International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:8, 2012

A note on the convergence of the generalized AOR iterative method for linear systems

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Abstract—Recently, some convergent results of the generalized AOR iterative (GAOR) method for solving linear systems with strictly diagonally dominant matrices are presented in [Darvishi, M.T., Hessari, P.: On convergence of the generalized AOR method for linear systems with diagonally dominant cofficient matrices. Appl. Math. Comput. 176, 128-133 (2006)] and [Tian, G.X., Huang, T.Z., Cui, S.Y.: Convergence of generalized AOR iterative method for linear systems with strictly diagonally dominant cofficient matrices. J. Comp. Appl. Math. 213, 240-247 (2008)]. In this paper, we give the convergence of the GAOR method for linear systems with strictly doubly diagonally dominant matrix, which improves these corresponding results.

Keywords—Diagonally dominant matrix, GAOR method; Linear system, Convergence

I. Introduction

CONSIDER, the linear system

$$Hy = f \tag{1}$$

where

$$H = \begin{bmatrix} I - B_1 & D \\ C & I - B_2 \end{bmatrix},$$

is an invertible matrix. For example, in the generalized least squares problem [3], [4], we must solve the generalized least squares problem

$$\min_{x \in R^n} (Ax - b)^T W^{-1} (Ax - b),$$

where W, is the variance-covariance matrix [5]. If $I - B_i$ for $i = 1, 2$ are nonsingular, we can apply the regular SOR method, or the regular AOR method [6] to solve (1). However, $I - B_i$ for $i = 1, 2$ sometimes are singular. In fact, even if $I - B_i$ are nonsingular, it is also not easy to solve linear system (1) because we have to find the inverses of $I - B_i$ for $i = 1, 2$, or to solve two subsystems

$$(I - B_i)x_i = d_i, i = 1, 2.$$

Hence a generalized SOR (GSOR) method was proposed by Yuan to solve linear system (1) in [3], afterwards, Yuan and Jin [4] established a generalized AOR (GAOR) method to solve linear system (1) as follows.

$$y^{k+1} = G_{\omega,\gamma} y^k + \omega k, \tag{2}$$

where

$$G_{\omega,\gamma} = (1 - \omega)I + \omega J + \omega\gamma K,$$

$$k = \begin{bmatrix} I & 0 \\ -\gamma C & I \end{bmatrix} f,$$

$$J = \begin{bmatrix} B_1 & -D \\ -C & B_2 \end{bmatrix},$$

$$K = \begin{bmatrix} 0 & 0 \\ C(I - B_1) & CD \end{bmatrix} = \begin{bmatrix} 0 \\ C \end{bmatrix} \begin{bmatrix} I - B_1 & D \end{bmatrix}.$$

From the above, we know that the GAOR method does not need any inverse of $I - B_i$ for $i = 1, 2$. It is easy to check that the GAOR method is the GSOR method when $\omega = \gamma$; the generalized Jacobi method when $\gamma = 0$; and the regular AOR method [5] when $B_1 = B_2 = 0$.

Throughout this paper, we shall employ the same notations as in [1], [2]. For instance, $N \triangleq \{1, 2, \ldots, n\}$, denote the class of all complex matrices by $C^{n,n}$, and denote $\rho(G_{\omega,\gamma})$ by the spectral radius of iterative matrix $G_{\omega,\gamma}$.

For $A = (a_{ij}) \in C^{n,n}$, let

$$R_i(A) = \sum_{i \neq j} |a_{ij}|.$$

Recall that A is said to be strictly diagonally dominant ($A \in SD$), if

$$|a_{ii}| > R_i(A), \forall i \in N,$$

and if

$$|a_{ii}||a_{jj}| > R_i(A)R_j(A), \forall i, j \in N, i \neq j.$$

we call that A is strictly doubly diagonally dominant ($A \in SDD$). Obviously, $SD \subseteq SDD$.

In [1], [2], the following main results are presented:

Theorem 1.1 ([1]) Let $H \in SD$, then $\rho(G_{\omega,\gamma})$ satisfies the following inequality

$$\begin{aligned} &|\omega - 1| + \min_i\{|\omega|J_i + |\omega\gamma|K_i\} \leq \rho(G_{\omega,\gamma}) \leq \\ &|\omega - 1| + \max_i\{|\omega|J_i + |\omega\gamma|K_i\}, \end{aligned} \tag{3}$$

where J_i and K_i are the i-row sums of the modulus of the entries of J and K, respectively.

Theorem 1.2 ([2]) Let $H \in SD$, then $\rho(G_{\omega,\gamma})$ satisfies the following inequality

$$\begin{aligned} &\min_i\{|\omega - 1| - |\omega|(J + \gamma K)_i\} \leq \rho(G_{\omega,\gamma}) \leq \\ &\max_i\{|\omega - 1| + |\omega|(J + \gamma K)_i\}. \end{aligned} \tag{4}$$

where $(J + \gamma K)_i$ denotes the i-row sums of the modulus of the entries of matrix $J + \gamma K$.

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International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:8, 2012

In this note, we will continue to study this problem and obtain some new inequalities which improves the corresponding results in [1], [2].

The paper is organized as follows. In Section 2, based on our results [8], [9], we obtain new upper and lower bounds for the spectral radius of $G_{\omega,\gamma}$ when $H \in SDD$, which is better than one of Theorem 1.1 and Theorem 1.2. In Section 3, we discuss the convergence of the GAOR method for SDD. In Section 4, we present numerical examples to show that our results are better than Theorem 1.1 and 1.2.

II. Upper and lower bounds for $\rho(G_{\omega,\gamma})$

Theorem 2.1 Let $H \in SDD$. Then $\rho(G_{\omega,\gamma})$ satisfies the following inequality

$$\rho(G_{\omega,\gamma}) \geq \max\left\{0, \min_{i\neq j}\{|\omega-1| - |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}\right\}, \tag{5}$$

or

$$\rho(G_{\omega,\gamma}) \leq \max_{i\neq j}\left\{|\omega-1| + |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}\right\}. \tag{6}$$

Proof. Let λ be an arbitrary eigenvalue of iterative matrix $G_{\omega,\gamma}$, then

$$det(\lambda I - G_{\omega,\gamma}) = 0. \tag{7}$$

We can show that Eq.(7) holds if and only if

$$det((\lambda+\omega-1)I - \omega J - \omega\gamma K) = 0.$$

If we take the parameter γ,ω and λ in order that

$$(\lambda+\omega-1)I - \omega J - \omega\gamma K \in SDD,$$

i.e., for any $i, j \in N, i \neq j$,

$$\omega^2((J+\gamma K)_i - (J+\gamma K)_{ii})((J+\gamma K)_j - (J+\gamma K)_{jj}) < |\lambda+\omega-1-\omega(J+\gamma K)_{ii}||\lambda+\omega-1-\omega(J+\gamma K)_{jj}|$$

then λ is not an eigenvalue of $G_{\omega,\gamma}$, where $(J+\gamma K)_{ii}$ denotes the diagonal element of matrix $J+\gamma K$.

Obviously, especially when

$$\omega^2(J+rK)_i(J+\gamma K)_j < |\lambda+\omega-1|^2, \forall i,j \in N, i \neq j,$$

i.e.,

$$|\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j} < |\lambda+\omega-1|$$

then λ can not an eigenvalue of $G_{\omega,\gamma}$. Hence if λ is an eigenvalue of $G_{\omega,\gamma}$, we must have

$$|\lambda+\omega-1| \leq |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j},$$

especially,

$$||\lambda| - |\omega-1|| \leq |\omega|\sqrt{(J+\gamma K)_i(J+rK)_j},$$

or

$$|\omega-1| - |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j} \leq |\lambda| \leq |\omega-1| + |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}.$$

i.e.,

$$\rho(G_{\omega,\gamma}) \geq \max\left\{0, \min_{i\neq j}\{|\omega-1| - |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}\right\},$$

or

$$\rho(G_{\omega,\gamma}) \leq \max_{i\neq j}\left\{|\omega-1| + |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}\right\}.$$

So the assertion holds. The proof is completed. □

Remark 2.1 The results (5) and (6) of Theorem 2.1 are better than ones of Theorem 1.1 and Theorem 1.2, since for any $i, j \in N, i \neq j$

$$\min_i\{(J+\gamma K)_i\} \leq \sqrt{(J+\gamma K)_i(J+\gamma K)_j} \leq \max_i\{(J+\gamma K)_i\}. \tag{8}$$

In addition, for some values of γ and ω, the GAOR method reduces to the well-known methods, i.e.,

1) The GAOR method reduces to the GSOR method when $\omega = \gamma$, thus

$$\rho(G_{\omega,\omega}) \geq \min_{i\neq j}\left\{0, |\omega-1| - |\omega|\sqrt{(J+\omega K)_i(J+\omega K)_j}\right\},$$

$$\rho(G_{\omega,\omega}) \leq \max_{i\neq j}\left\{|\omega-1| + |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j}\right\}.$$

2) The GAOR method reduces to the generalized Jacobi method when $\gamma = 0$, thus

$$\rho(G_{\omega,0}) \geq \min_{i\neq j}\left\{|\omega-1| - |\omega|\sqrt{J_iJ_j}\right\},$$

$$\rho(G_{\omega,0}) \leq \max_{i\neq j}\left\{|\omega-1| + |\omega|\sqrt{J_iJ_j}\right\}.$$

III. Convergence of the GAOR method

Theorem 3.1 Let $H \in SDD$ and assume that γ and ω satisfy

$$\max_{i\neq j}(J+\gamma K)_i(J+\gamma K)_j < 1$$

and

$$0 < \omega < \frac{2}{1+\sqrt{\max_{i\neq j}(J+\gamma K)_i(J+\gamma K)_j}}, \forall i,j \in N.$$

then the GAOR is convergent.

Proof. By Theorem 2.1, we see that $\rho(G_{\omega,\gamma}) < 1$ if

$$|\omega-1| + |\omega|\sqrt{(J+\gamma K)_i(J+\gamma K)_j} < 1, \forall i,j \in N, i \neq j.$$

Hence, ω must satisfy $0 < \omega < 2$.

Next, we consider the following two cases:

Case 1: If $0 < \omega \leq 1$, i.e.,

$$(J+\omega K)_i(J+\omega K)_j < 1, \forall i,j \in N.$$

Case 2: If $1 < \omega < 2$, i.e.,

$$0 < \omega < \frac{2}{1+\sqrt{(J+\gamma K)_i(J+\gamma K)_j}}, \forall i,j \in N, i \neq j.$$

which implies

$$(J+\omega K)_i(J+\omega K)_j < 1, \forall i,j \in N, i \neq j.$$

Combining Case 1 with 2, we get

$$\max_{i\neq j}(J+\gamma K)_i(J+\gamma K)_j < 1$$

and

$$0 < \omega < \frac{2}{1+\sqrt{\max\limits_{i\neq j}(J+\gamma K)_i(J+\gamma K)_j}}, \forall i,j \in N, i \neq j.$$

Hence the assertion holds. The proof is completed. $\square$

According to the inequality (8), our results are obviously better than the ones in [1], [2]. In addition, some other conclusions in [1], [2] may be also obtained similarly. Here, we can not describe these results in detail.

IV. Numerical Example

The following two simple examples show that the results of Theorem 2.1 are better than ones of Theorem 1.1 and 1.2.

Example 4.1 Let

$$H = \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & 1 & \frac{1}{8} \\ \hline \frac{1}{4} & \frac{1}{3} & 1 \end{array}\right] = \left[\begin{array}{cc} I-B_1 & D \\ C & I-B_2 \end{array}\right].$$

Clearly, $H \in SD$. Meanwhile, $H \in SDD$. For convenient, supposing that $\omega = \gamma = 1.0$. By Theorem 2.1, we have $0 \leq \rho(l_{\omega,\gamma}) \leq\ 0.5046$, but we have, by Theorem 1.2, $0 \leq \rho(l_{\omega,\gamma}) \leq\ 0.8333$. In fact, $\rho(l_{\omega,\gamma}) = 0.2392$. These show that our results are better than ones of Theorem 1.1 and 1.2.

Example 4.2 Let

$$H = \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & 1 & \frac{1}{8} \\ \hline \frac{1}{2} & \frac{2}{3} & 1 \end{array}\right] = \left[\begin{array}{cc} I-B_1 & D \\ C & I-B_2 \end{array}\right].$$

Obviously, $H \in SDD$, but $H \notin SD$, therefore Theorem 1.1 and Theorem 1.2 are not valid. For convenient, supposing that $\omega = \gamma = 1.0$. By Theorem 2.1, we have

$$0 \leq \rho(l_{\omega,\gamma}) \leq\ \frac{\sqrt{35}}{6} < 1,$$

which shows that our conclusions are valid.

Acknowledgment

This work is in part supported by the National Natural Science Foundation of China (11026085, 11101071), the Fundamental Research Funds for the Central Universities (ZYGX2009J103).

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