(y_2)\}$

Let x, y be arbitrarily chosen from R. Putting $x_1 = x, x_2 = 0, y_1 = y, y_2 = 0$ and noting that $\lambda_A(0) \leq \lambda_A(r)$ for all $r \in R$ we get $\lambda_A(x\gamma y) \leq \lambda_A(y)$.
Let $x_1, a_1, b_1, z_1, x_2, a_2, b_2, z_2 \in R$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$, and $x_2 + a_2 + z_2 = b_2 + z_2$, then $(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2)$ and hence
$\min\{\mu_A(x_1), \mu_A(x_2)\}$
$= (\mu_A \times \mu_A)(x_1,x_2)$
$\geq \min\{(\mu_A \times \mu_A)(a_1,a_2), (\mu_A \times \mu_A)(b_1,b_2)\}$
$= \min\{\min\{\mu_A(a_1), \mu_A(a_2)\}, \min\{\mu_A(b_1), \mu_A(b_2)\}\}$

Let x be arbitrarily chosen from R. Putting $x_1 = x, x_2 = 0, a_1 = a, a_2 = 0, b_1 = b, b_2 = 0, z_1 = z, z_2 = 0$ and noting that $\mu_A(0) \geq \mu_A(r)$ for all $r \in R$, we obtain

for all those $x, a, b, z \in R$ with $x + a + z = b + z$, $\mu_A(x) \geq \min\{\mu_A(a), \mu_A(b)\}$
Also,
$\max\{\lambda_A(x_1), \lambda_A(x_2)\}$
$= (\lambda_A \times_1 \lambda_A)(x_1, x_2)$
$\leq \max\{(\lambda_A \times_1 \lambda_A)(a_1, a_2), (\lambda_A \times \lambda_A)(b_1, b_2)\}$
$= \max\{\max\{\lambda_A(a_1), \lambda_A(a_2)\}, \max\{\lambda_A(b_1), \lambda_A(b_2)\}\}$

Let x be arbitrarily chosen from R. Putting $x_1 = x, x_2 = 0, a_1 = a, a_2 = 0, b_1 = b, b_2 = 0, z_1 = z, z_2 = 0$ and noting that $\lambda_A(0) \leq \lambda_A(r)$ for all $r \in R$, we obtain for all those $x, a, b, z \in R$ with $x + a + z = b + z$, $\lambda_A(x) \leq \max\{\lambda_A(a), \lambda_A(b)\}$.
Hence $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left h-ideal of R. ■

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