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A Single-Period Inventory Problem with Resalable Returns: A Fuzzy Stochastic Approach

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***Abstract*—**In this paper, a single period inventory model with resalable returns has been analyzed in an imprecise and uncertain *mixed* environment. Demand has been introduced as a fuzzy random variable. In this model, a single order is placed before the start of the selling season. The customer, for a full refund, may return purchased products within a certain time interval. Returned products are resalable, provided they arrive back before the end of the selling season and are found to be undamaged. Products remaining at the end of the season are salvaged. All demands not met directly are lost. The probabilities that a sold product is returned and that a returned product is resalable, both imprecise in a real situation, have been assumed to be fuzzy in nature.

***Keywords*—**Fuzzy random variable, Modified graded mean integration, Internet mail order, Inventory.

I. INTRODUCTION

CURRENT business scenario has given rise to a unique phenomenon where the customer is empowered with the freedom to return a purchased product within a specified time frame. The money subsequently is partially or totally refunded. The product so returned can persist in the business flow in the form of reselling provided its quality is undamaged and it is still in demand. This phenomenon is in vogue especially among the catalogue/internet mail order companies owing to multiple reasons.

Such companies carry on their business through the process of 'distance shopping' where customer makes the purchase via a catalogue or the internet. Now as the customer forms his idea of the product based on merely an image of it displayed in the catalogues, it often, after delivery, turns out to be different from his initial assessment or expectation. Naturally the customer then returns the product via a similarly anonymous process, contributing to high return rates. Since the company has the option of reselling the same product, it has to take into consideration this phenomenon while accepting orders. The present paper analyses the 'single period' problem, with the order arriving before the start of the selling season, and it attempts to determine the optimum quantity for a single order, since large ordering lead times and short selling seasons compel retailers to place the entire order before the season commences.

The model of Vlachos and Dekker [16] first considered such a concept but suffers from two restrictive assumptions. The first, assuming the return of a fixed percentage of the sold products, thus excludes the prospect of variability in the net demand. The second assuming a single reselling episode for each returned product, does not recognize the fact that high return rates will lead to multiple reselling of products. The analysis of the 'newsboy problem' with resalable returns, sans the above mentioned restrictions, by Mostard, Koster and Teunter [13,14], assumes a certain probability of return for each sold product and a certain probability for the resale of a returned product. The removal of these restrictions implies that an undamaged returned product can be resold multiple times within the duration of the selling season, provided the demand for it exists.

To incorporate two different types of uncertainty in the demand, which is often the case in reality; the demand has been assumed to be a fuzzy random variable in this paper. Implementing this modified approach in the 'newsboy' problem transforms the product demand from being normally distributed [13, 14] into a fuzzy random variable with imprecise probabilities, since the probability of a fuzzy event is a fuzzy number [1]. Recently, fuzzy random variable demand has also been considered in [5] by Dutta et. al. The probabilities of return of a sold product and resale of a returned product have been assumed to be fuzzy numbers, instead of crisp quantities. The associated cost viz. the salvage costs, shortage costs and collection costs have also been taken to be fuzzy in nature.

In Section 2 of this paper, a result involving an infinite series of fuzzy numbers has been proposed and deduced. Next, a fuzzy random variable and its fuzzy expectation have been defined. Later a brief outline of a "modified" graded mean integration representation of a triangular fuzzy number, developed in this paper, has been discussed. Next, in section 3, the mathematical model has been presented, the assumptions discussed and the problem formulated. Section 4 deals with a numerical example. The conclusion has been made in section 5.

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II. Some useful results

A. Infinite series of fuzzy numbers: a proposition

Proposition: If $\tilde{A} = (\underline{a}, a, \overline{a})$ is a triangular normal fuzzy number, where $\underline{a}, a, \overline{a}$ are positive real numbers, then the following relation holds:

$$1 + \tilde{A} + \tilde{A}^2 + \tilde{A}^3 + ... = (1 - \tilde{A})^{-1} \text{ for } |\tilde{A}| < 1$$

Proof: Let $\tilde{A} = (\underline{a}, a, \overline{a})$ be fuzzy number [3,4,21]. Then,

$\tilde{A}^2 = (\underline{a}, a, \overline{a}) \odot (\underline{a}, a, \overline{a}) = (\underline{a}^2, a^2, \overline{a}^2)$

$\tilde{A}^3 = (\underline{a}^2, a^2, \overline{a}^2) \odot (\underline{a}, a, \overline{a}) = (\underline{a}^3, a^3, \overline{a}^3)$ and so on.

Therefore,

$$\begin{aligned} \text{L.H.S.} &= 1 + \tilde{A} + \tilde{A}^2 + \tilde{A}^3 + ... \\ &= (1,1,1) + (\underline{a}, a, \overline{a}) + (\underline{a}^2, a^2, \overline{a}^2) + ... \\ &= (1 + \underline{a} + \underline{a}^2 + \cdots, 1 + a + a^2 + \cdots, 1 + \overline{a} + \overline{a}^2 + \cdots) \\ &= \left(\frac{1}{1-\underline{a}}, \frac{1}{1-a}, \frac{1}{1-\overline{a}} \right)_{TFN} \text{ for } |\underline{a}| < 1, |a| < 1, |\overline{a}| < 1 \end{aligned}$$

i.e., for $|\tilde{A}| < 1$

$$\begin{aligned} \text{R.H.S.} &= (1 - \tilde{A})^{-1} \\ &= ((1,1,1) - (\underline{a}, a, \overline{a}))^{-1} \\ &= (1 - \overline{a}, 1 - a, 1 - \underline{a})^{-1} \\ &= \left(\frac{1}{1-\underline{a}}, \frac{1}{1-a}, \frac{1}{1-\overline{a}} \right)_{TFN} \end{aligned}$$

Thus L.H.S. = R.H.S.
Hence,

$$1 + \tilde{A} + \tilde{A}^2 + \tilde{A}^3 + ... = (1 - \tilde{A})^{-1} \text{ for } |\tilde{A}| < 1.$$

B. Fuzzy Random Variable and its Fuzzy Expectation

Kwakernaak [10] first introduced fuzzy random variables. Puri and Ralescu [15] and Gil, Lopez-Diaz [7] also discussed this concept in later years. Here the definition given in [6] has been considered.

Let us consider the p-dimensional Euclidean space $\mathbb{R}^p$. $F(\mathbb{R}^p)$ denotes the class of upper semi continuous function in $[0,1]^{\mathbb{R}^p}$ with compact closure of the support. Then, for the one-dimensional case, $F_C(\mathbb{R})$ is the sub-class of convex sets of $F(\mathbb{R})$. Given a probability space $(\Omega, \mathrm{A}, \mathrm{P})$, a mapping $\chi : \Omega \to F_C(\mathbb{R})$ is said to be a fuzzy random variable if for all $\alpha \in [0,1]$, the two real-valued mappings $\inf \chi_\alpha : \Omega \to \mathbb{R}$ and $\sup \chi_\alpha : \Omega \to \mathbb{R}$ (defined so that for all $\omega \in \Omega$ we have $\chi_\alpha(\omega) = [\inf(\chi(\omega))_\alpha, \sup(\chi(\omega))_\alpha]$ are real-valued random variables.

The fuzzy expectation of a fuzzy random variable is a unique fuzzy number. It is defined as

$E\tilde{\mathrm{X}} = \int \tilde{\mathrm{X}} dP = \{(\int \mathrm{X}_\alpha^- dP, \int \mathrm{X}_\alpha^+ dP) \text{ for } 0 < \alpha < 1\}$,

where the fuzzy random variable is $[X]_\alpha = [\mathrm{X}_\alpha^-, \mathrm{X}_\alpha^+]$ for $\alpha \in [0,1]$. The α-cut of the fuzzy expectation is given by

$u_\alpha = [E\tilde{\mathrm{X}}]_\alpha = E[X]_\alpha = [E(\mathrm{X}_\alpha^-), E(\mathrm{X}_\alpha^+)]$ for $\alpha \in [0,1]$.

It is also proved [7] that $E\tilde{\mathrm{X}} \in F$ and $[E\tilde{\mathrm{X}}]_0 = \int_\Omega \mathrm{X}_0 dP = [E\mathrm{X}_0^-, E\mathrm{X}_0^+]$ for $\alpha = 0$.

C. Development of a "modified" Graded Mean Integration Representation of a Generalized Fuzzy Number

For achieving computational efficiency, we use the method of defuzzification of a generalized triangular fuzzy number by its graded mean integration representation. Let L^{-1} and R^{-1} be the inverse functions of the functions L and R, respectively. Then the graded mean h-level value of the fuzzy number $\tilde{\mathrm{A}} = (\underline{a}, a, \overline{a})_{TFN}$ is given by $h\left[L^{-1}(h) + R^{-1}(h)\right]/2$.

Therefore, the graded mean integration representation of a generalized triangular fuzzy number $\tilde{\mathrm{A}}$ with grade w, as proposed by Chen and Hseih [2], is given by

$$G(\tilde{\mathrm{A}}) = \frac{\int_0^w \left(h\left[L^{-1}(h) + R^{-1}(h)\right]/2 \right) dh}{\int_0^w h dh} \quad (1)$$

$$= \frac{\underline{a} + 4a + \overline{a}}{6} \quad (2)$$

where, h lies between 0 and w, $0 < w \le 1$.

It is to be noted here that in (1), equal weightage has been given to the left and right parts of the membership function. But the weightage actually depends on the attitude or optimism of the decision maker. So, the formula used, in this paper, as the graded mean h-level value of the fuzzy number $\tilde{\mathrm{A}} = (\underline{a}, a, \overline{a})_{TFN}$ is assumed to be of the form $h\left[\beta L^{-1}(h) + (1-\beta) R^{-1}(h)\right]$, where β is called the decision maker's attitude or optimism parameter. β can take values between 0 and 1 i.e., $0 \le \beta \le 1$. The value of β closer to 0 implies that the decision maker is more pessimistic while the value of β closer to 1 means that the decision maker is more optimistic.

Therefore, the formula (1) is modified as below:

$$G\left(\tilde{A}\right)=\frac{\int_0^w \left(h\left[\beta L^{-1}\left(h\right)+\left(1-\beta\right)R^{-1}\left(h\right)\right]\right)dh}{\int_0^w hdh} \quad (3)$$

Now,

$$L\left(u\right)=w\left(\frac{u-\underline{a}}{a-\underline{a}}\right),\ \underline{a}\le u\le a$$

$$R\left(u\right)=w\left(\frac{u-\overline{a}}{a-\overline{a}}\right),\ a\le u\le \overline{a}$$

Thus,

$$L^{-1}\left(h\right)=\underline{a}+\left(a-\underline{a}\right)h/w$$
$$R^{-1}\left(h\right)=\overline{a}-\left(\overline{a}-a\right)h/w$$

Now, using the formula (3), the graded mean integration representation of $\tilde{A}$ is given by

$$G\left(\tilde{A}\right)=\int_0^w h\left[\beta\underline{a}+\left(1-\beta\right)\overline{a}+\left\{a-\beta\underline{a}-\left(1-\beta\right)\overline{a}\right\}h/w\right]dh\Big/\int_0^w hdh$$
$$=\frac{\beta\underline{a}+2a+\left(1-\beta\right)\overline{a}}{3} \quad (4)$$

It is to be noted that when $\beta=0.5$ i.e., when equal weightage is given to the left and right parts of the membership function, then, the formula (4) reduces to the formula (2).

III. Methodology

A. Model and assumptions

A single-period inventory model with resalable returns has been considered here. Mostard and Teunter [13] assumed that there are no interdependencies between ordered items and hence analyzed a single item inventory model. The same reasoning has been followed here. It has been assumed that there is a single replenishment opportunity at which some quantity of products is ordered and those products arrive before the start of the selling season. The total number of products ordered during the season i.e., the gross demand, is denoted by D_G . The following assumptions have been made.

(i) Customers are allowed to return purchased products within a fixed time limit (usually 7-30 days [13]). If a sold product is returned, the customer gets a full refund for it. In [16], it is assumed that a fixed percentage of the sold products are always returned. This percentage is assumed to be a crisp number. In [13] and [14], it is assumed that each sold product is returned with a certain probability. This probability is assumed to be a crisp number. Let the probability that a sold product is returned be denoted by $\tilde{R}$.

(ii) An undamaged returned product is collected, tested and then put back on the shelf. But the entire process has to be completed before the end of the selling season for a possible resale. Moreover there should be sufficient demand to sell the returned products (assuming priority of resale over first sales [13]). In [16], it is assumed that a product is resold only once. This restrictive assumption is not considered in [13] and [14]. In [13] and [14], it is assumed that a product can be resold more than once and that there is a certain probability with which a returned product can be resold. This probability has been assumed to be a known fixed crisp number.

Now, the information regarding these probabilities is collected from various experts. Their opinions may well be expressed in linguistic terms. But, to use such information, quantification is required. Fuzzy set theory provides a powerful tool for dealing with such non-stochastic imprecision or vagueness of the data available, called "intrinsic fuzziness". Besides, abundance of information also leads to fuzziness, called "informational fuzziness" [22]. Prof. Zadeh, while referring to this informational fuzziness says that as the systems become more complex, it becomes increasingly difficult to discover underlying mathematical structures that are both meaningful and precise [20]. Thus, in order to develop a more realistic model, the above mentioned probabilities of return of a sold product and resale of a returned product have been assumed to be fuzzy numbers instead of crisp quantities.

It is to be noted here that, as remarked by Mostard and Teunter in [13], "for a practical case of the mail order retailer, the average time between a sale and a return plus the collection and test times is about 2-3 weeks and hence small relative to the length of the selling season (26 weeks). The (expected) number of demands is larger than the number of returns during almost all of the season (except for the last 4 weeks) for all products. So, almost all the returns that are back on the shelf before the season ends are indeed resalable." Thus, in the numerical example considered, a high probability for a resalable return has been justifiably assumed. Let the probability that a returned product is resold be denoted by $\tilde{K}$.

(iii) In [13] and [14], the demand has been assumed to have been normally distributed. Here, as mentioned earlier, we argue that the existing theory of probability, as employed in the stochastic approach, does not provide adequate representation of the real world inventory problem. This is because, while it does consider the stochastic uncertainty of the information, it does not take into account the vagueness or imprecision of the data. For example, if the demand information collected from experts contains an expression like "the demand is about 100", then such information can only be used in the stochastic approach, if some sort of approximation is done. But this leads to the loss of information. Also, as explained earlier, abundance of information could lead to “informational fuzziness”. Thus, here, we incorporate both non-stochastic imprecision and stochastic uncertainty into the demand and assume it to be a fuzzy random variable involving imprecise probabilities.

B. Problem formulation

The objective of this paper is to determine the order quantity that maximizes the expected profit. The relevant cost and revenue parameters (all per unit) are the selling price $\tilde{P}$, the cost price $\tilde{C}$ and the salvage value $\tilde{S}$, the loss of goodwill / shortage cost $\tilde{G}$ and the collection cost $\tilde{D}$. In a real case scenario, it is difficult to determine these costs precisely and hence these costs have been assumed to be fuzzy in nature. It has further been assumed that all returns are undamaged and hence a single salvage value has been used. Let the net demand denote the total number of (gross) demanded products that are either not returned or returned but not resalable, assuming that all demands are met. Let $\tilde{P}_G$ be the unit expected revenue of satisfying a gross demand, including salvage revenue if the sold product is returned but not resalable.

i.e., $\tilde{P}_G$ = total selling price – total collection cost + total salvage cost

$$= \left(1-\tilde{R}\right)\tilde{P} - \tilde{R}\tilde{D} + \tilde{R}\left(1-\tilde{K}\right)\tilde{S}$$

Let $\tilde{P}_N$ be the unit expected revenue of satisfying a net demand i.e., of repeatedly selling a product until it is either not returned or returned but not resalable, including salvage revenue if the sold product is returned but not resalable. If a product is sold, returned and resold again and again, then

$$\tilde{P}_N = \left(1+\tilde{R}\tilde{K}+\left(\tilde{R}\tilde{K}\right)^2+\left(\tilde{R}\tilde{K}\right)^3+.......\right)\tilde{P}_G = \frac{\tilde{P}_G}{1-\tilde{R}\tilde{K}},$$ using the proposition in section 2.1.

Let $\tilde{G}_N$ be the expected net shortage cost of not satisfying a net demand even on repeated resale. Then,

$$\tilde{G}_N = \left(1+\tilde{R}\tilde{K}+\left(\tilde{R}\tilde{K}\right)^2+\left(\tilde{R}\tilde{K}\right)^3+.....\right)\tilde{G} = \frac{\tilde{G}}{1-\tilde{R}\tilde{K}},$$ using the proposition in section 2.1.

It has been assumed that the net demand $\tilde{N}$ is a fuzzy random variable with the given set of data $\left\{(\tilde{y}_1,\tilde{p}_1),(\tilde{y}_2,\tilde{p}_2),....(\tilde{y}_n,\tilde{p}_n)\right\}$. The data being imprecise with fuzzy probability, for the sake of simplicity, the data set and its corresponding probabilities have been considered to be triangular fuzzy numbers i.e., $\tilde{y}_i$ and $\tilde{p}_i$ have been represented as $\left(\underline{y}_i, y_i, \overline{y}_i\right)$ and $\left(\underline{p}_i, p_i, \overline{p}_i\right)$ for

i = 1 to n, respectively.

If $\tilde{y}_k$ items are procured at the beginning of the selling season, then the profit function $\tilde{P}$ is given by

$$\tilde{P}\left(\tilde{y}_k,\tilde{N}\right) = \tilde{P}_N\tilde{y}_i - \tilde{C}\tilde{y}_k + \tilde{S}\left(\tilde{y}_k-\tilde{y}_i\right),\ \tilde{y}_i \lesssim \tilde{y}_k$$

$$= \tilde{P}_N\tilde{y}_k - \tilde{C}\tilde{y}_k - \tilde{G}_N\left(\tilde{y}_i-\tilde{y}_k\right),\ \tilde{y}_i \gtrsim \tilde{y}_k$$

for some i = 1 to n.

As the demand $\tilde{N}$ is a fuzzy random variable, its profit function is also a fuzzy random variable. Hence its total expected value $E\tilde{P}$ is a unique fuzzy number $E\tilde{P} = \left(E\underline{P}, EP, E\overline{P}\right)$ Therefore the fuzzy total expected profit function is determined by

$$E\tilde{P} = E\tilde{P}\left(\tilde{y}_k,\tilde{N}\right)$$
$$= \sum_{i=1}^{k}\left[\tilde{P}_N\tilde{y}_i - \tilde{C}\tilde{y}_k + \tilde{S}\left(\tilde{y}_k-\tilde{y}_i\right)\right]\tilde{p}_i + \sum_{i=k+1}^{n}\left[\left(\tilde{P}_N-\tilde{C}\right)\tilde{y}_k - \tilde{G}_N\left(\tilde{y}_i-\tilde{y}_k\right)\right]\tilde{p}_i$$
$$= \sum_{i=1}^{k}\left[\left(\tilde{P}_N-\tilde{S}\right)\tilde{y}_i - \left(\tilde{C}-\tilde{S}\right)\tilde{y}_k\right]\tilde{p}_i + \sum_{i=k+1}^{n}\left[\left(\tilde{P}_N-\tilde{C}+\tilde{G}_N\right)\tilde{y}_k - \tilde{G}_N\tilde{y}_i\right]\tilde{p}_i$$

where $EP = E\left[\tilde{P}_{\alpha=1}\right]$

$$= \sum_{i=1}^{k}\left[\left(P_N-S\right)y_ip_i - \left(C-S\right)y_kp_i\right] + \sum_{i=k+1}^{n}\left[\left(P_N-C+G_N\right)y_kp_i - G_Ny_ip_i\right]$$

$$E\underline{P} = E\left[P^-_{\alpha=0}\right]$$
$$= \sum_{i=1}^{k}\left[\left(\underline{P}_N-\underline{S}\right)\underline{y}_i\underline{p}_i - \left(\overline{C}-\overline{S}\right)\overline{y}_k\overline{p}_i\right] + \sum_{i=k+1}^{n}\left[\left(\underline{P}_N-\underline{C}+\underline{G}_N\right)\underline{y}_k\underline{p}_i - \overline{G}_N\overline{y}_k\overline{p}_i\right]$$

$$E\overline{P} = E\left[P^+_{\alpha=0}\right]$$
$$= \sum_{i=1}^{k}\left[\left(\overline{P}_N-\overline{S}\right)\overline{y}_i\overline{p}_i - \left(\underline{C}-\underline{S}\right)\underline{y}_k\underline{p}_i\right] + \sum_{i=k+1}^{n}\left[\left(\overline{P}_N-\overline{C}+\overline{G}_N\right)\overline{y}_k\overline{p}_i - \underline{G}_N\underline{y}_i\underline{p}_i\right]$$

Now, using the method of representation of generalized fuzzy number based on the integral value of graded mean h-level, as discussed in section 2.4 in this paper, we find a defuzzified representative of the unique fuzzy number $E\tilde{P}$ as

$$G\left(E\tilde{P}\right) = \frac{\beta E\underline{P} + 2EP + \left(1-\beta\right)E\overline{P}}{3}$$

Further for the optimal order quantity $\tilde{y}_k$, we must have

$$G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_k,\tilde{N}\right)\right)\right) - G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_{k-1},\tilde{N}\right)\right)\right) > 0 \tag{5}$$

$$G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_k,\tilde{N}\right)\right)\right) - G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_{k+1},\tilde{N}\right)\right)\right) > 0 \qquad (6)$$

The conditions, given by equations (5) and (6), give us the following inequalities:

$$\begin{aligned}
&(1-\beta)\left(\underline{P}_N - \underline{C} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=1}^{k-1}\underline{p}_i \\
&+\beta\left(\underline{P}_N - \underline{C} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=k}^{n}\underline{p}_i \\
&+\beta\left(\overline{P}_N - \overline{C} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=1}^{k-1}\overline{p}_i \\
&+(1-\beta)\left(\overline{P}_N - \overline{C} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=k}^{n}\overline{p}_i \\
&+(1-2\beta)\left(\underline{S} - \underline{C} + \underline{G}_N\right)\underline{y}_k\underline{p}_k \\
&+2\left(P_N - C + G_N\right)\left(y_k - y_{k-1}\right)\sum_{i=1}^{n}p_i > \\
&(1-\beta)\left(\underline{P}_N - \underline{S} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=1}^{k-1}\underline{p}_i \\
&+\beta\left(\overline{P}_N - \overline{S} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=1}^{k-1}\overline{p}_i \\
&+(1-2\beta)\left(\overline{S} - \overline{C} + \overline{G}_N\right)\overline{y}_k\overline{p}_k \\
&+2\left(P_N - S + G_N\right)\left(y_k - y_{k-1}\right)\sum_{i=1}^{k-1}p_i \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\beta)\left(\underline{P}_N - \underline{C} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=1}^{k}\underline{p}_i \\
&+\beta\left(\underline{P}_N - \underline{C} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=k+1}^{n}\underline{p}_i \\
&+\beta\left(\overline{P}_N - \overline{C} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=1}^{k}\overline{p}_i \\
&+(1-\beta)\left(\overline{P}_N - \overline{C} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=k+1}^{n}\overline{p}_i \\
&+(1-2\beta)\left(\underline{S} - \underline{C} + \underline{G}_N\right)\underline{y}_{k+1}\underline{p}_{k+1} \\
&+2\left(P_N - C + G_N\right)\left(y_k - y_{k-1}\right)\sum_{i=1}^{n}p_i < \\
&(1-\beta)\left(\underline{P}_N - \underline{S} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=1}^{k}\underline{p}_i \\
&+\beta\left(\overline{P}_N - \overline{S} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=1}^{k}\overline{p}_i \\
&+(1-2\beta)\left(\overline{S} - \overline{C} + \overline{G}_N\right)\overline{y}_{k+1}\overline{p}_{k+1} \\
&+2\left(P_N - S + G_N\right)\left(y_k - y_{k-1}\right)\sum_{i=1}^{k}p_i \qquad (8)
\end{aligned}$$

We thus find the optimal order quantity for a single-period inventory model with resalable returns. The calculation part of the conditions (5) and (6) has been given as an appendix. Next, a numerical example illustrating the methodology discussed in this section has been given.

IV. Numerical Example

Let the purchase cost per item be $C = 18$ and the selling price $P = 30$. The collection cost is $\tilde{D} = (3.00, 4.25, 6.00)$.

The loss of goodwill or shortage cost is $\tilde{G} = (20, 25, 29)$.

The salvage cost is $\tilde{S} = (4.00, 5.00, 6.75)$.

Probability that a product is returned is $\tilde{R} = (0.43, 0.45, 0.50)$. Probability that a returned product is resalable is $\tilde{K} = (0.92, 0.94, 0.98)$.

Unit expected revenue for satisfying gross demand is

$$\begin{aligned}
\tilde{P}_G &= (1-\tilde{R})P - \tilde{R}\tilde{D} + \tilde{R}(1-\tilde{K})\tilde{S} \\
&= (12.0344, 14.7225, 16.08)
\end{aligned}$$

Unit expected revenue for satisfying net demand is

$$\begin{aligned}
\tilde{P}_N &= P_G \big/ (1-\tilde{R}\tilde{K}) \\
&= \frac{(12.0344, 14.7225, 16.08)}{(0.51, 0.577, 0.6044)} \\
&= (19.91, 25.51, 31.53)
\end{aligned}$$

Expected net shortage cost for not satisfying a net demand is

$$\begin{aligned}
\tilde{G}_N &= \tilde{G} / (1 - (\tilde{R} \odot \tilde{K})) \\
&= \frac{(20, 25, 29)}{(0.51, 0.577, 0.6044)} \\
&= (33.09, 43.33, 56.86)
\end{aligned}$$

The demand information is given in the following tabular forming in Table I:

Table I

Demand	**Probabilities**
(13,15,17)	(0.045,0.05,0.055)
(18,20,22)	(0.180,0.20,0.225)
(23,25,27)	(0.275,0.30,0.325)
(28,30,32)	(0.155,0.20,0.250)
(33,35,37)	(0.120,0.15,0.175)
(38,40,42)	(0.055,0.10,0.125)

It is important to mention here that the rest of the calculations in the numerical example, as shown in a tabular form in Table 2, have been done assuming $\beta = 0.5$ i.e. assuming that equal weightage is being given to both the left and right reference functions. In Table 2, A and B have been taken to be the right hand side and left hand side of the inequality (8). Table 2 is given below:

TABLE II

$\tilde{y}_k$	**(13,15,17)**	**(18,20,22)**	**(23,25,27)**	**(28,30,32)**	**(33,35,37)**
$(\underline{P}_N - \underline{S} + \underline{G}_N)\times(\underline{y}_{k+1} - \underline{y}_k)\sum_{i=1}^{k}\underline{p}_i$	11.025	55.125	122.5	160.475	189.875
$(\overline{P}_N - \overline{S} + \overline{G}_N)\times(\overline{y}_{k+1} - \overline{y}_k)\sum_{i=1}^{k}\overline{p}_i$	22.451	114.296	246.961	349.011	420.446
$4(P_N - S + G_N)\times(y_{k+1} - y_k)\sum_{i=1}^{k} p_i$	63.84	319.2	702.24	957.6	1149.12
A	97.316	488.621	1071.701	1467.086	1759.441
$(\underline{P}_N - \underline{C} + \underline{G}_N)\times(\underline{y}_{k+1} - \underline{y}_k)\sum_{i=1}^{n}\underline{p}_i$	145.25	145.25	145.25	145.25	145.25
$(\overline{P}_N - \overline{C} + \overline{G}_N)\times(\overline{y}_{k+1} - \overline{y}_k)\sum_{i=1}^{n}\overline{p}_i$	406.50225	406.50225	406.50225	406.50225	406.50225
$(\overline{P}_N - \overline{C} + \overline{G}_N)\times(y_{k+1} - y_k)\sum_{i=1}^{n} p_i$	1016.8	1016.8	1016.8	1016.8	1016.8
B	1568.55225	1568.55225	1568.55225	1568.55225	1568.55225

as seen from the table II above A>B for $\tilde{y}_k = (33,35,37)$ i.e., the inequality (8) is satisfied. Also, this value of the order quantity satisfies the inequality (7) So, we conclude that, since $\tilde{y}_k = (33,35,37)$ satisfies both the optimality conditions, it is the required optimal order quantity.

V. CONCLUSION

The incorporation of fuzzy random variable as demand and the assumption of the relevant costs as being fuzzy in nature, as done in this paper, provide a more realistic model of the single period inventory problem with resalable returns. This amalgamation of the concepts of randomness and imprecision may be extended for other inventory models for better representation of real life problems.

APPENDIX

The optimal order quantity may be defined as

$$G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_k,\tilde{N}\right)\right)\right) - G\left(\mathrm{E}\left(\tilde{P}\left(\tilde{y}_{k-1},\tilde{N}\right)\right)\right) > 0$$

The left-hand side of the above inequality in the extended form reads like

$$=1/3\,[\,\beta\sum_{i=1}^{k}\{(\underline{P}_N - \underline{S})\underline{y}_i\underline{p}_i - (\overline{C} - \overline{S})\overline{y}_k\overline{p}_i\}$$

$$+\beta\sum_{i=k+1}^{n}\{(\underline{P}_N - \underline{C} + \underline{G}_N)\underline{y}_k\underline{p}_i - \overline{G}_N\overline{y}_i\overline{p}_i\}$$

$$2\sum_{i=1}^{k}\{(P_N - S)y_i p_i - (C - S)y_k p_i\}$$

$$+2\sum_{i=k+1}^{n}\{(P_N - C + G_N)y_k p_i - G_N y_i p_i\}$$

$$+(1-\beta)\sum_{i=1}^{k}\{(\overline{P}_N - \overline{S})\overline{y}_i\overline{p}_i - (\underline{C} - \underline{S})\underline{y}_k\underline{p}_i\}$$

$$+(1-\beta)\sum_{i=k+1}^{n}\{(\overline{P}_N - \overline{C} + \overline{G}_N)\overline{y}_k\overline{p}_i - \underline{G}_N\underline{y}_i\underline{p}_i\}\,]$$

$$-1/3\,[\,\beta\sum_{i=1}^{k-1}\{(\underline{P}_N - \underline{S})\underline{y}_i\underline{p}_i - (\overline{C} - \overline{S})\overline{y}_{k-1}\overline{p}_i\}$$

$$+\beta\sum_{i=k}^{n}\{(\underline{P}_N - \underline{C} + \underline{G}_N)\underline{y}_{k-1}\underline{p}_i - \overline{G}_N\overline{y}_i\overline{p}_i\}$$

$$+2\sum_{i=1}^{k-1}\{(P_N - S)y_i p_i - (C - S)y_k p_i\}$$

$$+2\sum_{i=k}^{n}\{(P_N - C + G_N)y_k p_i - G_N y_i p_i\}$$

$$+(1-\beta)\sum_{i=1}^{k-1}\{(\overline{P}_N - \overline{S})\overline{y}_i\overline{p}_i - (\underline{C} - \underline{S})\underline{y}_{k-1}\underline{p}_i\}$$

$$+(1-\beta)\sum_{i=k}^{n}\{(\overline{P}_N - \overline{C} + \overline{G}_N)\overline{y}_{k-1}\overline{p}_i - \underline{G}_N\underline{y}_i\underline{p}_i\}\,]$$

which implies

$$1/3\,[\,\beta\left(\underline{P}_N - \underline{C} + \underline{G}_N\right)\left(\underline{y}_k - \underline{y}_{k-1}\right)\sum_{i=k}^{n}\underline{p}_i$$

$$+(1-\beta)\left(\overline{P}_N - \overline{C} + \overline{G}_N\right)\left(\overline{y}_k - \overline{y}_{k-1}\right)\sum_{i=k}^{n}\overline{p}_i$$

$$+(1-2\beta)\{(\underline{S}-\underline{C}+\underline{G}_N)\underline{y}_k\underline{p}_k-(\overline{S}-\overline{C}+\overline{G}_N)\overline{y}_k\overline{p}_k\}$$
$$-(1-\beta)(\underline{C}-\underline{S})(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k-1}\underline{p}_i$$
$$-\beta(\overline{C}-\overline{S})(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k-1}\overline{p}_i$$
$$+2(P_N-C+G_N)(y_k-y_{k-1})\sum_{i=1}^{n}p_i$$
$$-2(P_N-S+G_N)(y_k-y_{k-1})\sum_{i=1}^{k-1}p_i\ ]$$

Equals

$$1/3\,[(1-\beta)(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k-1}\underline{p}_i$$
$$+\beta(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=k}^{n}\underline{p}_i$$
$$+\beta(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k-1}\overline{p}_i$$
$$+(1-\beta)(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=k}^{n}\overline{p}_i$$
$$+(1-2\beta)\{(\underline{S}-\underline{C}+\underline{G}_N)\underline{y}_k\underline{p}_k-(\overline{S}-\overline{C}+\overline{G}_N)\overline{y}_k\overline{p}_k\}$$
$$-(1-\beta)(\underline{P}_N-\underline{S}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k-1}\underline{p}_i$$
$$-\beta(\overline{P}_N-\overline{S}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k-1}\overline{p}_i$$
$$+2(P_N-C+G_N)(y_k-y_{k-1})\sum_{i=1}^{n}p_i$$
$$-2(P_N-S+G_N)(y_k-y_{k-1})\sum_{i=1}^{k-1}p_i\ ]$$

Now from equation (5) we have,

$$(1-\beta)(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k-1}\underline{p}_i$$
$$+\beta(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=k}^{n}\underline{p}_i$$
$$+\beta(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k-1}\overline{p}_i$$
$$+(1-\beta)(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=k}^{n}\overline{p}_i$$
$$+(1-2\beta)(\underline{S}-\underline{C}+\underline{G}_N)\underline{y}_k\underline{p}_k$$
$$+2(P_N-C+G_N)(y_k-y_{k-1})\sum_{i=1}^{n}p_i >$$
$$(1-\beta)(\underline{P}_N-\underline{S}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k-1}\underline{p}_i$$
$$+\beta(\overline{P}_N-\overline{S}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k-1}\overline{p}_i$$
$$+(1-2\beta)(\overline{S}-\overline{C}+\overline{G}_N)\overline{y}_k\overline{p}_k$$
$$+2(P_N-S+G_N)(y_k-y_{k-1})\sum_{i=1}^{k-1}p_i \qquad (7)$$

Similarly from equation (6) we have,

$$(1-\beta)(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k}\underline{p}_i$$
$$+\beta(\underline{P}_N-\underline{C}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=k+1}^{n}\underline{p}_i$$
$$+\beta(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k}\overline{p}_i$$
$$+(1-\beta)(\overline{P}_N-\overline{C}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=k+1}^{n}\overline{p}_i$$
$$+(1-2\beta)(\underline{S}-\underline{C}+\underline{G}_N)\underline{y}_{k+1}\underline{p}_{k+1}$$
$$+2(P_N-C+G_N)(y_k-y_{k-1})\sum_{i=1}^{n}p_i <$$
$$(1-\beta)(\underline{P}_N-\underline{S}+\underline{G}_N)(\underline{y}_k-\underline{y}_{k-1})\sum_{i=1}^{k}\underline{p}_i$$
$$+\beta(\overline{P}_N-\overline{S}+\overline{G}_N)(\overline{y}_k-\overline{y}_{k-1})\sum_{i=1}^{k}\overline{p}_i$$
$$+(1-2\beta)(\overline{S}-\overline{C}+\overline{G}_N)\overline{y}_{k+1}\overline{p}_{k+1}$$
$$+2(P_N-S+G_N)(y_k-y_{k-1})\sum_{i=1}^{k}p_i \qquad (8)$$

REFERENCES

[1] D. Chakraborty., "Redefining chance-constrained programming in fuzzy environment", *Fuzzy Sets and Systems* vol. 125, no. 3, pp. 327-333.

[2] S H Chen, C H Hseih, "Graded Mean Integration Representation of Generalized Fuzzy Number", *Journal of Chinese Fuzzy Systems Association* vol. 5, pp. 1-7, 1999.

[3] S J Chen, S M Chen, "Fuzzy Risk Analysis Based on Similarity Measures of Generalized Fuzzy Numbers", *IEEE Transactions on Fuzzy Systems* vol. 11, pp. 45-56, 2003.

[4] D Dubois, H Prade, *Fuzzy Sets and Systems, Theory and applications*, Academic Press, 1980.

[5] P Dutta, D Chakraborty, A R Roy, "A Single-Period Inventory Model with Fuzzy Random Variable Demand", *Mathematical and Computer Modeling* vol. 41, pp. 915-922, 2005.

[6] M A Gil, M Lopez-Diaz, D A Ralescu, "Overview on the Development of Fuzzy Random Variables", *Fuzzy Sets and Systems* (Available online), 2006.

[7] M A Gil, M Lopez-Diaz, "The λ-average value of the expected value of a fuzzy Random Variable", *Fuzzy Sets and Systems*, vol. 99, pp. 347-391, 1998.

[8] H Ishii, T Konno, "A stochastic inventory problem with fuzzy shortage cost", *European Journal of Operational Research*, vol. 106, pp. 90-94, 1998.

[9] C Kao, W K Hsu, "A single period inventory model with fuzzy demand", *Computers And Mathematics with Applications*, vol. 43, pp. 841-848, 2002.
[10] H Kwakernaak, "Fuzzy Random Variables: Definitions and Theorems", *Information Sciences*, vol. 15, pp. 1- 29, 1978.
[11] L Li, S N Kabadi, K P K Nair, "Fuzzy models for single-period inventory problem", *Fuzzy Sets and Systems*, vol. 132, pp. 273-289, 2002.
[12] M. K.Luhandjula, "Linear programming under randomness and fuzziness", *Fuzzy Sets and Systems*,vol.10,pp.45-55,1983.
[13] J Mostard, R Teunter, "The newsboy problem with resalable returns: a single period model and case study", *European Journal of Operational Research*, vol. 169, pp. 81-96, 2006.
[14] J Mostard, R Koster, R Teunter, "The distribution-free newsboy problem with resalable returns", *International Journal of Production Economics*, vol. 97, pp. 329-342, 2005.
[15] M L Puri, D A Ralescu, "Fuzzy random variables",*Journal of Mathematical Analysis and Applications*, vol. 114, pp. 409-422, 1986.
[16] D Vlachos, R Dekker, "Return handling options and order quantities for single period products", *European Journal of Operational Research*, vol. 151, pp. 38-52, 2003.
[17] G Wang, Q Zhong, "Linear programming with fuzzy random variable coefficients", *Fuzzy Sets and Systems*, vol. 57, pp. 295-311, 1993.
[18] G Wang, Q Zhong, "On solutions and distribution problems of the linear programming with fuzzy random variable coefficients", *Fuzzy Sets and Systems*, vol. 58, pp. 155-170, 1993.
[19] G Wang, Y Zhang, "The theory of fuzzy stochastic processes", *Fuzzy Sets and Systems*, vol. 51, pp. 161-178,1992.
[20] L. A. Zadeh, "Outline of a new approach to the analysis of complex systems and decision process", *IEEE Transactions on Systems, Man and Cybernetics*, vol. 3, pp. 28-44, 1973.
[21] H J Zimmermann, *Fuzzy Set Theory and its applications, Second Edition*, Allied Publishers Limited, 1996.
[22] H. J. Zimmermann, P Zysno, "Quantifying Vagueness in Decision Models", *European Journal of Operational Research*, vol. 22, pp. 148-158, 1985.