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# A Special Algorithm to Approximate the Square Root of Positive Integer

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*Abstract*—The paper concerns a special approximate algorithm of the square root of the specific positive integer, which is built by the use of the property of positive integer solution of the Pell's equation, together with using some elementary theorems of matrices, and then takes it to compare with general used the Newton's method and give a practical numerical example and error analysis; it is unexpected to find its special property: the significant figure of the approximation value of the square root of positive integer will increase one digit by one. It is well useful in some occasions.

*Keywords*—Special approximate algorithm, square root, Pell's equation, Newton's method; error analysis.

## I. Introduction

There are many algorithms for approximating the square root of the positive number published in many literature. For instance, F. Karakuş has introduced a history of Babylonian square root method and what is the method [1]. Babylonian method is helpful for lower grade mathematics lessons for calculating square root was used to provide some basis of the limits and infinity. In fact, this method is a variational algorithm based on the Newton's method. H. Davenport introduces that how to use methods of continuous fraction to approximate the square root of positive integer [2]. R. Garver also discusses similar problem and puts forward some announcements in the [3]. E.B. Escott has used skillfully some primary identities to structure an interesting and valid approximation algorithm of the square root which is convergent rapidly [4]. He states that his method for extracting square root is probably the most rapid method yet discovered. As it gives the square root in the form of an infinite product, it is especially well adapted to using with a computing machine. In computing a table of the square roots by the method of differences it is important to have an independent method of computing an occasional value and this method is very good for that purpose. However, an accurate approximate method established by using binomial series of the specific function skillfully, is accounted in the [5]. Then there are some iterative methods for approximating the square root and some cautions which are based on the Newton's method and also including its variations, we can find them in the papers [6] [9] [10] [19]. Many other distinctive iterative methods to approximate square root of positive integer have been developed well, reading relative papers in [11] [13] [14] [15] [16] [17] [18], here we should point out approximate method stated in the [17] is an inspiration for me. As a whole, these methods to approximate the square root of the specific positive integer are based on iteration idea which is used to approximation computing usually. We always hope the convergence speed of approximation algorithm to become fast. But we don't hope so all along. Sometimes, we need to find the approximation algorithm, which can make effective value of approximation increase one digit by one. This algorithm structured in this paper is useful for us to make a table of the square root of the positive integers.

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### A. Main results and proofs

The main idea of this paper stems from a problem of [20], also including its solutions. we review the problem without giving the solutions as follows:

**Proposition 1** If we consider these fractional sequences: $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \cdots, \frac{p_n}{q_n}, \cdots$, where $p_{n+1} = p_n + 2q_n,\ q_{n+1} = p_n + q_n$, then

(1) then $\frac{p_n}{q_n}$ are irreducible fractions for all $n$ $(n = 1, 2, \ldots)$;

(2) and $\left|\frac{p_n}{q_n} - \sqrt{2}\right|$ can become arbitrarily small, also we prove that signs of error by using it to approximate $\sqrt{2}$ will alternate between positive and negative.

It is not hard to find that all points, which are structured by above fractional sequences, $(1, 1), (3, 2), (7, 5), \cdots$ are integer solutions of the Pell's equation $x^2 - 2y^2 = \pm 1$. It inspires me to explore the relationship between the square root of positive integer and integer solutions of the Pell's equation.

It is well-known that a special Diophantus's equation $x^2 - dy^2 = \pm 1$, which is named by the Pell's equation, there $d$ is a non-square and positive integer number [12]. We denote these positive integer solutions of the Pell's equation $x^2 - dy^2 = 1$ by the $(x_1, y_1), (x_3, y_3), \cdots, (x_{2n-1}, y_{2n-1})$, then as well as denote these positive integer solutions of the Pell's equation $x^2 - dy^2 = -1$ by the $(x_0, y_0), (x_2, y_2), \cdots, (x_{2n}, y_{2n})$. At last, we arrange them as follow:

$$(x_0, y_0), (x_1, y_1), \cdots, (x_{2n-1}, y_{2n-1}), (x_{2n}, y_{2n}), \tag{1}$$

We assume $(x_0, y_0)$ is a known positive integer solution of the $x_0^2 - dy_0^2 = -1$, then

$$x_0^2 - dy_0^2 = (x_0 - \sqrt{d}y_0)(x_0 + \sqrt{d}y_0) = -1, \tag{2}$$

we let $(x, y)$ be a solution of the $x^2 - dy^2 = 1$,

$$(x - \sqrt{d}y)(x_0 - \sqrt{d}y_0)(x + \sqrt{d}y)(x_0 + \sqrt{d}y) = \pm 1, \tag{3}$$

then

$$\begin{aligned}&[((x_0x + dy_0y) - (\sqrt{d}y_0x + \sqrt{d}x_0y))\cdot \\ &((x_0x + dy_0y) + (\sqrt{d}y_0x + \sqrt{d}x_0y))] = \pm 1,\end{aligned} \tag{4}$$

or equivalently,

$$(x_0x + dy_0y)^2 - d(y_0x + x_0y)^2 = \pm 1. \tag{5}$$

It turns out that $(x_0x + dy_0y, y_0x + x_0y)$ is a positive integer solution of the $x^2 - dy^2 = \pm 1$.

It suggests that if the $(x_0, y_0)$ is a positive integer solution of the $x^2 - dy^2 = -1$, then regardless of the $(x, y)$ is a solution of the $x^2 - dy^2 = \pm 1$, we always obtain that the $(x_0x + dy_0y, y_0x + x_0y)$ is a solution of the $x^2 - dy^2 = \pm 1$. Let $(x_{n-1}, y_{n-1})$ be the solution of $x^2 - dy^2 = \pm 1$, then for every $(x_n, y_n)$, which is contented with following recursion relation:

$$\begin{cases} x_n = x_0x_{n-1} + dy_0y_{n-1}, \\ y_n = y_0x_{n-1} + x_0y_{n-1}, \end{cases} \tag{6}$$

where $n = 1, 2, \cdots$, are also solutions of the $x^2 - dy^2 = \pm 1$. we rewrite it by a type of matrix relationship:

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} x_0 & dy_0 \\ y_0 & x_0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \cdots \\ &= \begin{pmatrix} x_0 & dy_0 \\ y_0 & x_0 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \end{aligned} \tag{7}$$

if we use $A = \begin{pmatrix} x_0 & dy_0 \\ y_0 & x_0 \end{pmatrix}$, then the question that remains is how do we compute $A^n$.

According to the [8], we considering the characteristic polynomial of matrix $A$, then let

$$f(\lambda) = \begin{vmatrix} \lambda - x_0 & -dy_0 \\ -y_0 & \lambda - x_0 \end{vmatrix} = (\lambda - x_0)^2 - dy_0^2 = 0, \tag{8}$$

and finde out two different roots of the characteristic polynomial as follows:

$$\lambda_1 = x_0 + \sqrt{d}y_0 \ , \lambda_2 = x_0 - \sqrt{d}y_0. \tag{9}$$

They are also two eigenvalues of matrix $A$, then to find eigenvectors associated with two eigenvalues: one eigenvector associate with the eigenvalue $\lambda_1 = x_0 + \sqrt{d}y_0$ is $X_1 = \begin{pmatrix} \sqrt{d} \\ 1 \end{pmatrix}$, similarly, the other eigenvector associated with the eigenvalue $\lambda_2 = x_0 - \sqrt{d}y_0$ is $X_2 = \begin{pmatrix} -\sqrt{d} \\ 1 \end{pmatrix}$, if we denote $T = \begin{pmatrix} \sqrt{d} & -\sqrt{d} \\ 1 & 1 \end{pmatrix}$, then $|T| = \begin{vmatrix} \sqrt{d} & -\sqrt{d} \\ 1 & 1 \end{vmatrix} = 2\sqrt{d} \neq 0$, so the $T$ is nonsingular, $T^{-1} = \frac{1}{2\sqrt{d}} \begin{pmatrix} 1 & \sqrt{d} \\ -1 & \sqrt{d} \end{pmatrix}$ is the inverse matrix of $T$. Then we obtain

$$\begin{aligned} B = T^{-1}AT &= \begin{pmatrix} \sqrt{d} & -\sqrt{d} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_0 & dy_0 \\ y_0 & x_0 \end{pmatrix} \cdot \\ &\begin{pmatrix} \sqrt{d} & -\sqrt{d} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}, \end{aligned}$$

and combine with $A^n = TB^nT^{-1}$ and

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = TB^nT^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \frac{1}{2\sqrt{d}} \begin{pmatrix} \sqrt{d}[(x_0 + \sqrt{d}y_0)\lambda_1^n + (x_0 - \sqrt{d}y_0)\lambda_2^n] \\ (x_0 + \sqrt{d}y_0)\lambda_1^n - (x_0 - \sqrt{d}y_0)\lambda_2^n \end{pmatrix} \\ &= \frac{1}{2\sqrt{d}} \begin{pmatrix} \lambda_1^{n+1} + \lambda_2^{n+1} \\ \lambda_1^{n+1} - \lambda_2^{n+1} \end{pmatrix}. \end{aligned}$$

meanwhile, we notice that $\lambda_1\lambda_2 = -1$:

$$\begin{aligned} \lim_{n\to\infty} \frac{x_n}{y_n} &= \lim_{n\to\infty} \sqrt{d}\frac{\lambda_1^{n+1} + \lambda_2^{n+1}}{\lambda_1^{n+1} - \lambda_2^{n+1}} \\ &= \sqrt{d}\frac{\lambda_1^{n+1} + (\frac{-1}{\lambda_1})^{n+1}}{\lambda_1^{n+1} - (\frac{-1}{\lambda})^{n+1}} \\ &= \begin{cases} \sqrt{d}, & n = 2m, \\ \sqrt{d}, & n = 2m+1. \end{cases} \end{aligned}$$

Where $m = 1, 2, \ldots$, once we give a specific value of natural number $n$, so long as we can compute $A^n$ easily, then it is not difficult to get the value of $\frac{x_n}{y_n}$. Along with value of $n$ become increasingly large, the $\frac{x_n}{y_n}$ will be tightly close to $\sqrt{d}$. For instance, using previous method to approximate $\sqrt{2}$ by its asymptotic approximation fractional sequences as follows:

$$d = 2, \quad \begin{cases} x_0 = 7, \\ y_0 = 5, \end{cases} \quad \begin{cases} x_n = 7x_{n-1} + 10y_{n-1}, \\ y_n = 5x_{n-1} + 7y_{n-1}, \end{cases}$$

then fractional sequences will approximate to$\sqrt{2}$. we conclude the previous method as a new iterative method named the Matrix methods, some theorems will be drawn as follows:

**Theorem 1** If we suppose that the $(x_0, y_0)$ is a known positive integer solution of the $x^2 - dy^2 = -1$, and $(x_{n-1}, y_{n-1})(n = 1, 2, \cdots)$ is positive integer solutions of the $x^2 - dy^2 = \pm 1$, and then $(x_n, y_n)$ which satisfies the following recurrence relations:

$$\begin{cases} x_n = x_0x_{n-1} + dy_0y_{n-1}, \\ y_n = y_0x_{n-1} + x_0y_{n-1}, \end{cases}$$

will become the solution of the Pell's equation $x^2 - dy^2 = \pm 1$, along with the other conclusion is induced, it is $\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{d}$.

However, according to [7], it is unfortunate to know our hypothesis that $(x_0, y_0)$ is a known positive integer solution of the Pell's equation $x^2 - dy^2 = -1$ is probably wrong, because the $x^2 - dy^2 = -1$ possibly has no solutions. When the $x^2 - dy^2 = -1$ have no positive integer solutions. To take another kind of measures to approximate to square root; it is similar to above method. Give a theorem as follows without proof,

**Theorem 2** If we suppose that the $(x_0, y_0)$ is a known positive integer solution of the $x^2 - dy^2 = 1$, and $(x_{n-1}, y_{n-1})(n = 1, 2, \cdots)$ is positive integer solutions of the $x^2 - dy^2 = 1$, and then the $(x_n, y_n)$ which satisfies the following recurrence relations:

$$\begin{cases} x_n = x_0x_{n-1} + dy_0y_{n-1}, \\ y_n = y_0x_{n-1} + x_0y_{n-1}, \end{cases}$$

will become the solution of the Pell's equation $x^2 - dy^2 = 1$, along with the other conclusion is also induced, it is $\lim_{n\to\infty} \frac{x_n}{y_n} = \sqrt{d}$.

### *B. Numerical Examples and Error Analysis*

First of all, it is necessary to compare the previous matrix method with the frequently-used Newton's method, which it is used to approximate square root by using following recurrence relations,

$$u_{n+1} = \frac{u_n^2 + d}{2u_n} = \frac{1}{2}(u_n + \frac{d}{u_n}), \ n = 0, 1, 2, \ldots, \tag{10}$$

where $d$ is any arbitrary positive integer.

Next, we use the "ITS", "MM" and "NM" to present the iterations, Matrix methods and Newton's methods respectively, and compute the approximate value of $\sqrt{5}$ ($\sqrt{5} = 2.2360679774997896\cdots$) as a general example to compare two methods in the Table I.

TABLE I
NUMERICAL RESULTS FOR TWO METHODS.

| ITS | MM | NM |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 9/4=2.25 | 9/4=2.25 |
| 3 | 38/17 =2.235294117... | 161/72 =2.236111111... |
| 4 | 161/72=2.236111111... | 51841/23184=2.236067977... |
| 5 | 682/305=2. 236065573... | $\cdots$ |
| 6 | 2889/1292=2.236068111... | $\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ |

It is not hard to find a regularity from Table I, when using Matrix method to compute the approximate value of the square root of the specific positive integer, its significant figure will increase one digit by one, providedly, we do once recursion operation. It is in favor of us to get all kinds of accuracy of approximate value, and not to lose any information of the approximate value of the square root. However, if we use Newton's method to compute the same approximate value, but we fail to obtain all kinds of approximate value of square root, although we will show that the Newton's method has a quadratic convergence rate, i.e., its speed of convergence is quicker than the Matrix method. Many useful information of approximate value will be lost at the price. At times, we need more information of the approximate value of the square root to do some count. Then matrix method will show its great advantage for the calculation. For example, it is very convenient for us to make a table of the square root of the positive integers to help lower grade students to learn relative knowledge.

Moreover, we can also find that approximation fractional sequences of using the Newton's method are part of the approximation fractional sequences of the Matrix method. According to this statement,

**Remark** If we let initial iterative value of Newton's method be a solution of the Pell's equation $x^2 - dy^2 = \pm 1$, then the approximation fractional sequences of using the Newton's method become a proper subset of the approximation fractional sequences of above Matrix method.

In fact, it is not complex to prove above assumption. Let initial iterative value of Newton Method be $\frac{x_0}{y_0}$, there $(x_0, y_0)$ is a specific solution of the $x^2 - dy^2 = \pm 1$, then the first iterative value is

$$\frac{x_1}{y_1} = \frac{1}{2}\left(\frac{x_0}{y_0} + \frac{dy_0}{x_0}\right) = \frac{x_0^2 + dy_0^2}{2x_0y_0},$$

as well as we consider

$$x_1^2 - dy_1^2 = (x_0^2 + dy_0^2)^2 - d(2x_0y_0)^2 = (x_0^2 - dy_0^2)^2 = 1, \quad (11)$$

we can mimic previous strategy to obtain

$$\begin{aligned} x_n^2 - dy_n^2 &= (x_{n-1}^2 + dy_{n-1}^2)^2 - d(2x_{n-1}y_{n-1})^2 \\ &= (x_{n-1}^2 - dy_{n-1}^2)^2 = 1, \end{aligned} \quad (12)$$

So we have proved above assumption. Other method has similar property as above.

we should make an error analysis for the Matrix methods as following: its absolute error is ($x_n^2 - dy_n^2 = \pm 1$ )

$$\begin{aligned} \left|\frac{x_n}{y_n} - \sqrt{d}\right| &= \frac{|x_n - \sqrt{d}y_n|}{y_n} = \frac{1}{y_n(x_n + \sqrt{d}y_n)} \\ &< \frac{1}{\sqrt{d}y_n^2}, \end{aligned} \quad (13)$$

and the absolute error is

$$\begin{aligned} \frac{1}{\sqrt{d}}\left|\frac{x_n}{y_n} - \sqrt{d}\right| &= \frac{|x_n - \sqrt{d}y_n|}{\sqrt{d}y_n} = \frac{1}{\sqrt{d}y_n(x_n + \sqrt{d}y_n)} \\ &< \frac{1}{dy_n^2}, \end{aligned} \quad (14)$$

It has indicated straightforward that the convergent speed of this algorithm is a little slower than the Newton's method, which has quadratic convergence speed. However, along with $n$ becoming increasingly large, $x_n, y_n$ will also become more and more larger, then the accuracy of $\sqrt{d}$ get increasingly high.

## II. CONCLUSION

In conclusion, a special approximate method for computing the square root of the positive integer has been structured in this article. It is different from other methods presented, the significant figure of the approximation of the square root of the specific positive integer will increase one digit by one by the use of previous matrix method, we can obtain more information of the approximation value of the square root because of these asymptotic approximation fractions in above statement. Numerical example and remark have showed Matrix method keep most information of approximate than the Newton method, it is useful for us at sometimes, and also showed that speed of convergence of matrix method is slower than the Newton's method. It often seems unavoidable to make a alternative.

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