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A Numerical Algorithm for Positive Solutions of Concave and Convex Elliptic Equation on $\mathbb{R}^2$

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Abstract—In this paper we investigate numerically positive solutions of the equation $-\Delta u = \lambda u^q + u^p$ with Dirichlet boundary condition in a boundary domain Ω for $\lambda > 0$ and $0 < q < 1 < p < 2^*$, we will compute and visualize the range of λ, this problem achieves a numerical solution.

Keywords—positive solutions; concave-convex; sub-supersolution method; pseudo arclength method.

I. Introduction

IN this paper we use pseudo arclength method to obtain approximation solutions of the following concave-convex nonlinear problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \quad (1a) \\ u > 0, & x \in \Omega, \quad (1b) \\ u = 0, & x \in \partial\Omega, \quad (1c) \end{cases}$$

where $\lambda > 0$ is a real parameter, Δ is the Laplace operator and $0 < q < 1 < p < 2^*$, in which $2^* = \infty$ because Ω is a bounded domain in $\mathbb{R}^2$ here. In this paper, the solutions will be solved are classical solutions, which satisfy Eq.(1) pointwise. For the solution u of Eq.(1), we consider the energy functional $I_\lambda(u)$, for each $u \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$,

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\|u\|_{q+1}^{q+1} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \quad (2)$$

where $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$, $\|u\|_{q+1} = (\int_\Omega |u|^{q+1} dx)^{\frac{1}{q+1}}$, $\|u\|_{p+1} = (\int_\Omega |u|^{p+1} dx)^{\frac{1}{p+1}}$.

If we consider $\lambda = 0$, Eq.(1) turns into Lane-Emden(-Fowler) equation,

$$\begin{cases} -\Delta u = u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (3)$$

The above equation was introduced by Lane in the mid-19th century, as a model of the distribution of clusters of stars in Astrophysics. It was in 1931 that Fowler [1] solved completely the problem of finding radially symmetric equations $u = u(|x|)$. From then on, Eq.(3) has been extensively investigated, see for example [2,3] for a survey.

Under the assumption that $\lambda > 0$ is small enough, our Eq.(1) can be regarded as a perturbation problem of Eq.(3). To emphasize the dependence on λ, Eq.(1) is often regarded as Eq.$(1)_\lambda$.

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The existence of positive solutions of concave and convex nonlinearities have been studied by many researchers, we mention [4,5] to name a few.

Theorem 1.1 For all $0 < q < 1 < p < 2^*$ there exists $\Lambda \in \mathbb{R}, \Lambda > 0$, such that

1. for all $\lambda \in (0, \Lambda)$ problem $(1)_\lambda$ has a minimal solution u_λ such that $I_\lambda(u_\lambda) < 0$. Moreover u_λ is increasing with respect to λ;
2. for all $\lambda = \Lambda$ problem $(1)_\lambda$ has at least one weak solution $u \in H \cap L^{p+1}$;
3. for all $\lambda > \Lambda$ problem $(1)_\lambda$ has no solution.

The above theorem is proved in [6]. We now provide sketch of the proof: The smallest eigenvalue of

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \varphi = 0, & x \in \partial\Omega. \end{cases} \quad (4)$$

is denoted by $\widetilde{\lambda}$, $\widetilde{\varphi}$ denotes the corresponding eigenfunction satisfying $\widetilde{\varphi} > 0$ in Ω and $\|\widetilde{\varphi}\|_2 = 1$.

1. Define $\Lambda = sup\{\lambda > 0 : (1)_\lambda \text{ has a solution}\}$, we can prove $0 < \Lambda < \infty$ by sub-supersolution method. We can also show that, for all $0 < \lambda < \Lambda$ Eq.$(1)_\lambda$ has a solution u_λ, then we illustrate that the solution u_λ is a minimal solution of $(1)_\lambda$. In the end of the proof, we also use sub-supersolution method to show that

$$u_{\lambda_1} < u_{\lambda_2}, \qquad whenever \qquad \lambda_1 < \lambda_2.$$

2. We can prove that for λ_n being a sequence such that $\lambda_n \uparrow \Lambda$, and at the same time, for $u_n = u_{\lambda_n}$, there exists $u^* \in H$ and $u^* > 0$ such that $u_n \to u^*$ a.e. in Ω strongly in L^{p+1} and weakly in H. Such a u^* is thus a weak solution of $(1)_\lambda$.
3. This follows from the definition of Λ in the proof of 1.

Theorem 1.2 Let $0 < q < 1 < p < 2^*$. Then for all $\lambda \in (0, \Lambda)$ problem $(1)_\lambda$ has a second solution $v_\lambda > u_\lambda$.

Proof C.f.[6].

Remark We will compute and visualize solutions of Eq.$(1)_\lambda$ to illustrate Theorem 1.1 and Theorem 1.2 in III.

II. Numerical algorithm

The method discussed in this section, which is used to find zeros of function f, is Newton's method with the following equation:

$$x^{n+1} = x^n - \frac{f(x^n)}{f'(x^n)}. \quad (5)$$

The following theorem is used to show sufficient conditions under which Newton's method converges to a solution(c.f.[7]).

Theorem 2.1 Let $D \subset \mathbb{R}^n$ be open and convex and let $f : D \to \mathbb{R}^n$ be continuously differentiable. Assume that for some norm $\|\cdot\|$ on $\mathbb{R}^n$ and $x_0 \in D$ the following condition hold:

(a). f satisfies

$$\|f'(x) - f'(y)\| \le \gamma\|x - y\|$$

for all $x, y \in D$ and some constant $\gamma > 0$.

(b). The Jacobian matrix $f'(x)$ is nonsingular for all $x \in D$, and there exists a constant $\beta > 0$ such that

$$\|f'(x)^{-1}\| \le \beta, \quad x \in D.$$

(c). For the constants

$$\alpha := \|f'(x_0)^{-1}f(x_0)\| \quad and \quad q := \alpha\beta\gamma$$

the inequality

$$q < \frac{1}{2}$$

is satisfied.

(d). For $r := 2\alpha$ the closed ball $B[x_0, r] := \{x : \|x - x_0\| \le r\}$ is contained in D.

Then f has a unique zero x^* in $B[x_0, r]$. Starting with x_0 the Newton iteration

$$x^{n+1} = x^n - \frac{f(x^n)}{f'(x^n)}, \quad n = 0, 1, \cdots$$

is well-defined. The sequence $\{x^n\}$ converges to the zero x^* of f, and we have the error estimate

$$\|x^n - x^*\| \le 2\alpha q^{2^n - 1}, \quad n = 0, 1, \cdots$$

For Eq.(1a), we define

$$f(u) = \Delta u + \lambda u^q + u^p \tag{6}$$

as function f in (5). To keep the solution away from the trivial solution $u \equiv 0$, we choose $\widetilde{\varphi}$ as initial value because $\widetilde{\varphi} > 0$(this initial value is important to the convergence of Newton's method). Besides, to keep the solution on the "right" way, we can't directly regard Eq.(6) as iterative function. So we let

$$L = \Delta + \widetilde{\lambda},$$
$$X = \{u | u \in C^2(\Omega), u|_{\partial\Omega = 0}\},$$
$$Y = \{u | u \in C^0(\Omega)\}.$$

From Equations above we know $L : X \to Y$ is a Fredholm operator, and $N(L) = span\{\widetilde{\varphi}\}$, we define inner product in Y by

$$\langle u, v\rangle = \int_\Omega uv dx. \tag{7}$$

Split spaces X, Y into

$$X = N(L) \oplus M, \qquad Y = N(L^*) \oplus R(L), \tag{8}$$

where $M = N(L)^\perp \cap X$, L^* is an adjoint operator and $R(L)$ is the range of L. From Eq.(8), we get $u = \tau\widetilde{\varphi} + \omega$, where $\langle\widetilde{\varphi}, \omega\rangle = 0$.

Then Eq.(1) changes into system (9)

$$\begin{cases} -\Delta(\tau\widetilde{\varphi} + \omega) = \lambda(\tau\widetilde{\varphi} + \omega)^q + (\tau\widetilde{\varphi} + \omega)^p, & x \in \Omega, \quad (9a) \\ \omega > 0, & x \in \Omega, \quad (9b) \\ \omega = 0, & x \in \partial\Omega, \quad (9c) \\ \langle\widetilde{\varphi}, \omega\rangle = 0. & (9d) \end{cases}$$

where τ, ω are unknown. We use

$$f(\tau, \omega) = \begin{pmatrix} \Delta(\tau\widetilde{\varphi} + \omega) + \lambda(\tau\widetilde{\varphi} + \omega)^q + (\tau\widetilde{\varphi} + \omega)^p \\ \langle\widetilde{\varphi}, \omega\rangle \end{pmatrix} \tag{10}$$

as iterative function to keep the solution on the "right" way, then we can solve the untrivial solution of Eq.(1).

We have done enough for Theorem 1.1(we will visualize it with an example in III). To find the second solution v_λ of (1), we have something to prepare.

As mentioned above, we've got the untrivial solution u_{λ_1} for Eq.$(1)_{\lambda_1}$, Let $(u_1, \lambda_1) = (u_{\lambda_1}, \lambda_1)$ be a solution on the solution branch of $f(u_\lambda, \lambda) = 0$, where $f(u_\lambda, \lambda)$ is Eq.(6). To get the solution branch, the most obvious parameter is the control variable λ. Let (u_1, λ_1) be the predictor point of $f(u_2, \lambda_2) = 0$, we can get the solution (u_2, λ_2) of Eq.$(1)_{\lambda_2}$. See Fig.1.

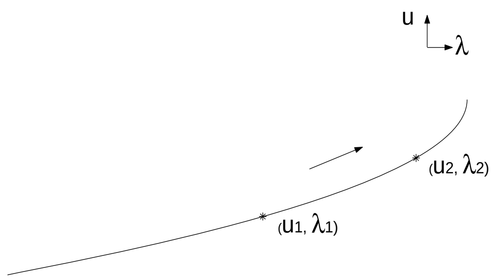

Fig.1. *Predictor-Corrector* method to calculate the untrivial solution branch with parameter λ.

While parameter λ has the advantage of having practical significance, it encounters difficulties at turning point, where the pulling direction is normal to the branch. See (u_3, λ_3) in Fig.2.

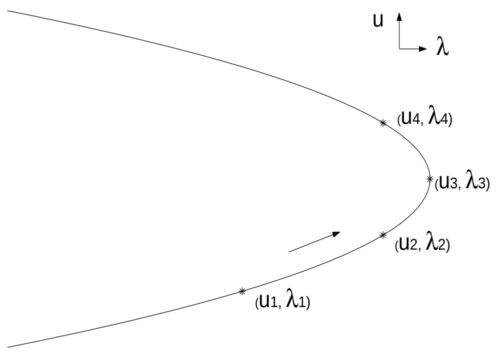

Fig.2. *Pseudo Arclength* method to calculate the untrivial solution branch with curve parameter s to cross the turning point (u_3, λ_3).

Solution branch of $f(u_\lambda, \lambda) = 0$ can be parameterized by curve parameter. A general curve parameter is called s. A parameterization by s means that the solution of $f(u_\lambda, \lambda) = 0$ depends on s:

$$u = u(s), \qquad \lambda = \lambda(s).$$

This means pulling the imaginary particle in the direction tangent to the branch; turning point does not pose problems. In [8] "pseudo arclength" was proposed, that is:

$$N(u, \lambda, s) = \dot{u}^{*T}(u - u^*) + \dot{\lambda}^*(\lambda - \lambda^*) - (s - s^*) = 0, \qquad (11)$$

where(u^*, λ^*) is the solution previously calculated, and $\dot{u}^{*T}, \dot{\lambda}^*$ can be calculated by

$$\dot{u}^{*T} = \beta v, \beta \in \mathbb{R},$$
$$\dot{\lambda}^* = \beta,$$

where v, β satisfy

$$f_u^* v = -f_\lambda^*,$$
$$\beta = \frac{\pm 1}{\sqrt{1 + \|v\|^2}}.$$

From equation $f(u_\lambda, \lambda) = 0$ and Eq.(11), we get the system:

$$F(u, \lambda, s) = \begin{pmatrix} f(u(s), \lambda(s)) \\ \\ N(u(s), \lambda(s), s) \end{pmatrix} = 0. \qquad (12)$$

We can solve the solution branch by Theorem 2.1. By this way, we can complete the proof of Theorem 1.2(we will visualize it with an example in III).

Algorithm

1. **Define** region Ω and step size $\delta s = (s - s^*)$.
2. **Let** (u_1, λ_1) be the initial value(this initial value is important to convergence of Newton's method).
   Begin loop $i = 0$ to n(decided by step size).
   Begin loop $j = 0$ to m.
   2.1 Calculate the derivative$F'(u, \lambda, s)$ of Eq.(12).
   2.2 Using Eq.(5) to calculate (u_{k+1}, λ_{k+1}). Repeat step 2.2 until convergence criteria are met:
   $$\|u_{k+1} - u_k\| \to 0.$$
   End loop j.
   End loop i.

III. Numerical results

We consider Eq.(1) with $q = 0.5, p = 3$ and let $\Omega = [-1, 1] \times [-1, 1]$, we want to obtain a numerical solution of the problem

$$\begin{cases} -\Delta u = \lambda u^{1/2} + u^3, & x \in \Omega, \quad (13a) \\ u > 0, & x \in \Omega, \quad (13b) \\ u = 0, & x \in \partial\Omega. \quad (13c) \end{cases}$$

General domain Ω is divided on an equidistant mesh, and then the five-point difference scheme is used to discrete Laplace operator Δ , where we choose $N = 50, h = 1/50$. Using pseudo arclength method, we have solution branch of Eq.(13).(See Fig.3)

From Fig.3, we can know that point B is a turning point

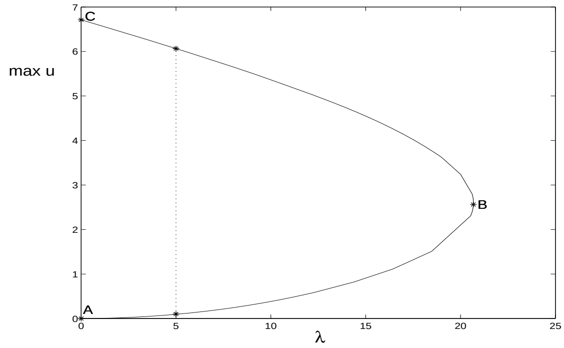

Fig.3. solution branch of Eq.(13) with different λ.

of solution branch of (13). For $\lambda < x_B \approx 20.68$, Eq.(13) has two solutions which illustrates Theorem 1.2. We also know that for $\lambda > x_B \approx 20.68$, Eq.(13) has no solution which agrees with (3) in Theorem 1.1. Take $\lambda = 5$ for example, we visualize the solutions of (13)(see Fig.4, Fig.5). Fig.4 denotes the solution u of (13) on the branch A to B(which is the minimal solution) and Fig.5 denotes the solution u of (13) on the branch B to C.

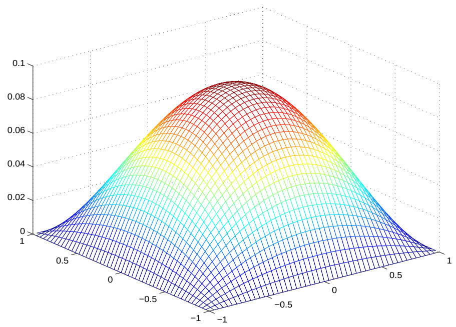

Fig.4. the solution u with $\lambda = 5$ of (13) on the branch A to B in Fig.3 (which is the minimal solution).

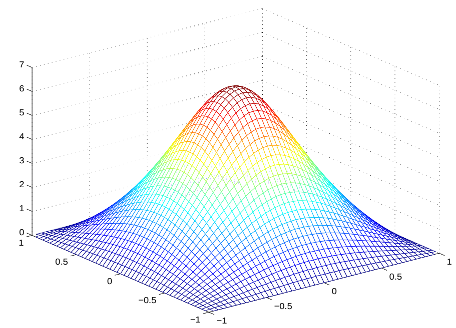

Fig.5. the solution u with $\lambda = 5$ of (13) on the branch B to C in Fig.3.

We present just some values of u on the solution branch A to B of Fig.3 in following tables for different λ:

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TABLE I
APPROXIMATION OF u FOR $\lambda = 5$ WITH $I_\lambda(u) = -0.0335 < 0$ AND $\varepsilon = 10^{-10}$, WHERE ε DENOTES THE RELATIVE CONVERGENCE ERROR $\varepsilon = \|u_{n+1} - u_n\|$.

x\y	-0.8	-0.4	0	0.4	0.8
-0.8	0.0121	0.0281	0.0333	0.0281	0.0121
-0.4	0.0281	0.0666	0.0794	0.0666	0.0281
0	0.0333	0.0794	0.0948	0.0794	0.0333
0.4	0.0281	0.0666	0.0794	0.0666	0.0281
0.8	0.0121	0.0281	0.0333	0.0281	0.0121

TABLE II
APPROXIMATION OF u FOR $\lambda = 10$ WITH $I_\lambda(u) = -0.5434 < 0$ AND $\varepsilon = 10^{-11}$

x\y	-0.8	-0.4	0	0.4	0.8
-0.8	0.0488	0.1132	0.1342	0.1132	0.0488
-0.4	0.1132	0.2684	0.3201	0.2684	0.1132
0	0.1342	0.3201	0.3826	0.3201	0.1342
0.4	0.1132	0.2684	0.3201	0.2684	0.1132
0.8	0.0488	0.1132	0.1342	0.1132	0.0488

TABLE III
APPROXIMATION OF u FOR $\lambda = 15$ WITH $I_\lambda(u) = -2.3602 < 0$ AND $\varepsilon = 10^{-10}$

x\y	-0.8	-0.4	0	0.4	0.8
-0.8	0.0983	0.2285	0.2712	0.2285	0.0983
-0.4	0.2285	0.5436	0.6495	0.5436	0.2285
0	0.2712	0.6495	0.7777	0.6495	0.2712
0.4	0.2285	0.5436	0.6495	0.5436	0.2285
0.8	0.0983	0.2285	0.2712	0.2285	0.0983

By comparing Table I to Table II and Table III, we see that u_λ is increasing with respect to λ, and find that $I_\lambda(u_\lambda) < 0$ for $\lambda = 5, 10, 15$. In fact, $I_\lambda(u_\lambda)$ of (13) computed are all negative on solution branch C to A except for point A, and $I_\lambda(u_\lambda)$ is increasing from C to A with $I_\lambda(u_{\lambda_C}) \approx -49.77$ and $I_\lambda(u_{\lambda_A}) = 0$.

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