

A Simplified Distribution for Nonlinear Seas

M. A. Tayfun, M. A. Alkhalidi

Abstract—The exact theoretical expression describing the probability distribution of nonlinear sea-surface elevations derived from the second-order narrowband model has a cumbersome form that requires numerical computations, not well-disposed to theoretical or practical applications. Here, the same narrowband model is re-examined to develop a simpler closed-form approximation suitable for theoretical and practical applications. The salient features of the approximate form are explored, and its relative validity is verified with comparisons to other readily available approximations, and oceanic data.

Keywords—Ocean waves, probability distributions, second-order nonlinearities, skewness coefficient, wave steepness.

I. INTRODUCTION

THE second-order narrowband model describing long-crested surface waves observed in time t at a fixed point in deep water is given by [1]

$$\eta = \eta_1 + \eta_2 = \xi \cos \chi + \frac{1}{2} \varepsilon \xi^2 \cos 2\chi. \quad (1)$$

η_1 is the linear Gaussian surface elevation, ξ amplitude or envelope of η_1 , and η_2 represents second-order nonlinear corrections, all scaled with the root-mean-square (*rms*) of η_1 . It is assumed that the spectral density of η_1 is unimodal and narrowband over frequencies and directions. The *rms* of η is also σ , correct to $O(\varepsilon)$. The dimensionless parameter $\varepsilon \ll 1$ and defined as $\varepsilon = \sigma k_m$, where k_m = spectral-mean wavenumber, corresponds to the *rms* gradient of η_1 . In the context of narrowband approximation, it serves as an integral measure wave steepness. Further, $\chi = \omega_m t + \phi(t)$ = total wave phase, ω_m = spectral-mean frequency such that $k_m = \omega_m^2 / g$, g = gravitational acceleration, and ϕ = wave phase. Finally, ξ is Rayleigh distributed while χ and therefore ϕ are uniformly random over an interval of 2π .

Representation (1) has been used successfully to describe the distributions of nonlinear surface elevations and various other surface features, including wave crest and trough amplitudes, wave heights, envelopes and phases in nonlinear seas successfully in [4]-[13] and a variety other practical and theoretical studies, with extension and generalizations to large waves and shallower water depths in [14], [15].

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Despite the apparent simplicity of η , its probability density function p_η cannot be obtained in an explicit closed form [1]. Therefore, its applications rely on either rather cumbersome numerical computations or require further approximations. One particular approximation that has been devised in [2] and often used in practical and theoretical studies has the form

$$p_\eta(z) \approx \frac{(1 - 7\varepsilon^2/8) \exp(-G^2/2\varepsilon^2)}{\sqrt{2\pi(1 + 3G + 2G^2)}}, \quad (2)$$

where $G = \sqrt{1 + 2\varepsilon z} - 1$ and $z > -3/8\varepsilon$. A second and more conventional alternative that follows from expanding p_η in a Gram-Charlier [3] series is given, correct to $O(\varepsilon)$, by

$$p_\eta(z) \approx \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \left[1 + \frac{\lambda_3}{6} z(z^2 - 3) \right], \quad (3)$$

where $|z| < \infty$ and $\lambda_3 = \langle \eta^3 \rangle$ is the skewness coefficient. For (1), $\lambda_3 = 3\varepsilon$ to $O(\varepsilon^2)$ so that $\varepsilon = \lambda_3/3$. In oceanic wind seas, including those generated by extreme storms $0 < \lambda_3 < 0.3$ typically. The preceding expression then suggests that $0 < \varepsilon < 0.1$ as a typical range.

Since the first-order linear η_1 is zero-mean Gaussian, its marginal probability density p and cumulative distribution P are given, respectively, by

$$p(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}; \quad |x| < \infty, \quad (4)$$

$$P(x) = \int_{-\infty}^x \frac{\exp(-u^2/2)}{\sqrt{2\pi}} du. \quad (5)$$

Similarly, the conjugate $\hat{\eta}_1 = \xi \sin \chi$ is also zero-mean Gaussian, with the marginal density and distribution identically described by (5) and (6), respectively. Finally, because $\hat{\eta}_1$ and η_1 are uncorrelated, their joint probability density reduces to a product of their marginal densities.

It is noted that while both (2) and (3) are simple enough, (2) has a somewhat restricted domain. In contrast, the domain over which (3) is non-negative is restricted to $\eta \geq -(\alpha^+ + \alpha^-)$, where $\alpha^\pm = \{ [1 \pm (1 - \varepsilon^2)^{1/2}] / \varepsilon \}^{1/3}$.

Evidently, the latter expression can also be expressed in terms of λ_3 since $\varepsilon = \lambda_3/3$. For instance, when $\lambda_3 = 0.3$, (3) is non-negative provided that $\eta \geq -3.08$ whereas (2) is valid

for $\eta > -3/8\varepsilon = -3.75$, a somewhat larger domain than that of (3). In addition, both (2) and (3) are not entirely zero-mean or normalized to unity over their restricted domains.

This study represents an attempt to develop a new simpler approximation for the density and cumulative distribution of (1), and explores its salient features relative to (2) and (3). Eventually, the new approximation is compared with (2), (3) and oceanic measurements to investigate its relative validity.

II. APPROXIMATE PROBABILITY STRUCTURE

For simplicity in notation, (1) is rewritten as

$$\eta = z(x, y) = x + \frac{1}{2}\varepsilon(x^2 - y^2), \tag{6}$$

where $x = \xi \cos \chi$ and $y = \xi \sin \chi$. It is recalled that the joint probability density of x and y is a product of two marginal standard Gaussian densities, each of the form (4). Consequently, the conditional density of η , given y , follows from a simple change of variables as

$$p_{\eta|y}(z|y) = p(x) \left| \frac{\partial x}{\partial z} \right|_{x=z^{-1}(x|y)}, \tag{7}$$

where

$$x = z^{-1}(x|y) = (-1 + \sqrt{1 + 2\varepsilon z + \varepsilon^2 y^2}) / \varepsilon, \tag{8}$$

and

$$\frac{\partial x}{\partial z} \Big|_{x=z^{-1}(x|y)} = (1 + \varepsilon x)^{-1} \Big|_{x=z^{-1}(x|y)}. \tag{9}$$

The approximation

$$x = z^{-1}(x|y) \approx (-1 + \sqrt{1 + 2\varepsilon z}) / \varepsilon = G / \varepsilon, \tag{10}$$

can now be used to rewrite (9) as

$$\frac{\partial x}{\partial z} \Big|_{x=z^{-1}(x|y)} \approx 1 / \sqrt{1 + 2\varepsilon z}. \tag{11}$$

Finally, the substitutions of (10) and (11) in (7) leads to an expression independent of y . As a result, (7) reduces to the approximation sought in the form

$$p_{\eta}(z) \approx \exp[-G^2/2\varepsilon^2] / (1+G)\sqrt{2\pi}, \tag{12}$$

The corresponding cumulative distribution is given by

$$P_{\eta}(z) \approx P(G/\varepsilon) - P(-1/\varepsilon). \tag{13}$$

Both expressions are defined for $z > -1/2\varepsilon$, a wider domain than either (2) or (3) in particular. The normalization of (12) and (13) requires that

$$\lim_{z \rightarrow \infty} P_{\eta}(z) = 1 - P(-1/\varepsilon) = P(1/\varepsilon) = 1' \tag{14}$$

For $1/\varepsilon \gg 1$, the asymptotic expansion of $P(1/\varepsilon)$ gives [16]

$$P(1/\varepsilon) = 1 - \varepsilon \exp(-1/2\varepsilon^2)(1 - \varepsilon^2 + \dots) / \sqrt{2\pi}. \tag{15}$$

Therefore,

$$P(1/\varepsilon) = 1 - O(\varepsilon e^{-1/2\varepsilon^2}). \tag{16}$$

The variation of $\beta = 1/P(1/\varepsilon) = 1/P(3/\lambda_3)$ with $\lambda_3 = 3\varepsilon$ is shown in Fig. 1. It is seen that that $\beta = 1$ for $0 < \lambda_3 < 0.6$, a much wider range than the typical range $0 < \lambda_3 < 0.3$ observed under oceanic conditions. For instance, if $\lambda_3 = 0.3$, then $\beta = 1 + O(10^{-16})$. For $\lambda_3 = 0.6$, $\beta = 1 + O(10^{-7})$. Therefore, there is essentially no need for introducing any normalization in (12) or (13) for oceanic applications.

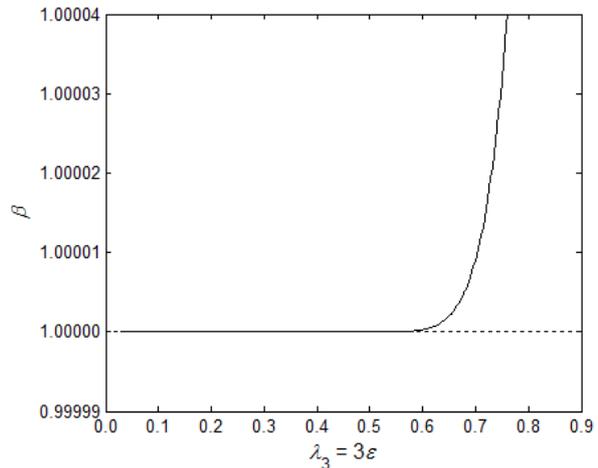


Fig. 1 Variation of normalization parameter β with $\lambda_3 = 3\varepsilon$

The mean of η implied by (13) is given by

$$\langle \eta \rangle = \varepsilon/2 + O(e^{-1/2\varepsilon^2}) \approx \varepsilon/2. \tag{17}$$

The approximate density (13) can now be centralized to zero mean by replacing z with $z + \varepsilon/2$ to obtain

$$p_{\eta}(z) \approx \exp[-H^2/2\varepsilon^2] / (1+H)\sqrt{2\pi}, \tag{18}$$

where $z > -(1 + \varepsilon^2) / 2\varepsilon$, and

$$H = \sqrt{1 + 2\varepsilon(z + \varepsilon/2)} - 1. \tag{19}$$

The corresponding cumulative distribution is obtained by

replacing G in (14) with H as

$$P_\eta(z) \approx P(H/\varepsilon) - P(-1/\varepsilon) \sim P(H/\varepsilon). \quad (20)$$

The mean-square of η follows after some algebra as

$$\langle \eta^2 \rangle = 1 + 3 \frac{\varepsilon^2}{4} + O(e^{-1/2\varepsilon^2}/\varepsilon) \approx 1 + O(\varepsilon^2). \quad (21)$$

So, the corresponding variance and *rms* are given by, respectively,

$$\text{var}(\eta) = 1 + \varepsilon^2/2 = 1 + O(\varepsilon^2), \quad (22)$$

and

$$\sigma_\eta = 1 + \varepsilon^2/4 = 1 + O(\varepsilon^2). \quad (23)$$

Finally, the skewness coefficient associated with the final form of the approximate density (19) is given by

$$\langle \eta^3 \rangle = 3\varepsilon + O(e^{-1/2\varepsilon^2}/\varepsilon^2) \approx \lambda_3, \quad (25)$$

as it is to be expected.

III. COMPARISONS

The comparisons between the final form of present approximation (18) with (2), (3) and (4) are shown in Fig. 2, where $\lambda_3 = 0.3$. Numerical comparisons indicate that the present approximation is by and large the same as the exact density such that the differences between the two cannot be differentiated in this figure. It is evident that the present approximation (18) is also the same as (2) except over large negative values at and beyond $z = -3/8\varepsilon = -3.75$, where (2) has a singularity. The differences between (2) and/or (18) and Gram-Charlier series (3) are more noticeable, particularly over the range of relatively large negative as well as positive values. Further, (3) becomes negative for $z < -3.08$. In contrast with (2) and/or (3), (18) is well-behaved and valid up to $z = -5.05$, where it also has a singularity. In oceanic applications where $0 < \lambda_3 < 0.3$, it has already been shown above that the domain $-(1 + \varepsilon^2)/2\varepsilon < z < \infty$ of (18) contains practically the whole probability mass over it.

The comparisons with oceanic data will consider two sets of wave measurements. These datasets represent oceanic waves measured during severe storms, one set comprising surface measurements collected by a wave staff from an offshore platform in the Gulf of Mexico during the passage of hurricane Camille in August, 1969, and the other measurements gathered by a Marex radar from the Tern platform in the northern North Sea in January, 1993. These data will be denoted as Camille and Tern93, respectively.

Both Tern93 and Camille are characterized with nonlinear distortions and rather large waves with heights nearly as large

as 25 m in Tern93 and 27 m in Camille. Tern93 is suggestive of a relatively steady sea state whereas Camille is conspicuously not so, judging from the temporal variation of σ from about 1.85 m almost linearly to about 3.5 m as the hurricane approached the measurement site.

There is no rigorous analytical proposition of practical value for analyzing the statistical structure of non-stationary random functions based on a single realization. To compensate for the general temporal variability of σ , surface displacements from the mean-zero level are estimated in both cases from 30-minute consecutive segments and scaled with the corresponding segmental estimates of σ . The parameters in Table I represent overall averages derived from the segmental estimates. The variations of segmental estimates of σ and λ_3 are displayed in Fig. 3.

The comparisons of the density estimates observed in Tern93 and Camille to the Gaussian form of (4) and the present approximation (18) are shown in Figs. 4 and 5, respectively. It appears that the present approximation does reasonably well in describing the observed data.

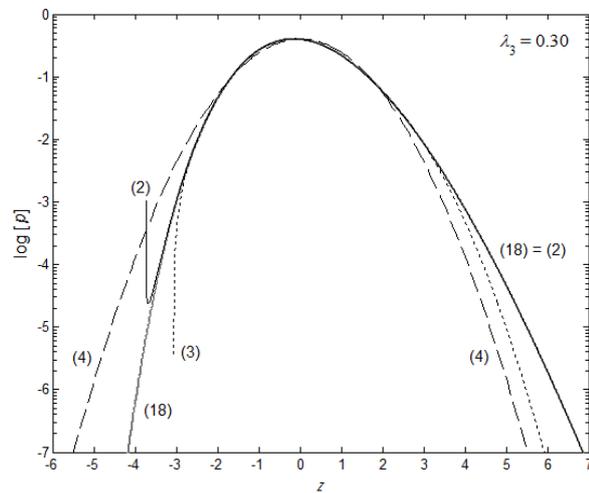


Fig. 2 Comparisons of probability densities predicted from (2), (3), present approximation (19) and Gaussian density (5) for $\lambda_3 = 0.30$

TABLE I
PARTICULARS OF OCEANIC MEASUREMENTS

description	Tern93	Camille
duration (h)	9.00	6.30
Sampling rate (Hz)	5.12	10.0
σ_{avg} (m)	3.02	2.61
λ_3 avg	0.17	0.15
$\varepsilon \sim \lambda_3 \text{ avg} / 3$	0.06	0.05

IV. CONCLUSIONS

The second-order narrowband model describing long-crested deep-water waves was used to develop a simple and yet fairly accurate approximation to the probability structure of sea-surface elevations.

Comparisons with the two sets of measurements representative of nonlinear storm seas indicate that the approximation developed works fairly well in representing the

observed data. The comparisons of the approximation with two presently popular approximations also suggest that it has advantages over these in terms of its relative simplicity and, in particular, its validity over a wider range of surface elevations.

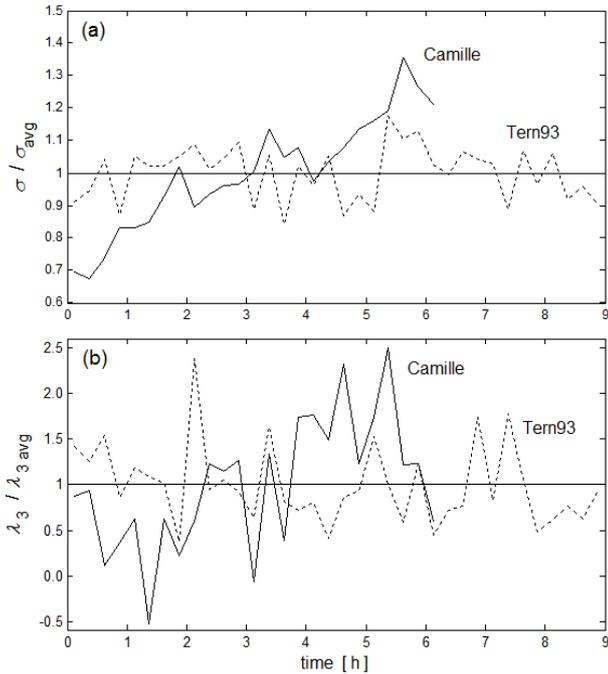


Fig. 3 Segmental variations of (a) σ and (b) λ_3 in Tern93 and Camille

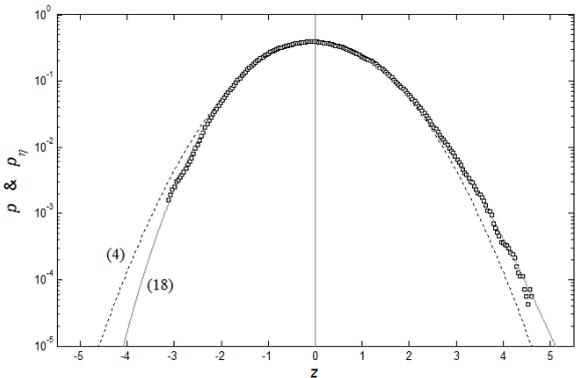


Fig. 4 Tern93: the density estimates observed (points) compared to Gaussian density (5) and present approximation (19)

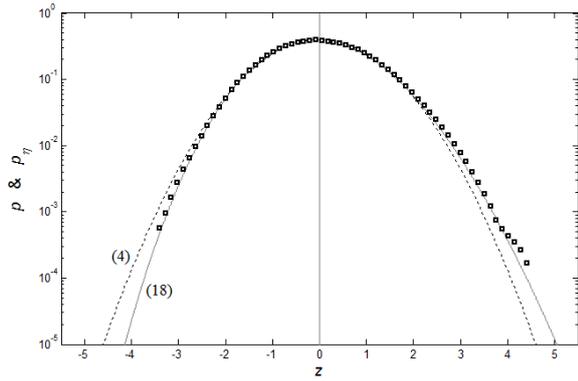


Fig. 5 Same as Fig 4 except for Camille

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