A Refined Nonlocal Strain Gradient Theory for Assessing Scaling-Dependent Vibration Behavior of Microbeams

Xiaobai Li, Li Li, Yujin Hu, Weiming Deng, Zhe Ding

Abstract—A size-dependent Euler–Bernoulli beam model, which accounts for nonlocal stress field, strain gradient field and higher order inertia force field, is derived based on the nonlocal strain gradient theory considering velocity gradient effect. The governing equations and boundary conditions are derived both in dimensional and dimensionless form by employed the Hamilton principle. The analytical solutions based on different continuum theories are compared. The effect of higher order inertia terms is extremely significant in high frequency range. It is found that there exists an asymptotic frequency for the proposed beam model, while for the nonlocal strain gradient theory the solutions diverge. The effect of strain gradient field in thickness direction is significant in low frequencies domain and it cannot be neglected when the material strain length scale parameter is considerable with beam thickness. The influence of each of three size effect parameters on the natural frequencies are investigated. The natural frequencies increase with the increasing material strain gradient length scale parameter or decreasing velocity gradient length scale parameter and nonlocal parameter.

Keywords—Euler-Bernoulli Beams, free vibration, higher order inertia, nonlocal strain gradient theory, velocity gradient.

I. INTRODUCTION

IN recent years, various investigations have been carried out to study wave propagation [1]–[3], bending [4], [5], buckling [6]–[9] and free vibration [5], [10]–[13] of the micro/nano structures in which the continuum mechanic theories have been employed. For these micro structures, such as actuators, sensors, microscopes, micro/nano-electro-mechanical systems (MEMS/NEMS), the effects of a very small scale over which the neighboring material particles or constituents interact should be considered. Beams are core structures widely used in these micro systems and the dynamic properties are closely related to the microstructures. To accurately evaluate the dynamic behaviors of MEMS/NEMS, it's necessary to study the dynamic properties of microbeams.

In small scales, the vibration behavior of microbeams cannot be predicted accurately by employing the classical elasticity theory. Recently, a series of nonclassical mechanics theories considering the size-effect, including nonlocal elasticity theory

Li Li, Yujin Hu, Weiming Deng and Zhe Ding are with State Key Laboratory of Digital Manufacturing Equipment and Technology, School of Mechanical Science and Engineering, Huazhong University of Science and Technology, Wuhan 430074, China.
Xiaobai Li is with State Key Laboratory of Digital Manufacturing Equipment and Technology, School of Mechanical Science and Engineering, Huazhong University of Science and Technology, Wuhan 430074, China (e-mail: lixiaobai@hust.edu.cn).

[14], strain gradient theory [15], [16] and nonlocal strain gradient theory [17], were developed and employed to simulate the dynamic behaviors of micro structures. The nonlocal elasticity theory (NET) states that the total stress at one point (a reference point) is a function of the strain at all points in the domain. Some studies [18]–[23] showed that the nonlocal size effect plays an important role in dynamic properties especially in high frequency range. However, NET cannot always be accurate when predicting the size-dependent dynamic behavior of microbeams. For instance, its capability of identifying size-dependent stiffness is limited as illustrated in several articles [24]–[27].

Yang et al. [28] have proposed a modified couple stress theory by considering the effects of the curvature tensor conjugated with couple stress tensor. Thus, a new higher order equilibrium equation (moments of couples equilibrium equation) is considered as a supplementary of classical equilibrium equations. The strain gradient theory (SGT) state that not only the classical strain tensor but also the derivatives of the strain tensor should be taken into consideration. Fleck and Hutchinson [29], [30] used and simplified Mindlin's formulations by only considering the first derivative of the strain tensor. Compared with the couple stress theory, the SGT contains some additional higher order stress components beside the classical stress and couple stress. It means that the couple stress theory is a special case of the SGT.

The higher order inertia term, namely velocity gradients, which are introduced by Mindlin [16], should be included in the governing equations, enlightened by the strain gradient theory (SGT), especially for microbeams [31]–[33]. Under this assumption, the kinetic energy depends not only on the velocity but also on the velocity gradient. In a series of studies [34]–[36], considering the higher order inertia effect in wave motion and the related dispersion problems gave results in consistent with those of atomic-lattice models. It was found that the higher order inertia effect is indispensable for wave propagating at high frequencies.

Lim et al. [17] considering both nonlocal elasticity effect and the effect of the gradient of strain tensor, proposed a nonlocal strain gradient theory (NSGT) to study the mechanics behaviors of solids at micro and nano scales. To the best knowledge of authors, the NSGT is probably the most successful theory to study the static and dynamic behavior considering the size effects. Li et al. [1]–[3], [5], [7], [12], [13] have studied mechanical and dynamical behaviors of microbeams, such as wave propagation properties, buckling

behaviors and free vibration characteristics based on NSGT. However, to the best knowledge of the authors, the studies considering size-effect in both length direction and other directions are rare. Therefore, there is a desperate need to discuss the influence of size-effect in thickness direction of a micro plane beam structure.

In the present work, the higher order inertia effect as well as the strain gradient through thickness direction is considered in the size-dependent Euler–Bernoulli beams models based on the NSGT (i.e. the nonlocal elasticity effect and strain gradient effect are also considered). In this model, microbeams are assumed to be thin and long enough. Besides, the nonlocal elasticity effect, strain gradient effect and the higher order inertia effect are combined. In Section II, Hamilton principle is employed to derive the balance equations and the corresponding boundary conditions. The dimensionless formulation and boundary conditions are explicitly expressed. In Section III, a kind of simply supported boundary condition is given and the analytical solutions of natural frequencies are obtained. The influences of size effects in thickness direction and higher order inertia are discussed respectively. Section IV compared the solution based on different kinds of continuum theory (namely classical elasticity theory, nonlocal elasticity theory, strain gradient theory, nonlocal strain gradient theory and the proposed theory). The effects of different length scale parameters are discussed respectively. Conclusions are drawn in Section V.

II. EQUATIONS OF MOTION OF SIZE-DEPENDENT EULER–BERNOULLI BEAMS

In MEMS/NEMS engineering, there are many beam-like structures used as key components of the micro-/nano-systems. Usually, these beam-like components are treated as micro-beam structures during dynamic/mechanic behaviors analysis. In these studies, the structures are assumed as lines or curves and the mechanical quantities, such as kinetic and potential energies, are simplified from 3-dimensional(3D) form by integrating on cross-section. In this section, the 3D equations of kinetic and potential energy are expressed explicit, firstly. The basic assumption of Euler–Bernoulli beam displacement fields is employed to simplifying the energy equations. Then, Hamilton's principle is employed to obtain the equilibrium equations and governing equations. Furthermore, the dimensionless form of governing equations and corresponding boundary conditions are given.

A. Variational Formulation of the Nonlocal Gradient Theory

The strain energy U of the continuum body based on nonlocal strain gradient theory has been expressed in 3D form as [17]:

$$U = \int_V \left(\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\tau} \vdots \boldsymbol{\eta} \right) \mathrm{d}V$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are nonlocal Cauchy stress tensor and classical strain tensor, respectively. $\boldsymbol{\eta} = \nabla \boldsymbol{\varepsilon}$ denotes the first gradient of strain tensor which is work-conjugate to higher order nonlocal

stress tensor $\boldsymbol{\tau}$. " : " and " $\vdots$ " denote, respectively, double dot product and triple dot product. V denotes the region occupied by the material body. The classical symmetric strain component ε_{ij} and strain gradient component η_{kij} can be expressed as

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$
$$\eta_{kij} = \varepsilon_{ij,k}$$

here coma denotes the partial derivative. By considering the nonlocal assumption, the constitutive relations can be expressed as

$$\boldsymbol{\sigma} = \int_V \alpha_0\left(\mathbf{x}', \mathbf{x}, e_0 a\right) \mathbf{C} : \boldsymbol{\varepsilon}' \mathrm{d}V'$$
$$\boldsymbol{\tau} = l^2 \int_V \alpha_1\left(\mathbf{x}', \mathbf{x}, e_1 a\right) \mathbf{C} : \boldsymbol{\eta}' \mathrm{d}V'$$
$$= l^2 \int_V \alpha_1\left(\mathbf{x}', \mathbf{x}, e_1 a\right) \mathbf{C} : \nabla \boldsymbol{\varepsilon}' \mathrm{d}V'$$

where $\mathbf{C}$ is fourth order material elasticity tensor and its component can be expressed as $C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right)$. λ and μ are Lame constants and δ_{ij} denotes the Kronecker delta function. α_0 and α_1 are attenuation kernel function related to nonlocal elasticity effect in terms of the distance between the point $\mathbf{x}$ and $\mathbf{x}'$. ∇ denotes the 3D gradient operator. $e_0 a$ and $e_1 a$ are nonlocal parameters introduced to consider the nonlocal stress field effect. l is a material length scale parameter with units of m^2 introduced to consider the strain gradient effect.

According to the nonlocal strain gradient theory, the total stress tensor accounts for the nonlocal stress tensor as well as the strain gradient stress tensor and has been given as

$$\mathbf{t} = \boldsymbol{\sigma} - \nabla \boldsymbol{\tau}$$

where the nonlocal Cauchy stress tensor and the higher order stress tensor can be expressed in gradient form as

$$\left[1 - (e_0 a)^2 \nabla^2\right] \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$$
$$\left[1 - (e_1 a)^2 \nabla^2\right] \boldsymbol{\tau} = l^2 \mathbf{C} : \boldsymbol{\eta} = l^2 \mathbf{C} : \nabla \boldsymbol{\varepsilon}$$

Thus, considering homogeneous material, the total stress tensor can be expressed as

$$\left[1 - (e_1 a)^2 \nabla^2\right]\left[1 - (e_0 a)^2 \nabla^2\right] \mathbf{t} = \mathbf{C} : \left[1 - (e_1 a)^2 \nabla^2\right] \boldsymbol{\varepsilon}$$
$$-l^2 \mathbf{C} : \left[1 - (e_0 a)^2 \nabla^2\right] \nabla^2 \boldsymbol{\varepsilon}$$

where ∇^2 is the Laplacian operator and is denoted by $\nabla^2 = \left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2\right)$.

By assuming $e = e_0 = e_1$, the general constitutive equation for the size-dependent continuum can be simplified as

$$\left[1 - (ea)^2 \nabla^2\right] \mathbf{t} = \mathbf{C} : \boldsymbol{\varepsilon} - l^2 \mathbf{C} : \nabla^2 \boldsymbol{\varepsilon} \qquad (1)$$

Finally, the variation of strain energy U in terms of components can be expressed as

$$\delta U = \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + \tau_{kij}\delta\eta_{kij}\right) \mathrm{d}V \qquad (2)$$

The higher order inertia effect, as is introduced by Mindlin [16], is considered. According to the assumption, the kinetic energy of the continuum body depends not only on the velocity but also on the velocity gradient and can be expressed as

$$K = \int_V \left(\frac{1}{2}\rho\dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{1}{2}l_1^2\rho\nabla\dot{\mathbf{u}} : \nabla\dot{\mathbf{u}} \right) \mathrm{d}V$$

where $\mathbf{u}$ denotes the displacement vector of a reference point, "$\bullet$" denotes the time derivative of "$\bullet$", thus $\dot{\mathbf{u}}$ is the velocity vector. ∇ denotes the 3D gradients operator. "$\cdot$" is dot product and "$:$" denotes double dot product. ρ denotes the mass density of the material and l_1 is a material characteristic parameter (one kind of length scale parameter) with units of m^2 introduced to consider the higher order inertia effect [31]. Polyzos et al. [37], [38] investigated and obtained the coefficient l_1 as well as the strain gradient coefficient l by using lattice model. The Kinetic energy which considering the velocity gradient can be reduced to the classical form by setting $l_1 = 0$. The variation of kinetic energy can be expressed in terms of displacement components as

$$\delta K = \int_V \rho \left(\dot{u}_i\delta\dot{u}_i + l_1^2\dot{u}_{i,j}\delta\dot{u}_{i,j} \right) \mathrm{d}V$$

B. Constitutive Relation of Nonlocal Strain Gradient Euler–Bernoulli Beams

According to Euler–Bernoulli beam theory, the displacement field can be expressed as

$$\begin{aligned} u\left(x,z,t\right) &= u\left(x,t\right) - z\frac{\partial w}{\partial x} \\ v\left(x,z,t\right) &= 0 \\ w\left(x,z,t\right) &= w\left(x,t\right) \end{aligned} \tag{3}$$

where x, y z are the Cartesian coordinates, $u\left(x,t\right)$ is the axial displacement of the cross-section in x direction, $w\left(x,t\right)$ is the transverse displacement of the cross-section in the z direction.

The nonzero linear strain-displacement and strain gradient-displacement relations can be expressed according to (3) as

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\ \eta_{xxx} = \varepsilon_{xx,x} &= \frac{\partial^2 u}{\partial x^2} - z\frac{\partial^3 w}{\partial x^3} \\ \eta_{zxx} = \varepsilon_{xx,z} &= -\frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{4}$$

By substituting (4) into (1) and neglecting the effect of Poisson's ratio, the constitutive relations can be obtained as

$$\begin{aligned} \left[1 - (ea)^2\nabla^2\right]\sigma_{xx} &= E\varepsilon_{xx} \\ \left[1 - (ea)^2\nabla^2\right]\tau_{xxx} &= El^2\eta_{xxx} \\ \left[1 - (ea)^2\nabla^2\right]\tau_{zxx} &= El^2\eta_{zxx} \end{aligned} \tag{5}$$

Here, the conjugate pair τ_{zxx} and η_{zxx} takes the strain gradient effect in the thickness direction into account by cross-section integrating in next subsection.

C. Equilibrium Equations in Terms of Hamilton Principle

The first variation of strain energy functional U can be expressed, by considering (2), (4) and (5) as

$$\begin{aligned} \delta U &= \int_V \left(\sigma_{xx}\delta\varepsilon_{xx} + \tau_{xxx}\delta\eta_{xxx} + \tau_{zxx}\delta\eta_{zxx}\right)\mathrm{d}V \\ &= \int_L \left(N\delta\frac{\partial u}{\partial x} + N_h\delta\frac{\partial^2 u}{\partial x^2} \right. \\ &\quad \left. - (M+P)\,\delta\frac{\partial^2 w}{\partial x^2} - M_h\delta\frac{\partial^3 w}{\partial x^3} \right)\mathrm{d}x \end{aligned} \tag{6}$$

Here, the following stress resultant (including classical axial force N, higher order axial force N_h and P, classical bending moment M, higher order bending moment M_h) are defined as

$$\begin{aligned} N &= \int_A \sigma_{xx}\mathrm{d}A, \quad N_h = \int_A \tau_{xxx}\mathrm{d}z, \quad P = \int_A \tau_{zxx}\mathrm{d}A, \\ M &= \int_A z\sigma_{xx}\mathrm{d}A, \quad M_h = \int_A z\tau_{xxx}\mathrm{d}A. \end{aligned} \tag{7}$$

where P is the term considering strain gradient effect through thickness direction. According to constitutive equations (1) and the above resultants (7), the resultants of the total nonlocal stresses read

$$\begin{aligned} \vec{N} &= N - \frac{\partial N_h}{\partial x} \\ \vec{M} &= (M+P) - \frac{\partial M_h}{\partial x} \end{aligned}$$

Using strain-displacement relations (4), nonlocal constitutive equations (5), the relations between resultants (7) and displacement are as:

$$\begin{aligned} \left[1 - (ea)^2\nabla^2\right] N &= EA\frac{\partial u}{\partial x}, \\ \left[1 - (ea)^2\nabla^2\right] N_h &= EAl^2\frac{\partial^2 u}{\partial x^2}, \\ \left[1 - (ea)^2\nabla^2\right] M &= -EI\frac{\partial^2 w}{\partial x^2}, \\ \left[1 - (ea)^2\nabla^2\right] M_h &= -EIl^2\frac{\partial^3 w}{\partial x^3} \\ \left[1 - (ea)^2\nabla^2\right] P &= -EAl^2\frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{8}$$

where A denotes the area of cross section, I denotes the inertia moment and can be expressed as $I = \int_A z^2 dA$. According to definition (7), we can clearly observe that the resultants of stresses are functions of x because of the integrating on cross-section. Thus, the operator ∇^2 of nonlocal terms can be reduced to one-dimension form as $\nabla^2 = \partial^2/\partial x^2$.

The first variation of kinetic energy K, while both the axial displacement and the transverse motions are considered, can

be given by

$$
\delta K = \int_L \left\{ \rho A \left(\frac{\partial u}{\partial t} \delta \left(\frac{\partial u}{\partial t} \right) + \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) \right) \right.
$$
$$
\left. + \rho I \left(\frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial^2 w}{\partial x \partial t} \right) \right) \right\} \mathrm{d}x
$$
$$
+ l_1^2 \int_L \left\{ \rho A \left(\frac{\partial^2 u}{\partial x \partial t} \delta \left(\frac{\partial^2 u}{\partial x \partial t} \right) + 2 \frac{\partial^2 w}{\partial x \partial t} \delta \left(\frac{\partial^2 w}{\partial x \partial t} \right) \right) \right.
$$
$$
\left. + \rho I \left(\frac{\partial^3 w}{\partial x^2 \partial t} \delta \left(\frac{\partial^3 w}{\partial x^2 \partial t} \right) \right) \right\} \mathrm{d}x
$$

(9)

According to Hamilton's principle, we have

$$
\int_{t_1}^{t_2} (\delta K - \delta U + \delta W) \, dt = 0 \tag{10}
$$

where t_1 and t_2 denote, respectively, the initial and final time of motion.

For free vibration problems, there is no external load. Thus the first variation of work done by external loads $\delta W = 0$. By substituting (6) and (9) and using partial integration, the above equation can be rewritten as

$$
0 = \int_L \left[\rho A \frac{\partial u}{\partial t} \delta u + l_1^2 \rho A \frac{\partial^2 u}{\partial x \partial t} \delta \left(\frac{\partial u}{\partial x} \right) \right]_{t_1}^{t_2} \mathrm{d}x
$$
$$
+ \int_L \left[\begin{array}{c} \rho A \frac{\partial w}{\partial t} \delta w + l_1^2 \rho I \frac{\partial^3 w}{\partial x^2 \partial t} \delta \left(\frac{\partial^2 w}{\partial x^2} \right) \\ + \left(\rho I \frac{\partial^2 w}{\partial x \partial t} + 2 l_1^2 \rho A \frac{\partial^2 w}{\partial x \partial t} \right) \delta \left(\frac{\partial w}{\partial x} \right) \end{array} \right]_{t_1}^{t_2} \mathrm{d}x
$$
$$
+ \int_{t_1}^{t_2} \int_L \left(\frac{\partial \hat{N}}{\partial x} - \rho A \frac{\partial^2 u}{\partial t^2} \right) \delta u \, \mathrm{d}x \mathrm{d}t
$$
$$
+ \int_{t_1}^{t_2} \int_L \left(\frac{\partial \hat{Q}}{\partial x} - \rho A \frac{\partial^2 w}{\partial t^2} \right) \delta w \, \mathrm{d}x \mathrm{d}t
$$
$$
- \int_{t_1}^{t_2} \left[\hat{Q} \delta w - \hat{M} \delta \left(\frac{\partial w}{\partial x} \right) - M_h \delta \left(\frac{\partial^2 w}{\partial x^2} \right) \right]_0^L \mathrm{d}t
$$
$$
- \int_{t_1}^{t_2} \left[\hat{N} \delta u + N_h \delta \left(\frac{\partial u}{\partial x} \right) \right]_0^L \mathrm{d}t
$$

(11)

where the total tension force, bending moment, shear force of the cross-section in which the higher-order inertia terms are considered as inertial forces can be expressed as

$$
\hat{N} = \vec{N} + l_1^2 \rho A \frac{\partial^3 u}{\partial x \partial t^2}
$$
$$
\hat{M} = \vec{M} - l_1^2 \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}
$$
$$
\hat{Q} = \frac{\partial \hat{M}}{\partial x} + \rho I \frac{\partial^3 w}{\partial x \partial t^2} + 2 l_1^2 \rho A \frac{\partial^3 w}{\partial x \partial t^2}
$$

Assuming that the initial and final state are prescribed as

$$
\delta u = \delta \left(\frac{\partial u}{\partial x} \right) = 0, \ \delta w = \delta \left(\frac{\partial w}{\partial x} \right) = \delta \left(\frac{\partial^2 w}{\partial x^2} \right) = 0
$$

Then, the equilibrium equation can be obtained as

$$
\delta u : \frac{\partial \hat{N}}{\partial x} - \rho A \frac{\partial^2 u}{\partial t^2} = 0
$$
$$
\delta w : \frac{\partial \hat{Q}}{\partial x} - \rho A \frac{\partial^2 w}{\partial t^2} = 0
$$

(12)

with the corresponding boundary conditions divided into classical and non-classical parts. The classical boundary conditions can be expressed as

specify u or $\hat{N}$

specify w or $\hat{Q}$

specify $\frac{\partial w}{\partial x}$ or $\hat{M}$

and the non-classical boundary conditions are given as

specify $\frac{\partial u}{\partial x}$ or N_h

specify $\frac{\partial^2 w}{\partial x^2}$ or M_h

By substituting (8) into equilibrium equations (12), the governing equations of the beams can be obtained as

$$
\delta u : EA \frac{\partial^2 u}{\partial x^2} - EAl^2 \frac{\partial^4 u}{\partial x^4}
$$
$$
- (ea)^2 \left(l_1^2 \rho A \frac{\partial^6 u}{\partial x^4 \partial t^2} - \rho A \frac{\partial^4 u}{\partial x^2 \partial t^2} \right)
$$
$$
+ l_1^2 \rho A \frac{\partial^4 u}{\partial x^2 \partial t^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0
$$
$$
\delta w : - EI \frac{\partial^4 w}{\partial x^4} - EAl^2 \frac{\partial^4 w}{\partial x^4} + EIl^2 \frac{\partial^6 w}{\partial x^6}
$$
$$
+ 2 l_1^2 \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} - l_1^2 \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2}
$$
$$
+ \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - (ea)^2 \left(2 l_1^2 \rho A \frac{\partial^6 w}{\partial x^4 \partial t^2} \right.
$$
$$
\left. - l_1^2 \rho I \frac{\partial^8 w}{\partial x^6 \partial t^2} - \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} + \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} \right) = 0
$$

(13)

with N, N_h, P, M, M_h explicitly expressed in V-A and $\hat{N}$, $\hat{M}$, $\hat{Q}$ as following.

$$
\hat{N} = EA \frac{\partial u}{\partial x} - EAl^2 \frac{\partial^3 u}{\partial x^3} + l_1^2 \rho A \frac{\partial^3 u}{\partial x \partial t^2}
$$
$$
- (ea)^2 \left(l_1^2 \rho A \frac{\partial^5 u}{\partial x^3 \partial t^2} - \rho A \frac{\partial^3 u}{\partial x \partial t^2} \right)
$$
$$
\hat{M} = -EI \frac{\partial^2 w}{\partial x^2} - EAl^2 \frac{\partial^2 w}{\partial x^2} + EIl^2 \frac{\partial^4 w}{\partial x^4}
$$
$$
- l_1^2 \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - (ea)^2 \left(2 l_1^2 \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} \right.
$$
$$
\left. - l_1^2 \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} \right)
$$
$$
\hat{Q} = -EI \frac{\partial^3 w}{\partial x^3} - EAl^2 \frac{\partial^3 w}{\partial x^3} + EIl^2 \frac{\partial^5 w}{\partial x^5}
$$
$$
- l_1^2 \rho I \frac{\partial^5 w}{\partial x^3 \partial t^2} + \rho I \frac{\partial^3 w}{\partial x \partial t^2} + 2 l_1^2 \rho A \frac{\partial^3 w}{\partial x \partial t^2}
$$
$$
- (ea)^2 \left(2 l_1^2 \rho A \frac{\partial^5 w}{\partial x^3 \partial t^2} - l_1^2 \rho I \frac{\partial^7 w}{\partial x^5 \partial t^2} \right.
$$
$$
\left. - \rho A \frac{\partial^3 w}{\partial x \partial t^2} + \rho I \frac{\partial^5 w}{\partial x^3 \partial t^2} \right)
$$

D. Dimensionless Equations

To analyze the dynamic characters of the nonlinear size-dependent beams regardless of their material properties, while neglecting the fast dynamics (i.e. the axial motion), the following dimensionless variables are introduced:

$$X = \frac{x}{L}, \ W = \frac{w}{L}, \ \alpha = \frac{ea}{L}, \ \zeta = \frac{l}{L},$$

$$\xi = \frac{l_1}{L}, \ \gamma = \frac{AL^2}{I}, \ \Omega = \omega\sqrt{\frac{\rho L^2}{E}}$$

$$\tilde{Q} = \frac{\hat{Q}L^2}{EI}, \ \tilde{M} = \frac{\hat{M}L}{EI}, \ \tilde{M}_h = \frac{M_h}{EI}$$

Thus, we have

$$\frac{\partial^n w}{\partial x^n} = \frac{1}{L^{n-1}} \frac{\partial^n W}{\partial X^n}$$

and dimensionless shear force and bending moments can be obtained as

$$\tilde{Q} = \left(2\alpha^2\xi^2\gamma\Omega^2 + \alpha^2\Omega^2 + \xi^2\Omega^2 - \zeta^2\gamma - 1\right)\frac{\partial^3 W}{\partial X^3}$$
$$+ \left(\zeta^2 - \alpha^2\xi^2\Omega^2\right)\frac{\partial^5 W}{\partial X^5}$$
$$- \left(\alpha^2\gamma\Omega^2 + \Omega^2 + 2\xi^2\gamma\Omega^2\right)\frac{\partial W}{\partial X}$$

$$\tilde{M} = \left(2\alpha^2\xi^2\gamma\Omega^2 + \alpha^2\Omega^2 + \xi^2\Omega^2 - \zeta^2\gamma - 1\right)\frac{\partial^2 W}{\partial X^2}$$
$$+ \left(\zeta^2 - \alpha^2\xi^2\Omega^2\right)\frac{\partial^4 W}{\partial X^4} - \alpha^2\gamma\Omega^2 W$$

$$\tilde{M}_h = \frac{1}{\alpha^2 + \alpha^2\zeta^2\gamma - \zeta^2}\left[\left(\alpha^2\zeta^4 - \xi^2\alpha^4\zeta^2\Omega^2\right)\frac{\partial^5 W}{\partial X^5}\right.$$
$$- \gamma\alpha^4\zeta^2\Omega^2\frac{\partial W}{\partial X} + \left(\alpha^2 - \zeta^2 + 2\gamma\xi^2\alpha^4\Omega^2\right.$$
$$\left.-\gamma\alpha^2\zeta^2 + \alpha^4\Omega^2\right)\zeta^2\frac{\partial^3 W}{\partial X^3}\bigg]$$

(14)

and the governing equation of the Euler–Bernoulli–Rayleigh beams can be expressed as dimensionless form in frequency domain

$$\left(\zeta^2 - \alpha^2\xi^2\Omega^2\right)\frac{\partial^6 W}{\partial X^6} + \left(\xi^2\Omega^2 + 2\alpha^2\xi^2\gamma\Omega^2\right.$$
$$\left.+\alpha^2\Omega^2 - \gamma\zeta^2 - 1\right)\frac{\partial^4 W}{\partial X^4}$$
$$- \left(2\xi^2\gamma\Omega^2 + \alpha^2\gamma\Omega^2 + \Omega^2\right)\frac{\partial^2 W}{\partial X^2} + \gamma\Omega^2 W = 0$$

(15)

with the boundary conditions at $X = 0$ or 1:

$$\text{specify } W \text{ or } \tilde{Q}$$
$$\text{specify } \frac{\partial W}{\partial X} \text{ or } \tilde{M}$$
$$\text{specify } \frac{\partial^2 W}{\partial X^2} \text{ or } \tilde{M}_h$$

Particularly, we can give a typical form of simply supported boundary condition as

$$W = 0, \ \tilde{M} = 0, \ \frac{\partial^2 W}{\partial X^2} = 0$$

III. Free Vibration of Simply Supported Nonlocal Strain Gradient Beams

This section presents the analytical solutions for free vibrations of nonlocal strain gradient Euler–Bernoulli–Rayleigh beams under typical simply supported boundary condition.

A. A Type of Simply Supported Boundary Conditions

In this section, some common a probable kind of simply supported boundary condition, for nonlocal strain gradient Rayleigh beams containing higher-order terms are discussed. Here, we use the end at $x = 0$ to discuss the boundary condition. It should be noted that only one term of each of the three pairs ought to be specified on one end of the beam.

For simply supported end, as mentioned above, the displacement and bending moment at the end point of beam are specified. Thus, the lower order boundary conditions are

$$W = 0, \ \tilde{M} = 0$$

One kind of the higher order boundary conditions is considered in this paper as

$$\frac{\partial^2 W}{\partial X^2} = 0$$

By substituting (14), the above boundary conditions can be expressed as

$$W = 0, \ \frac{\partial^4 W}{\partial X^4} = 0, \ \frac{\partial^2 W}{\partial X^2} = 0$$

B. General Solutions

In order to find the natural frequencies of the nonlocal strain gradient Euler–Bernoulli–Rayleigh beams, the harmonic solution in frequency domain can be assumed as $W = Ce^{kX}$. Substituting into the sixth order ordinary differential equation (15), the characteristic equation can be obtained as

$$ar^3 + br^2 + cr + d = 0$$

where

$$r = k^2, \ a = \zeta^2 - \alpha^2\xi^2\Omega^2,$$
$$b = \xi^2\Omega^2 + 2\alpha^2\xi^2\gamma\Omega^2 + \alpha^2\Omega^2 - \gamma\zeta^2 - 1,$$
$$c = -\left(2\xi^2\gamma\Omega^2 + \alpha^2\gamma\Omega^2 + \Omega^2\right), \ d = \gamma\Omega^2$$

The corresponding solution can be obtained as (16), where
$$\theta = \frac{-1 + \sqrt{3}i}{2}, \ m = \frac{c}{a} - \frac{b^2}{3a^2}, \ n = \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}$$
Thus, the general solution of the dimensionless governing equation, can be given as

$$W = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \end{bmatrix}$$
$$\cdot \begin{bmatrix} e^{k_1 X} & e^{-k_1 X} & e^{k_2 X} & e^{-k_2 X} & e^{k_3 X} & e^{-k_3 X} \end{bmatrix}^T$$

(17)

$$k_1 = \sqrt{\sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \frac{b}{3a}}$$

$$k_2 = \sqrt{\theta\sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} + \theta^2\sqrt[3]{-\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \frac{b}{3a}} \quad (16)$$

$$k_3 = \sqrt{\theta 2\sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} + \theta\sqrt[3]{-\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \frac{b}{3a}}$$

C. Analytical Solution for Simply Supported Beams

Considering a nonlocal strain gradient beam simply supported at ends $X = 0$ and $X = 1$, the characteristic equation can be obtained by substituting (17) into the boundary conditions as

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ k_1^2 & k_1^2 & k_2^2 & k_2^2 & k_3^2 & k_3^2 \\ k_1^4 & k_1^4 & k_2^4 & k_2^4 & k_3^4 & k_3^4 \\ e^{k_1} & e^{-k_1} & e^{k_2} & e^{-k_2} & e^{k_3} & e^{-k_3} \\ e^{k_1}k_1^2 & e^{-k_1}k_1^2 & e^{k_2}k_2^2 & e^{-k_2}k_2^2 & e^{k_3}k_3^2 & e^{-k_3}k_3^2 \\ e^{k_1}k_1^4 & e^{-k_1}k_1^4 & e^{k_2}k_2^4 & e^{-k_2}k_2^4 & e^{k_3}k_3^4 & e^{-k_3}k_3^4 \end{vmatrix} = 0$$

The above characteristic equation can be simplified as

$$e^{-k_1-k_2-k_3}\left(e^{2k_1} - 1\right)\left(e^{2k_2} - 1\right)\left(e^{2k_3} - 1\right)$$
$$\left(k_1^2 - k_2^2\right)^2\left(k_1^2 - k_3^2\right)^2\left(k_2^2 - k_3^2\right)^2 = 0$$

A possible solution of the normalized deflection W may read

$$W = \sin\left(n\pi X\right) \quad (18)$$

where n is the mode number.

Substituting (18) into the governing equation (15) yield the analytical solution of the dimensionless mode frequency:

$$\Omega = \sqrt{\frac{(n\pi)^6\zeta^2 + \gamma\zeta^2(n\pi)^4 + (n\pi)^4}{\left(1 + \alpha^2(n\pi)^2\right)\left(\xi^2(n\pi)^4 + (n\pi)^2 + 2\xi^2\gamma(n\pi)^2 + \gamma\right)}} \quad (19)$$

As can be seen, the analytical frequency solution implies that the dimensionless strain length scale parameter ζ increases the frequency, while the dimensionless inertia length scale parameter ξ and nonlocal parameter α decrease the frequency. When $n \to \infty$, the dimensionless asymptotic frequency can be obtained as

$$\Omega_{asym} = \frac{\zeta}{\alpha\xi}$$

It is worth discussing that some kinds of scale-effects influence on the solutions showed in (19), significantly.

1) (*Size-effect in thickness direction*): It is interesting that the solution can be reduced to the following form when the scale effect in thickness direction is neglected, which is applied in many articles. In this case, the gradient of strain η_{zxx} and the work-conjugate stress τ_{zxx} are vanished. Hence, the dimensionless natural frequencies can be reduced to

$$\Omega_{red} = \sqrt{\frac{(n\pi)^6\zeta^2 + (n\pi)^4}{\xi^2(n\pi)^4 + (n\pi)^2 + 2\xi^2\gamma(n\pi)^2 + \gamma}} \cdot \sqrt{\frac{1}{1 + \alpha^2(n\pi)^2}} \quad (20)$$

When the dimensionless parameters are satisfied with (21), the error will be significant (i.e. $(\Omega - \Omega_{red})/\Omega_{red} \times 100\% \geq 5\%$).

$$\sqrt{1 + \frac{\gamma}{(n\pi)^2 + 1/\zeta^2}} \geq 1.05 \quad (21)$$

It indicates that the increasing geometrical parameter γ and length-scale parameter ζ and decreasing order n of natural frequencies increase the error. When and only when the relation

$$\zeta^2 < \frac{0.1025}{\gamma - 0.1025 \times (N\pi)^2}$$

is satisfied for a specific microbeam structure, the size-effect in the thickness direction can be neglected. Here, N denotes the lowest order of natural frequency that is concerned about. Especially, while $N = 1$, it means the value of ζ satisfies the above inequality for all natural frequencies. In this case, the strain gradient effect through thickness direction can be neglected. Furthermore, for a specific material (i.e. l is a constant), only when the geometric parameters of microbeam structure satisfy

$$\frac{0.1025L^2}{\gamma - 0.1025 \times (N\pi)^2} > l^2$$

the size-effect in the thickness direction can be neglected. Thus, one can design the geometric properties according this inequality to avoid the size-effect through thickness direction. Also, it indicates that for higher frequencies than Nth order natural frequency this effect could be neglected. Thus, the strain gradient effect in thickness direction is more significant in lower frequency range.

2) (*Higher order inertia terms*): If the higher order inertia effect is neglected (i.e. $l_1 = 0$ and $\xi = 0$), the dimensionless natural frequencies can be reduced to

$$\Omega_{red} = \sqrt{\frac{(n\pi)^6 \zeta^2 + \gamma\zeta^2(n\pi)^4 + (n\pi)^4}{\left(1 + \alpha^2(n\pi)^2\right)\left((n\pi)^2 + \gamma\right)}} \quad (22)$$

When the dimensionless parameters are satisfied with (23), the error will be significant (i.e. $\dfrac{\Omega - \Omega_{red}}{\Omega} \times 100\% \geq 5\%$).

$$\sqrt{\frac{\xi^2}{1/(n\pi)^2 + 1/\gamma} + \xi^2(n\pi)^2 + 1} \geq 1.05 \quad (23)$$

As can be seen, the increasing order n, geometrical parameter γ and length-scale parameter ξ increase the error. By simplifying the inequality, one can point out that when the length scale parameter ξ satisfied

$$\xi^2 < \frac{0.0125\left(\gamma + (N\pi)^2\right)}{2\gamma(N\pi)^2 + (N\pi)^4}$$

the higher order inertia effect can be neglected. Here N denotes the highest order of natural frequencies that is concerned about. Further more, for a specific material (i.e. l_1 is a constant), only when the geometric parameters are designed satisfying

$$\frac{0.0125\left(\gamma + (N\pi)^2\right)L^2}{2\gamma(N\pi)^2 + (N\pi)^4} > l_1^2$$

the velocity gradient effect of the beam can be neglected. However, it can be obviously found that this condition can not always be satisfied, especially, for high orders of natural frequencies. Thus, the effect of higher order inertia should be considered in the free vibration analysis.

IV. RESULT AND DISCUSSION

This section presents the numerical results to evaluate the effect of three length scale parameters on natural frequencies. Since Euler–Bernoulli–Rayleigh assumption is considered, the ratio of length to diameter (or the height of cross-section) should be large enough.

A. Effect of Different Continuum Theories

In this section, we consider a microbeam with dimensionless parameter $\gamma = 4800$ and dimensionless scale parameters α, ξ and ζ varying from $0 \sim 0.05$. As is detected in [12], the natural frequencies for nonlocal elasticity theory (NET) are smaller than the ones for classical elasticity theory (CET) because of the effect of nonlocal parameter α on inertia terms. However, the natural frequencies for strain gradient theory (SGT) are larger than the ones for CET, because the length scale parameter ζ positively effects on dynamic stiffness of the micro-structure. For nonlocal strain gradient theory (NSGT), the natural frequencies are effected by both α and ζ. Fig. 1 shows the natural frequencies of the microbeam for

different continuum theories. As can be seen, it makes the natural frequencies increasing to consider the effect of strain gradient. On the contrast, it makes the natural frequencies decreasing to consider the effect of nonlocal elasticity and higher oder inertia. These effects are much more significant at higher order frequencies than at lower ones. As a consequence, the values of natural frequencies for nonlocal strain gradient theory considering higher order inertia (NSG-HI) are between those for NSGT and those for NET. It is of interest that, the solutions for CET, SGT, NSGT diverge with the increasing order, while the solutions for NET and NSG-HI converge. That means both for NET and NSG-HI there is an asymptotic value of frequencies. Especially, the analytical expression of the asymptotic frequency for NSG-HI beam has been given in Section III.
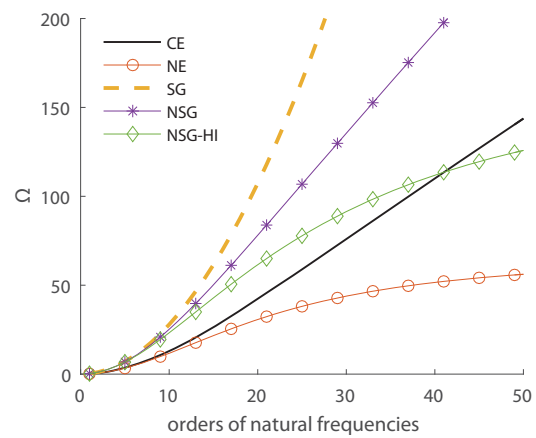

Fig. 1 Natural frequencies of simply supported microbeams for different continuum theories (including classical elasticity (CE) theory, nonlocal elasticity (NE) theory, strain gradient (SG) theory, nonlocal strain gradient (NSG) theory and nonlocal strain gradient theory with higher-order inertia effect (NSG-HI))
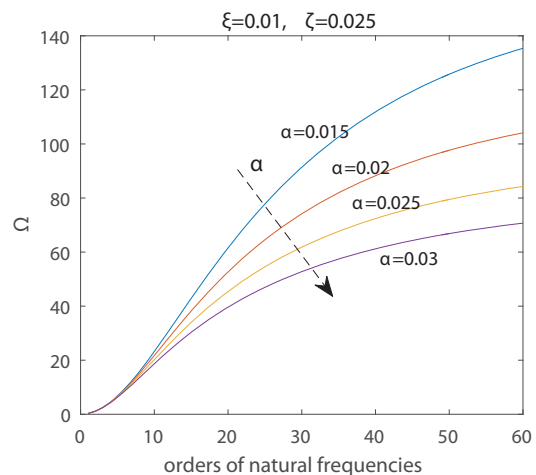

Fig. 2 Natural frequencies of simply supported microbeams for NSG-HI with dimensionless parameters: velocity gradient coefficient $\xi = 0.01$, strain gradient coefficient $\zeta = 0.025$ and various nonlocal parameter α
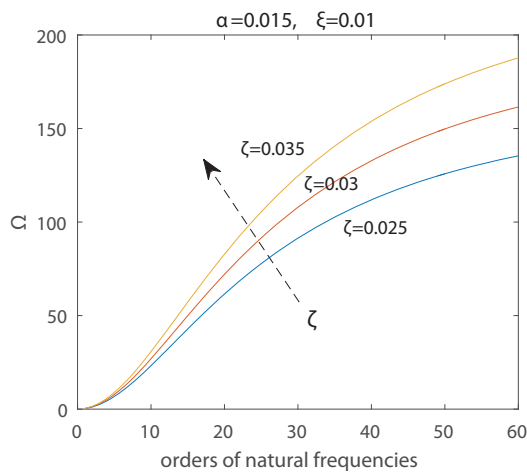
Fig. 3 Natural frequencies of simply supported microbeams for NSG-HI with dimensionless parameters: nonlocal parameter $\alpha = 0.015$, velocity gradient coefficient $\xi = 0.01$ and various strain gradient coefficient ζ
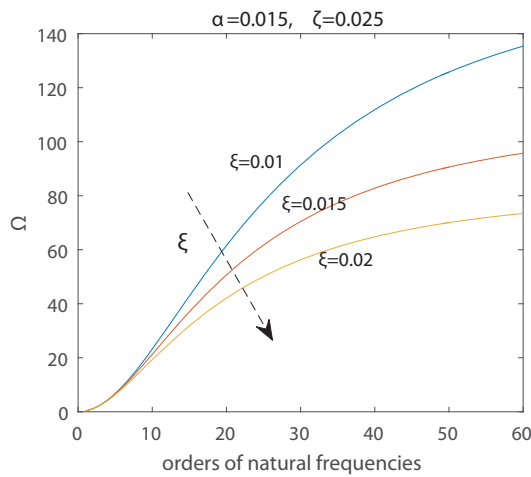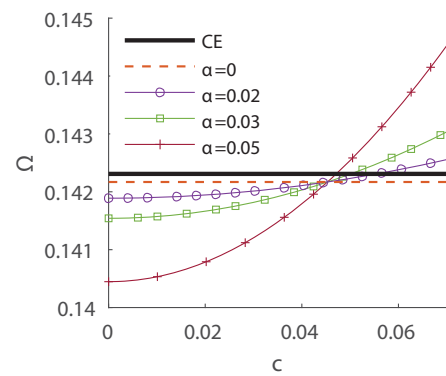

Fig. 4 Natural frequencies of simply supported microbeams for NSG-HI with dimensionless parameters: nonlocal parameter $\alpha = 0.015$, strain gradient coefficient $\zeta = 0.025$ and various velocity gradient coefficient ξ
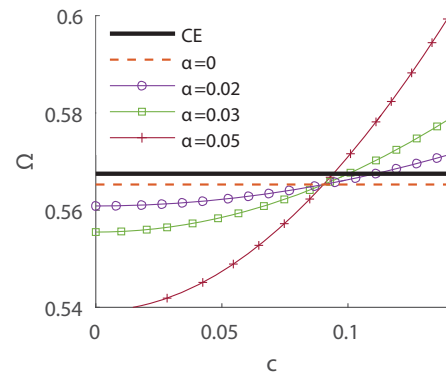

(a) First-order frequency

(b) Second-order frequency

(c) Third-order frequency

Fig. 5 Effect of dimensionless scale parameters on the first three natural frequencies of a simply supported microbeam with $\xi = 0.01$

B. Effect of Length Scale Parameters

Here, we consider a microbeam for NSG-HI with length scale parameter $\zeta = 0.025$ and $\xi = 0.01$. Fig. 2 shows the natural frequencies with varying values of nonlocal parameter α. As can be seen, the natural frequencies decrease with the increasing value of α. Similarly, Fig. 3 shows natural frequencies for NSG-HI with varying values of ξ while the length scale parameter $\alpha = 0.015$ and $\zeta = 0.025$. Fig. 4 shows natural frequencies for NSG-HI with varying values of ζ while the length scale parameter $\alpha = 0.015$ and $\xi = 0.01$. The values of natural frequencies increase with the increasing value of ζ and decreasing value of ξ.
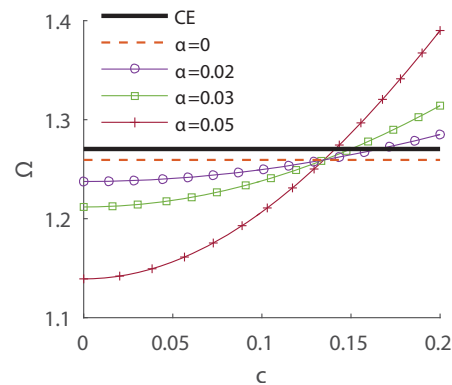
Fig. 5 shows the lowest three orders of natural frequencies of microbeam for NSG-HI with constant $\xi = 0.01$ versus the rate of other two scale parameters $c = \zeta/\alpha$. The dark solid lines plot the natural frequencies for CET. Dash lines plot the natural frequencies for $\xi = 0.01$ and $\alpha = \zeta = 0$. In each

subfigure, the curves for NSG-HI cross the point satisfied

$$c = \sqrt{\frac{(n\pi)^2}{(n\pi)^2 + \gamma}} \qquad (24)$$

It is worth mentioning that the above equation can meet the results by Li. et. al [12], [13] when the size-dependent behavior in the thickness direction is neglected.

For a constant $\alpha = 0.025$, the dimensionless natural frequencies of NSG-HI beam for various values of ζ and $\xi = c'\zeta$ are shown in Fig. 6. The dark solid lines plot the

(a) First-order frequency

(b) Second-order frequency
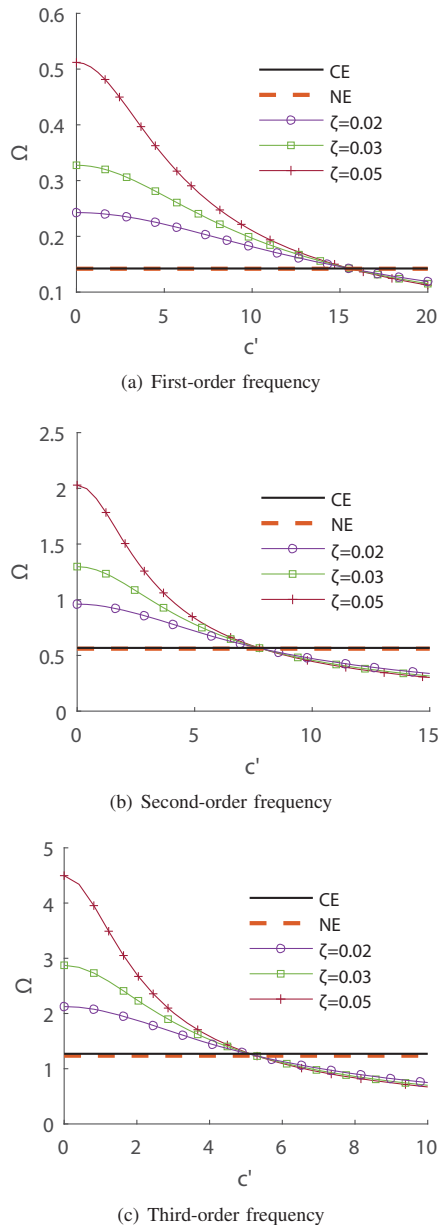

(c) Third-order frequency

Fig. 6 Effect of dimensionless scale parameters on the first three natural frequencies of a simply supported microbeam with $\alpha = 0.025$

solutions for CET, while the dash lines plot the solutions for NET. The intersection points of these curves and y-axis are solutions for NSGT.

In each subfigure, the curves for NSG-HI and the line for NET cross the point satisfied

$$c' = \frac{(n\pi)^2 + \gamma}{n\pi} \sqrt{\frac{1}{(n\pi)^2 + 2\gamma}} \qquad (25)$$

V. Conclusion

This paper combines the nonlocal elasticity effect, strain gradient effect and higher order inertia effect and proposes a new model of micro thin beam. Hamilton principle is employed to obtain the equilibrium equations and the corresponding boundary conditions of the nonlocal strain gradient Euler-Bernoulli beams with considering higher order inertia effect. Two additional material length scale parameters are employed to consider the significance of strain gradient stress field and higher order inertia force field. A nonlocal parameter is also employed to consider the significance of nonlocal elasticity stress field. For a more general study (to analyze the dynamic properties of microbeams), the dimensionless governing equation and the explicit boundary conditions are deduced. For simply supported boundary conditions, the analytical solutions of the proposed beam models for free vibration problems are derived. The influences of strain gradient in thickness direction and the higher order inertia force field are discussed. Besides, the influences of each size dependent parameters on the natural frequencies are investigated. The most important results are as follows:

1) The strain gradient field in thickness direction affect significantly on natural frequencies, especially at low frequency domain for thin beams. It's because, in this case (for thin microbeams), the length scale parameter l is considerable with the thickness h.
2) The effect of higher order inertia term on higher order frequencies is more significant than that on lower ones. Besides, with the decreasing of thickness h and length L, the effect of higher order inertia force field become significant.
3) For the Euler-Bernoulli beam models based on the nonlocal strain gradient theory considering the higher order inertia, similarly to that based on the nonlocal elasticity theory, there is an asymptotic value of natural frequencies, which is probably about the atomic vibration frequencies.
4) The vibration frequencies increase with the increasing material strain gradient length scale parameter ζ or decreasing higher order inertia length scale parameter ξ and nonlocal parameter α.

Appendix

A. Explicit Expression of Stress Resultants

According to (8), we have the following relations

$$N_h = l^2 \frac{\partial N}{\partial x}, \ M_h = l^2 \frac{\partial M}{\partial x}, \ P = \frac{l^2 A}{I} M$$

Thus, the explicit express of the stress resultants can be obtained as

$$N = EA \frac{\partial u}{\partial x} - \frac{EA l^2 ea^2}{ea^2 - l^2} \frac{\partial^3 u}{\partial x^3}$$
$$- \frac{(ea)^4}{ea^2 - l^2} \left(l_1^2 \rho A \frac{\partial^5 u}{\partial x^3 \partial t^2} - \rho A \frac{\partial^3 u}{\partial x \partial t^2} \right)$$
$$N_h = l^2 EA \frac{\partial^2 u}{\partial x^2} - \frac{EA l^4 ea^2}{ea^2 - l^2} \frac{\partial^4 u}{\partial x^4}$$
$$- \frac{(ea)^4 l^2}{ea^2 - l^2} \left(l_1^2 \rho A \frac{\partial^6 u}{\partial x^4 \partial t^2} - \rho A \frac{\partial^4 u}{\partial x^2 \partial t^2} \right)$$

$$M = -\left(\frac{ea^2 - l^2}{ea^2 I + ea^2 l^2 A - l^2 I}\right) EI^2 \frac{\partial^2 w}{\partial x^2}$$

$$- \frac{(ea)^2 l^2 EIA}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^2 w}{\partial x^2}$$

$$+ \frac{(ea)^2 EI^2 l^2}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^4 w}{\partial x^4}$$

$$- \frac{(ea)^4 I}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \left(2l_1^2 \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} \right.$$

$$\left. -l_1^2 \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} \right)$$

$$P = -\left(\frac{ea^2 - l^2}{ea^2 I + ea^2 l^2 A - l^2 I}\right) l^2 EIA \frac{\partial^2 w}{\partial x^2}$$

$$- \frac{(ea)^2 l^4 EA^2}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^2 w}{\partial x^2}$$

$$+ \frac{(ea)^2 EIAl^4}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^4 w}{\partial x^4}$$

$$- \frac{(ea)^4 l^2 A}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \left(2l_1^2 \rho A \frac{\partial^4 w}{\partial x^2 \partial t^2} \right.$$

$$\left. -l_1^2 \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} + \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} \right)$$

$$M_h = -\left(\frac{ea^2 - l^2}{ea^2 I + ea^2 l^2 A - l^2 I}\right) l^2 EI^2 \frac{\partial^3 w}{\partial x^3}$$

$$- \frac{(ea)^2 l^4 EIA}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^3 w}{\partial x^3}$$

$$+ \frac{(ea)^2 EI^2 l^4}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \frac{\partial^5 w}{\partial x^5}$$

$$- \frac{(ea)^4 l^2 I}{(ea)^2 I + (ea)^2 l^2 A - l^2 I} \left(2l_1^2 \rho A \frac{\partial^5 w}{\partial x^3 \partial t^2} \right.$$

$$\left. -l_1^2 \rho I \frac{\partial^7 w}{\partial x^5 \partial t^2} - \rho A \frac{\partial^3 w}{\partial x \partial t^2} + \rho I \frac{\partial^5 w}{\partial x^3 \partial t^2} \right)$$

REFERENCES

[1] L. Li, Y. Hu, and L. Ling, "Flexural wave propagation in small-scaled functionally graded beams via a nonlocal strain gradient theory," *Composite Structures*, vol. 133, pp. 1079–1092, 2015.

[2] L. Li, Y. Hu, and L. Ling, "Wave propagation in viscoelastic single-walled carbon nanotubes with surface effect under magnetic field based on nonlocal strain gradient theory," *Physica E: Low-dimensional Systems and Nanostructures*, vol. 75, pp. 118–124, 2016.

[3] L. Li and Y. Hu, "Wave propagation in fluid-conveying viscoelastic carbon nanotubes based on nonlocal strain gradient theory," *Computational Materials Science*, vol. 112, pp. 282–288, 2016.

[4] P. Khodabakhshi and J. Reddy, "A unified beam theory with strain gradient effect and the von kármán nonlinearity," *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 2016.

[5] L. Li and Y. Hu, "Nonlinear bending and free vibration analyses of nonlocal strain gradient beams made of functionally graded material," *International Journal of Engineering Science*, vol. 107, pp. 77–97, 2016.

[6] F. Ebrahimi and M. R. Barati, "Magneto-electro-elastic buckling analysis of nonlocal curved nanobeams," *The European Physical Journal Plus*, vol. 131, no. 9, p. 346, 2016.

[7] L. Li and Y. Hu, "Buckling analysis of size-dependent nonlinear beams based on a nonlocal strain gradient theory," *International Journal of Engineering Science*, vol. 97, pp. 84–94, 2015.

[8] F. Ebrahimi and M. R. Barati, "On nonlocal characteristics of curved inhomogeneous euler–bernoulli nanobeams under different temperature distributions," *Applied Physics A*, vol. 122, no. 10, p. 880, 2016.

[9] F. Ebrahimi and M. R. Barati, "A nonlocal strain gradient refined beam model for buckling analysis of size-dependent shear-deformable curved fg nanobeams," *Composite Structures*, vol. 159, pp. 174–182, 2017.

[10] S. Guo, Y. He, D. Liu, J. Lei, L. Shen, and Z. Li, "Torsional vibration of carbon nanotube with axial velocity and velocity gradient effect," *International Journal of Mechanical Sciences*, 2016.

[11] D. Zhang, Y. Lei, and Z. Shen, "Vibration analysis of horn-shaped single-walled carbon nanotubes embedded in viscoelastic medium under a longitudinal magnetic field," *International Journal of Mechanical Sciences*, vol. 118, pp. 219–230, 2016.

[12] L. Li, Y. Hu, and X. Li, "Longitudinal vibration of size-dependent rods via nonlocal strain gradient theory," *International Journal of Mechanical Sciences*, vol. 115, pp. 135–144, 2016.

[13] L. Li, X. Li, and Y. Hu, "Free vibration analysis of nonlocal strain gradient beams made of functionally graded material," *International Journal of Engineering Science*, vol. 102, pp. 77–92, 2016.

[14] A. C. Eringen, "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves," *Journal of Applied Physics*, vol. 54, no. 9, pp. 4703–4710, 1983.

[15] E. C. Aifantis, "On the role of gradients in the localization of deformation and fracture," *International Journal of Engineering Science*, vol. 30, no. 10, pp. 1279–1299, 1992.

[16] R. D. Mindlin, "Micro-structure in linear elasticity," *Archive for Rational Mechanics and Analysis*, vol. 16, no. 1, pp. 51–78, 1964.

[17] C. W. Lim, G. Zhang, and J. N. Reddy, "A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation," *Journal of the Mechanics and Physics of Solids*, vol. 78, pp. 298–313, 2015.

[18] Y. Zhang, C. Wang, and N. Challamel, "Bending, buckling, and vibration of micro/nanobeams by hybrid nonlocal beam model," *Journal of Engineering Mechanics*, vol. 136, no. 5, pp. 562–574, 2009.

[19] J. Zang, B. Fang, Y.-W. Zhang, T.-Z. Yang, and D.-H. Li, "Longitudinal wave propagation in a piezoelectric nanoplate considering surface effects and nonlocal elasticity theory," *Physica E*, vol. 63, pp. 147–150, 2014.

[20] B. Wang, Z. Deng, H. Ouyang, and J. Zhou, "Wave propagation analysis in nonlinear curved single-walled carbon nanotubes based on nonlocal elasticity theory," *Physica E*, vol. 66, pp. 283–292, 2015.

[21] M. Z. Nejad, A. Hadi, and A. Rastgoo, "Buckling analysis of arbitrary two-directional functionally graded euler–bernoulli nano-beams based on nonlocal elasticity theory," *International Journal of Engineering Science*, vol. 103, pp. 1–10, 2016. (Online). Available: http://www.sciencedirect.com/science/article/pii/S0020722516300180

[22] A. Daneshmehr, A. Rajabpoor, and A. Hadi, "Size dependent free vibration analysis of nanoplates made of functionally graded materials based on nonlocal elasticity theory with high order theories," *International Journal of Engineering Science*, vol. 95, pp. 23–35, 2015.

[23] F. Ebrahimi and M. R. Barati, "A nonlocal higher-order shear deformation beam theory for vibration analysis of size-dependent functionally graded nanobeams," *Arabian Journal Forence & Engineering*, vol. 41, no. 5, pp. 1679–1690, 2015.

[24] N. Challamel and C. Wang, "The small length scale effect for a non-local cantilever beam: a paradox solved," *Nanotechnology*, vol. 19, no. 34, p. 345703, 2008.

[25] J. Fernández-Sáez, R. Zaera, J. Loya, and J. Reddy, "Bending of euler–bernoulli beams using eringens integral formulation: a paradox resolved," *International Journal of Engineering Science*, vol. 99, pp. 107–116, 2016.

[26] N. Challamel, Z. Zhang, C. Wang, J. Reddy, Q. Wang, T. Michelitsch, and B. Collet, "On nonconservativeness of eringens nonlocal elasticity in beam mechanics: correction from a discrete-based approach," *Archive of Applied Mechanics*, vol. 84, no. 9-11, pp. 1275–1292, 2014.

[27] E. Benvenuti and A. Simone, "One-dimensional nonlocal and gradient elasticity: closed-form solution and size effect," *Mechanics Research Communications*, vol. 48, pp. 46–51, 2013.

[28] F. Yang, A. Chong, D. Lam, and P. Tong, "Couple stress based strain gradient theory for elasticity," *International Journal of Solids and Structures*, vol. 39, no. 10, pp. 2731–2743, 2002.

[29] N. Fleck, G. Muller, M. Ashby, and J. Hutchinson, "Strain gradient plasticity: theory and experiment," *Acta Metallurgica et Materialia*, vol. 42, no. 2, pp. 475–487, 1994.

[30] N. Fleck and J. Hutchinson, "A phenomenological theory for strain gradient effects in plasticity," *Journal of the Mechanics and Physics of Solids*, vol. 41, no. 12, pp. 1825 – 1857, 1993. (Online). Available: http://www.sciencedirect.com/science/article/pii/002250969390072N

[31] C. Polizzotto, "A gradient elasticity theory for second-grade materials and higher order inertia," *International Journal of Solids and Structures*, vol. 49, no. 15, pp. 2121–2137, 2012.

[32] C. Polizzotto, "A second strain gradient elasticity theory with second velocity gradient inertia–part i: Constitutive equations and quasi-static behavior," *International Journal of Solids and Structures*, vol. 50, no. 24, pp. 3749–3765, 2013.

[33] C. Polizzotto, "A second strain gradient elasticity theory with second velocity gradient inertia–part ii: Dynamic behavior," *International Journal of Solids and Structures*, vol. 50, no. 24, pp. 3766–3777, 2013.

[34] H. Askes and E. C. Aifantis, "Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results," *International Journal of Solids and Structures*, vol. 48, no. 13, pp. 1962–1990, 2011.

[35] H. Askes and E. C. Aifantis, "Gradient elasticity and flexural wave dispersion in carbon nanotubes," *Physical Review B*, vol. 80, no. 19, p. 195412, 2009.

[36] S. Papargyri-Beskou, D. Polyzos, and D. Beskos, "Wave dispersion in gradient elastic solids and structures: a unified treatment," *International Journal of Solids and Structures*, vol. 46, no. 21, pp. 3751–3759, 2009.

[37] D. Polyzos and D. Fotiadis, "Derivation of mindlins first and second strain gradient elastic theory via simple lattice and continuum models," *International Journal of Solids and Structures*, vol. 49, no. 3, pp. 470–480, 2012.

[38] S. N. Iliopoulos, D. G. Aggelis, and D. Polyzos, "Wave dispersion in fresh and hardened concrete through the prism of gradient elasticity," *International Journal of Solids and Structures*, vol. 78, pp. 149–159, 2016.