A Relationship between Two Stabilizing Controllers and Its Application to Two-Stage Compensator Design without Coprime Factorizability — Single-Input Single-Output Case —

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Abstract—In this paper, we first show a relationship between two stabilizing controllers, which presents an extended feedback system using two stabilizing controllers. Then, we apply this relationship to the two-stage compensator design. In this paper, we consider single-input single-output plants. On the other hand, we do not assume the coprime factorizability of the model. Thus, the results of this paper are based on the factorization approach only, so that they can be applied to numerous linear systems.

Keywords—Relationship among Compensators, Two-Stage Compensator Design, Parametrization of Stabilizing Controllers, Factorization Approach

I. INTRODUCTION

THE factorization approach to control systems has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc.[1], [2], [3], [4]. Hence the result given in this paper will be able to a number of models in addition to the multidimensional systems. In factorization approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. This approach leads to conceptually simple and computationally tractable solutions to many important and interesting problems[5]. A transfer function of this approach is considered as the ratio of two stable causal transfer functions. For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer functions, which is satisfied by transfer functions over the principal ideal domains or the Bézout domains.

In some design problems, one uses a so-called *two-state procedure* for selecting an appropriate stabilizing compensator[5]. Given a plant, the first stage consists of selecting a stabilizing compensator for the plant. The second stage consists of selecting a stabilizing controller for the new closed-loop system that also achieves some other design objectives such as decoupling, sensitivity minimization, etc. The rationale behind this procedure is that the design problems are often easier to solve when the plant is stable. It is known that there are

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models such that some stabilizable plants do not have coprime factorizations [6]. For some models of control systems, it is not known yet whether or not a stabilizable plant always has its doubly coprime factorization.

In this paper, we restrict our attention to single-input and single-output plants. Our first result is to give a relationship between two stabilizing controllers, which presents an extended feedback system using two stabilizing controllers. Then, we apply this relationship to the two-stage compensator design. This two-stage compensator design is different from the original two-stage compensator design[5]. The original one considered one input and one output. On the other hand, this two-stage compensator design uses two inputs and two outputs. Further, all stabilizing controllers can be obtained.

II. PRELIMINARIES

The stabilization problem considered in this paper follows that of [7], and [8], who consider the feedback system Σ [5, Ch.5, ig. 5.1] as in Fig. 1. For further details the reader is referred to [5], [7], [8], and [9].
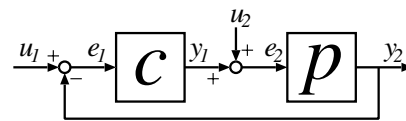

Fig. 1   Feedback system Σ

We consider that the set of stable causal transfer functions is an integral domain, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \neq 0\}$. This $\mathcal{F}$ is considered as the set of all possible transfer functions. Matrices over $\mathcal{F}$ are transfer matrices. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Define the subsets $\mathcal{P}$ and $\mathcal{P}_S$ of $\mathcal{F}$ as follows: $\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A} \backslash \mathcal{Z}\}$, $\mathcal{P}_S = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A} \backslash \mathcal{Z}\}$. Then, every transfer function in $\mathcal{P}$ ($\mathcal{P}_S$) is called *causal* (*strictly causal*). Analogously, if every entry of a transfer matrix is in $\mathcal{P}$ ($\mathcal{P}_S$), the transfer matrix is called *causal* (*strictly causal*).

Throughout the paper, the plant we consider has single-input and single-output, and its transfer function, which is also called a *plant* itself simply, is denoted by p and belongs to $\mathcal{P}$. We

can always represent p in the form of a fraction $p = nd^{-1}$, where $n \in \mathcal{A}$ and $d \in \mathcal{A}$ with nonzero d.

For $p \in \mathcal{P}$ and c, a matrix $H(p,c) \in \mathcal{F}^{2 \times 2}$ is defined as

$$H(p,c) := \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \quad (1)$$

provided that $1 + pc$ is a nonzero of $\mathcal{A}$. This $H(p,c)$ is the transfer matrix from $\begin{bmatrix} u_1^t & u_2^t \end{bmatrix}^t$ to $\begin{bmatrix} e_1^t & e_2^t \end{bmatrix}^t$ of the feedback system Σ. If $1 + pc$ is a nonzero of $\mathcal{A}$ and $H(p,c) \in \mathcal{A}^{2 \times 2}$, then we say that the plant p is *stabilizable*, p is *stabilized* by c, and c is a *stabilizing controller* of p. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [8].

It is known that $W(p,c)$ defined below is over $\mathcal{A}$ if and only if $H(p,c)$ is over $\mathcal{A}$:

$$W(p,c) := \begin{bmatrix} c(1+pc)^{-1} & -pc(1+cp)^{-1} \\ pc(1+pc)^{-1} & p(1+cp)^{-1} \end{bmatrix}. \quad (2)$$

This $W(p,c)$ is the transfer matrix from $\begin{bmatrix} u_1^t & u_2^t \end{bmatrix}^t$ to $\begin{bmatrix} y_1^t & y_2^t \end{bmatrix}^t$.

We employ the symbols used in [10] and [7]. Also we will denote by $\mathcal{S}(p)$ the set of stabilizing controllers of p.

III. A RELATIONSHIP BETWEEN TWO STABILIZING CONTROLLERS

Let p be a causal plant ($p \in \mathcal{P}$). Here we consider two stabilizing controllers c_0 and c_1 of p (c_0, $c_1 \in \mathcal{F}$). Using c_0 only we consider $w(p, c_0)$, a feedback system. The relationship we give here is that the this $w(p, c_0)$ can be stabilized by the following matrix:

$$\begin{bmatrix} p & 1 + 2pc_1 \\ 1 & c_1 \end{bmatrix}. \quad (3)$$

We note that c_0 and c_1 can be independently selected. Even so (3) is a stabilizing controller of $W(p,c)$.

This is stated as following theorem:

Theorem 1: Let p be a causal plant. Also let c_0 and c_1 of p be stabilizing controllers of p. Then $w(p, c_0)$ is stabilized by the matrix of (3).

We note that there is no restriction between c_0 and c_1 in this theorem.

Before starting the proof of this theorem, we present the following:

Theorem 2: ([11, Theorem 2.4]) Let F be a field. Let A is a square matrix of F with size $n_1 + n_2$ and is decomposed into as follows:

$$\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := A \end{array}$$

Assume that A_{11} is nonsingular. Then, A is nonsingular if and only if $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is also nonsingular.

In the case where two matrices A_{11} and $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are nonsingular, then we have the following:

(i) The $(1,1)$-block of A^{-1} is

$$A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}.$$

(ii) The $(1,2)$-block of A^{-1} is

$$-A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}.$$

(iii) The $(2,1)$-block of A^{-1} is

$$-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}.$$

(iv) The $(2,2)$-block of A^{-1} is

$$-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}.$$

Proof: First we denote by C_1 the matrix in (3). Thus we need to show that

(i) $(I_2 + C_1 W(p, c_0))$ (or equivalently $(I_2 + W(p, c_0)C_1)$) is well-defined.

(ii) $W(W(p, c_0), C_1)$ is over $\mathcal{A}$.

First we show (i) and then (ii).

(i). Let N and d be a matrix over $\mathcal{A}$ and an element of $\mathcal{A}$, respectively, such that $p = Nd^{-1}$. Also let A and b be a matrix over $\mathcal{A}$ and an element of $\mathcal{A}$, respectively, such that $C_1 = Ab^{-1}$. We consider the following matrix:

$$\begin{bmatrix} (1+pc_0)^{-1} & -2p(1+c_0p)^{-1} + p(1+c_1p)^{-1} \\ -c_0(1+pc_0)^{-1} & 2c_0(1+pc_0)^{-1}p + (1+c_1p)^{-1} \end{bmatrix}. \quad (4)$$

The determinant of (4) is $(1 + c_1 p)^{-1}$, which is nonsingular. Hence (4) has its inverse.

Now we compute $I_2 + C_1 W(p, c_0)$:

$$\begin{aligned} & I_2 + C_1 W(p, c_0) \\ = \; & I_2 + \begin{bmatrix} p & 1 + 2pc_1 \\ 1 & c_1 \end{bmatrix} \\ & \times \begin{bmatrix} c_0(1+pc_0)^{-1} & -c_0p(1+c_0p)^{-1} \\ pc_0(1+pc_0)^{-1} & p(1+c_0p)^{-1} \end{bmatrix} \\ = \; & \begin{bmatrix} 1 + 2(1+pc_1)pc_0(1+pc_0)^{-1} \\ (p + 2pc_1p - pc_0p)(1+c_0p)^{-1} \\ c_0(1+pc_1)(1+pc_0)^{-1} \\ (1+c_1p)(1+c_0p)^{-1} \end{bmatrix}. \quad (5) \end{aligned}$$

Now we consider the multiplication of (4) and (5).

$$\begin{bmatrix} (1+pc_0)^{-1} & -2p(1+c_0p)^{-1} + p(1+c_1p)^{-1} \\ -c_0(1+pc_0)^{-1} & 2c_0(1+pc_0)^{-1}p + (1+c_1p)^{-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 + 2(1+pc_1)pc_0(1+pc_0)^{-1} \\ (p + 2pc_1p - pc_0p)(1+c_0p)^{-1} \\ c_0(1+pc_1)(1+pc_0)^{-1} \\ (1+c_1p)(1+c_0p)^{-1} \end{bmatrix}. \quad (6)$$

This multiplication results the identity of the size 2×2. Now we see that the matrix $(I_2 + C_1 W(p, c_0))$ is well-defined and its inverse is the matrix of (4).

(ii). We now turn to show that the transfer matrix $W(W(p, c_0), C_1)$ is over $\mathcal{A}$, which means that $w(p, c_0)$ is stabilized by the matrix of (3). Decompose $W(W(p, c_0), C_1)$ into

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = W(W(p, c_0), C_1)$$

as follows

$$\begin{aligned} M_{11} &= C_1(I_2 + W(p, c_0)C_1)^{-1}, \\ M_{12} &= -C_1 W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1}, \\ M_{21} &= W(p, c_0)C_1(I_2 + W(p, c_0)C_1)^{-1}, \\ M_{22} &= W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1}. \quad (7) \end{aligned}$$

In the following, we show every M_{11} to M_{22} is over $\mathcal{A}$. First we consider M_{12}, which is as follows:

$$
\begin{aligned}
M_{11} &= C_1(I_2 + W(p, c_0)C_1)^{-1} \qquad (8)\\
&= (I_2 + C_1 W(p, c_0))^{-1} C_1\\
&= \begin{bmatrix} -p(1 + c_0 p)^{-1} + p(1 + c_1 p)^{-1} \\ (1 + pc_0)^{-1} + p(1 + c_1 p)^{-1} c_1 \\ c_0(1 + pc_0)^{-1} p + (1 + c_1 p)^{-1} \\ - c_0(1 + pc_0)^{-1} + (1 + c_1 p)^{-1} c_1 \end{bmatrix}.
\end{aligned}
$$

Then we consider M_{12}, which is as follows:

$$
\begin{aligned}
M_{12} &= -C_1 W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1}\\
&= -I_2 + I_2 - C_1 W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1}\\
&= -I_2 + (I_2 + C_1 W(p, c_0))^{-1}. \qquad (9)
\end{aligned}
$$

The first and the second terms of the right hand side is over $\mathcal{A}$. Hence M_{12} is over $\mathcal{A}$. Then we consider M_{21}, which is as follows:

$$
\begin{aligned}
M_{21} &= W(p, c_0)C_1(I_2 + W(p, c_0)C_1)^{-1}\\
&= W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1} C_1\\
&= W(p, c_0)M_{11}. \qquad (10)
\end{aligned}
$$

The last one is M_{22}, which is as follows:

$$
M_{22} = W(p, c_0)(I_2 + C_1 W(p, c_0))^{-1}. \qquad (11)
$$

Both $W(p, c_0)$ and $(I_2 + C_1 W(p, c_0))^{-1}$ are over $\mathcal{A}$.

Hence $W(W(p, c_0), C_1)$ is over $\mathcal{A}$. ∎

IV. TWO-STAGE COMPENSATOR DESIGN

In some design problems, one uses a so-called *two-stage procedure* for selecting an appropriate stabilizing compensator[5]. Given a plant p, the first stage consists of selecting a stabilizing compensator for p. Let $c_0 \in \mathcal{S}(p)$ denote this compensator (that is, an arbitrary but fixed compensator of p) and define $p_1 = p(1 + c_0 p)^{-1}$. The second stage consists of selecting a stabilizing controller for p_1 that also achieves some other design objectives such as decoupling, sensitivity minimization, etc. The rationale behind this procedure is that the design problems are often easier to solve when the plant is stable. The resulting configuration with its inner and outer loops is shown in Fig. 2.
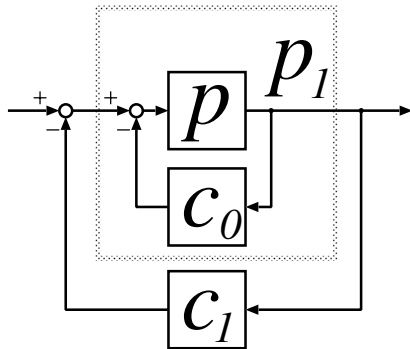
Fig. 2 Two-Stage Compensator Design (y_2 to u_2)

The following show that, in general, the two-stage compensator design based on Fig. 2 cannot give all stabilizing controllers.

Theorem 3 ([12]): Let p denote a causal plant of $\mathcal{P}$ and c_0 a causal stabilizing controller of p ($c_0 \in \mathcal{P}$). Further let p_1 be $p(1 + c_0 p)^{-1}$. Denote by $c_0 + \mathcal{S}(p_1)$ the following set:

$$
\{c_0 + c_1 \,|\, c_1 \in \mathcal{S}(p_1)\}.
$$

Then

$$
c_0 + \mathcal{S}(p_1) \subset \mathcal{S}(p), \qquad (12)
$$

with equality holding if and only if $c_0 \in \mathcal{A}$.

Theorem 4 ([12]): Let p, c_0, p_1 be as in Theorem 3. Let n, d, y, x be in $\mathcal{A}$ such that

$$
\begin{cases} p = nd^{-1}, & c_0 = yx^{-1}, \\ ny + dx = 1. \end{cases} \qquad (13)
$$

Then we have

$$
c_0 + \mathcal{S}(p_1) =
$$
$$
\{(x - rn)^{-1}(y + rd) \,|\, r = r_1 x^2, r_1 \in \mathcal{A}, (x - rn) \neq 0\}. \quad (14)
$$

By Theorem 3, we see that the sum of c_0 and a stabilizing controller of p_1, say c_1, is again a stabilizing controller of p. This sum, a stabilizing controller of p, is the parallel allocation of c_0 and c_1, as shown in Fig. 3. However, this cannot give all stabilizing controllers if $c_0 \notin \mathcal{A}$. To solve this problem we will apply Theorem 1 to the two-stage compensator design.
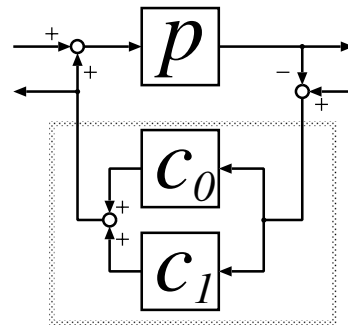
Fig. 3 Composite Stabilized Feedback with c_0 and c_1

V. APPLICATION TO TWO-STAGE COMPENSATOR DESIGN

This section investigates the full-feedback of Fig. 4, to which we apply the result of the last section.

Theorem 5:

$$
\mathcal{S}(p) = \{(1 + w_{12})^{-1} w_{11} |
$$

(a) $R_1 \in \mathcal{A}^{2 \times 2}$,

(b) $(I_2 - R_1 W(p, c_0))$ is nonsingular,

(c) $C_1 := (I_2 - R_1 W(p, c_0))^{-1} R_1$,

(d) $\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} := \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$
W(W(p, c_0), C_1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},
$$

(e) $1 + w_{12}$ is nonsingular}. \qquad (15)

Proof: "⊂"-part. Let c_0 and c_1 be stabilizing controllers of p (c_0, $c_1 \in \mathcal{S}(p)$). Let C_1 be

$$C_1 = \begin{bmatrix} p & 1 + 2pc_1 \\ 1 & c_1 \end{bmatrix}.$$

Then by Theorem 1, $W(W(p, c_0), C_1)$ is over $\mathcal{A}$. This $W(W(p, c_0), C_1)$ is calculated as follows:

$$W(W(p, c_0), C_1) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix},$$

where

$$\begin{aligned}
m_{11} &= -p(1 + c_0 p)^{-1} + p(1 + c_1 p)^{-1}, \\
m_{12} &= (1 + pc_0)^{-1} + pc_1(1 + pc_1)^{-1}, \\
m_{13} &= -1 + (1 + pc_0)^{-1}, \\
m_{14} &= -2p(1 + c_0 p)^{-1} + p(1 + c_1 p)^{-1}, \\
m_{21} &= c_0(1 + pc_0)^{-1}p + (1 + c_1 p)^{-1}, \\
m_{22} &= -c_0(1 + pc_0)^{-1} + (1 + c_1 p)^{-1}c_1, \\
m_{23} &= -c_0(1 + pc_0)^{-1}, \\
m_{24} &= -1 + 2c_0(1 + pc_0)^{-1}p + (1 + c_1 p)^{-1}, \\
m_{31} &= -c_0(1 + pc_0)^{-1}, \\
m_{32} &= c_0(1 + pc_0)^{-1}, \\
m_{33} &= c_0(1 + pc_0)^{-1}, \\
m_{34} &= -2c_0 p(1 + c_0 p)^{-1}, \\
m_{11} &= p(1 + c_1 p)^{-1}, \\
m_{42} &= p(1 + c_1 p)^{-1}c_1, \\
m_{43} &= 0, \\
m_{44} &= p(1 + c_1 p)^{-1}.
\end{aligned}$$

Consider Condition (c):

$$\begin{aligned}
&\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} W(W(p, c_0), C_1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} m_{22} + m_{32} + m_{23} + m_{33} & m_{24} + m_{34} \\ m_{42} + m_{43} & m_{44} \end{bmatrix} \quad (16)
\end{aligned}$$

Thus,

$$\begin{aligned}
w_{11} &= m_{22} + m_{32} + m_{23} + m_{33} \\
&= -c_0(1 + pc_0)^{-1} + (1 + c_1 p)^{-1}c_1 \\
&\quad + c_0(1 + pc_0)^{-1} - c_0(1 + pc_0)^{-1} \\
&\quad + c_0(1 + pc_0)^{-1} \\
&= (1 + c_1 p)^{-1}c_1, \quad (17)
\end{aligned}$$

$$\begin{aligned}
w_{12} &= m_{24} + m_{34} \\
&= -1 + 2c_0(1 + pc_0)^{-1}p \\
&\quad + (1 + c_1 p)^{-1} - 2c_0 p(1 + c_0 p)^{-1} \\
&= -1 + (1 + c_1 p)^{-1}, \quad (18)
\end{aligned}$$

$$\begin{aligned}
w_{21} &= m_{42} + m_{43} \\
&= p(1 + c_1 p)^{-1}c_1, \quad (19)
\end{aligned}$$

and

$$\begin{aligned}
w_{22} &= m_{44} \\
&= p(1 + c_1 p)^{-1}. \quad (20)
\end{aligned}$$

Hence, w_{11} to w_{22} are all in $\mathcal{A}$. Thus,

$$\begin{aligned}
1 + w_{12} &= 1 + (-1 + (1 + c_1 p)^{-1}) \\
&= (1 + c_1 p)^{-1} \quad (21)
\end{aligned}$$

This is nonsingular, by which Condition (e) in Theorem 5 is satisfied.

Further $(1 + w_{12})^{-1}w_{11}$ is now $((1 + c_1 p)^{-1})^{-1}((1 + c_1 p)^{-1}c_1)$, which is equal to c_1. That is, c_1 is an element of the right hand of (15).

Let $R_1 \in \mathcal{A}^{2 \times 2}$ of Condition (a) in Theorem 5 be

$$R_1 = C_1(I_2 + W(p, c_0)C_1)^{-1}. \quad (22)$$

This is equal to (8), so that this R_1 is over $\mathcal{A}$.

Now, $I_2 - R_1 W(p, c_0)$ is equal to $(I_2 + C_1 W(p, c_0))^{-1}$, which means that $I_2 - R_1 W(p, c_0)$ is nonsingular. Thus Condition (b) of Theorem 5 is satisfied. Also, from (22), we have Condition (c) of Theorem 5.

"⊃"-part. Let c_0 be stabilizing controllers of p (c_0, $c_1 \in \mathcal{S}(p)$). Let $R_1 \in \mathcal{A}^{2 \times 2}$, $C_1 \in \mathcal{F}^{2 \times 2}$ be arbitrary matrices that satisfy Conditions (a) to (e) of Theorem 5.

First we decompose r_1 as follows:

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} := r_1.$$

Then,

$$\begin{aligned}
&\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} W(W(p, c_0), C_1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (1 + c_0 p)^{-1}(-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22})(1 + pc_0)^{-1} \\ (1 + c_0 p)^{-1}(c_0 r_{11} c_0 + r_{21} c_0 - c_0 r_{12} - r_{22})p(1 + c_0 p)^{-1} \\ p(1 + c_0 p)^{-1}(-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22})(1 + pc_0)^{-1} \\ p(1 + c_0 p)^{-1}(c_0 r_{11} c_0 + r_{21} c_0 - c_0 r_{12} - r_{22})p(1 + c_0 p)^{-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} c_0(1 + pc_0)^{-1} & -c_0 p(1 + c_0 p)^{-1} \\ pc_0(1 + pc_0)^{-1} & p(1 + c_0 p)^{-1} \end{bmatrix}.
\end{aligned}$$

Now, we have

$$\begin{aligned}
w_{11} &= (1 + c_0 p)^{-1}(-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22}) \\
&\quad \times (1 + pc_0)^{-1} + c_0(1 + pc_0)^{-1}, \\
w_{12} &= (1 + c_0 p)^{-1}(c_0 r_{11} c_0 + r_{21} c_0 - c_0 r_{12} - r_{22}) \\
&\quad \times p(1 + c_0 p)^{-1} - c_0 p(1 + c_0 p)^{-1}, \\
w_{21} &= p(1 + c_0 p)^{-1}(-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22}) \\
&\quad \times (1 + pc_0)^{-1} + pc_0(1 + pc_0)^{-1}, \\
w_{22} &= p(1 + c_0 p)^{-1}(c_0 r_{11} c_0 + r_{21} c_0 - c_0 r_{12} - r_{22}) \\
&\quad \times p(1 + c_0 p)^{-1} + p(1 + c_0 p)^{-1}.
\end{aligned}$$

Let c_{New} be

$$
\begin{aligned}
c_{\text{New}} \\
&= (1 + w_{12})^{-1} w_{11} \\
&= ((c_0 r_{11} c_0 + r_{21} c_0 - c_0 r_{12} - r_{22}) p (1 + c_0 p)^{-1} + 1)^{-1} \\
&\quad ((-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22})(1 + p c_0)^{-1} + c_0) \quad (23)
\end{aligned}
$$

Further let $Q = -c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22}$. Then

$$
c_{\text{New}} = (-Qp(1 + c_0 p)^{-1} + 1)^{-1}(Q(1 + pc_0)^{-1} + c_0) \quad (24)
$$

In the following we show that c_{New} is in $\mathcal{S}(p)$. To do so, we show that $(1 + c_{\text{New}} p)^{-1}$, $c_{\text{New}}(1 + c_{\text{New}} p)^{-1}$, $p(1 + c_{\text{New}} p)^{-1}$ are in $\mathcal{A}$. Now

$$
\begin{aligned}
&(1 + c_{\text{New}} p)^{-1} \\
&= (1 + c_0 p)^{-1}(-Qp(1 + c_0 p)^{-1} + 1) \\
&= \underline{(1 + c_0 p)^{-1} c_0 r_{11} c_0 p (1 + c_0 p)^{-1}} \\
&\quad + \underline{(1 + c_0 p)^{-1} r_{21} c_0 p (1 + c_0 p)^{-1}} \\
&\quad - \underline{(1 + c_0 p)^{-1} c_0 r_{12} p (1 + c_0 p)^{-1}} \\
&\quad - \underline{(1 + c_0 p)^{-1} r_{22} p (1 + c_0 p)^{-1}} \\
&\quad + \underline{(1 + c_0 p)^{-1}}
\end{aligned}
$$

Every r_{ij} and every underlined expression above are in $\mathcal{A}$. Hence $(1 + c_{\text{New}} p)^{-1}$ is in $\mathcal{A}$.

$$
\begin{aligned}
&p(1 + c_{\text{New}} p)^{-1} \\
&= -p(1 + c_0 p)^{-1} Q p (1 + c_0 p)^{-1} + p(1 + c_0 p)^{-1} \\
&= -p(1 + c_0 p)^{-1}(-c_0 r_{11} c_0 - r_{21} c_0 + c_0 r_{12} + r_{22}) \\
&\quad \times p(1 + c_0 p)^{-1} + p(1 + c_0 p)^{-1} \\
&= \underline{p(1 + c_0 p)^{-1} c_0 r_{11} c_0 p (1 + c_0 p)^{-1}} \\
&\quad + \underline{p(1 + c_0 p)^{-1} r_{21} c_0 p (1 + c_0 p)^{-1}} \\
&\quad - \underline{p(1 + c_0 p)^{-1} c_0 r_{12} p (1 + c_0 p)^{-1}} \\
&\quad - \underline{p(1 + c_0 p)^{-1} r_{22} p (1 + c_0 p)^{-1}} \\
&\quad + \underline{p(1 + c_0 p)^{-1}}
\end{aligned}
$$

Every r_{ij} and every underlined expression above are in $\mathcal{A}$. Hence $p(1 + c_{\text{New}} p)^{-1}$ is in $\mathcal{A}$.

$$
\begin{aligned}
&(1 + c_{\text{New}} p)^{-1} c_{\text{New}} \\
&= (1 + c_0 p)^{-1}(-Qp(1 + c_0 p)^{-1} + 1) c_{\text{New}} \\
&= (1 + c_0 p)^{-1} Q(1 + pc_0)^{-1} + (1 + c_0 p)^{-1} c_0 \\
&= -\underline{(1 + c_0 p)^{-1} c_0 r_{11} c_0 (1 + pc_0)^{-1}} \\
&\quad - \underline{(1 + c_0 p)^{-1} r_{21} c_0 (1 + pc_0)^{-1}} \\
&\quad + \underline{(1 + c_0 p)^{-1} c_0 r_{12} (1 + pc_0)^{-1}} \\
&\quad + \underline{(1 + c_0 p)^{-1} r_{22} (1 + pc_0)^{-1}} \\
&\quad + \underline{(1 + c_0 p)^{-1} c_0}
\end{aligned}
$$

Every r_{ij} and every underlined expression above are in $\mathcal{A}$. Hence $(1 + c_{\text{New}} p)^{-1} c_{\text{New}}$ is in $\mathcal{A}$.

Hence c_{New} is in $\mathcal{S}(p)$. ∎

VI. CONCLUSION

In this paper, we have shown a relationship between two stabilizing controllers (Theorem 1). This result gives that for any stabilizing controller c_0 of p, the feedback system $W(p, c_0)$ of p and c_0 is stabilized by the matrix

$$
\begin{bmatrix} p & 1 + 2pc_1 \\ 1 & c_1 \end{bmatrix},
$$

where c_1 is also any stabilizing controller of p.

We have applied this relationship to the two-stage compensator design. This two-stage compensator design uses two inputs u_1 and u_2 and two outputs y_1 and y_2 (Theorem 5). This gives the parametrization of all stabilizing controllers.

The results of this paper are based on the factorization approach, so that they can be applied to numerous linear systems. In this paper, we do not have considered multi-input multi-output case, which should be investigated as a future work.

Fig. 4   Feedback System with c_0 and C_1

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