# A note on the convergence of the generalized AOR iterative method for linear systems 

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#### Abstract

Recently, some convergent results of the generalized AOR iterative (GAOR) method for solving linear systems with strictly diagonally dominant matrices are presented in [Darvishi, M.T., Hessari, P.: On convergence of the generalized AOR method for linear systems with diagonally dominant cofficient matrices. Appl. Math. Comput. 176, 128-133 (2006)] and [Tian, G.X., Huang, T.Z., Cui, S.Y.: Convergence of generalized AOR iterative method for linear systems with strictly diagonally dominant cofficient matrices. J. Comp. Appl. Math. 213, 240-247 (2008)]. In this paper, we give the convergence of the GAOR method for linear systems with strictly doubly diagonally dominant matrix, which improves these corresponding results.


Keywords-Diagonally dominant matrix, GAOR method; Linear system, Convergence

## I. Introduction

C ONSIDER, the linear system

$$
\begin{equation*}
H y=f \tag{1}
\end{equation*}
$$

where

$$
H=\left[\begin{array}{ll}
I-B_{1} & D \\
C & I-B_{2}
\end{array}\right]
$$

is an invertible matrix. For example, in the generalized least squares problem [3], [4], we must solve the generalized least squares problem

$$
\min _{x \in R^{n}}(A x-b)^{T} W^{-1}(A x-b)
$$

where $W$, is the variance-covariance matrix [5]. If $I-B_{i}$ for $i=1,2$ are nonsingular, we can apply the regular SOR method, or the regular AOR method [6] to solve (1). However, $I-B_{i}$ for $i=1,2$ sometimes are singular. In fact, even if $I-B_{i}$ are nonsingular, it is also not easy to solve linear system (1) because we have to find the inverses of $I-B_{i}$ for $i=1,2$, or to solve two subsystems

$$
\left(I-B_{i}\right) x_{i}=d_{i}, i=1,2
$$

Hence a generalized SOR (GSOR) method was proposed by Yuan to solve linear system (1) in [3], afterwards, Yuan and Jin [4] established a generalized AOR (GAOR) method to solve linear system (1) as follows.

$$
\begin{equation*}
y^{k+1}=G_{\omega, \gamma} y^{k}+\omega k \tag{2}
\end{equation*}
$$

where

$$
G_{\omega, \gamma}=(1-\omega) I+\omega J+\omega \gamma K
$$

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$$
\begin{gathered}
k=\left[\begin{array}{ll}
I & 0 \\
-\gamma C & I
\end{array}\right] \\
J=\left[\begin{array}{ll}
B_{1} & -D \\
-C & B_{2}
\end{array}\right] \\
K=\left[\begin{array}{ll}
0 & 0 \\
C\left(I-B_{1}\right) & C D
\end{array}\right]=\left[\begin{array}{l}
0 \\
C
\end{array}\right]\left[\begin{array}{ll}
I-B_{1} & D
\end{array}\right] .
\end{gathered}
$$

From the above, we know that the GAOR method does not need any inverse of $I-B_{i}$ for $i=1,2$. It is easy to check that the GAOR method is the GSOR method when $\omega=\gamma$; the generalized Jacobi method when $\gamma=0$; and the regular AOR method [5] when $B_{1}=B_{2}=0$.

Throughout this paper, we shall employ the same notations as in [1], [2]. For instance, $N \triangleq\{1,2, \ldots, n\}$, denote the class of all complex matrices by $C^{n, n}$, and denote $\rho\left(G_{\omega, \gamma}\right)$ by the spectral radius of iterative matrix $G_{\omega, \gamma}$.

For $A=\left(a_{i j}\right) \in C^{n, n}$, let

$$
R_{i}(A)=\sum_{i \neq j}\left|a_{i j}\right|
$$

Recall that $A$ is said to be strictly diagonally dominant $(A \in$ $S D)$, if

$$
\left|a_{i i}\right|>R_{i}(A), \forall i \in N
$$

and if

$$
\left|a_{i i}\right|\left|a_{j j}\right|>R_{i}(A) R_{j}(A), \forall i, j \in N, i \neq j
$$

we call that $A$ is strictly doubly diagonally dominant $(A \in$ $S D D)$. Obviously, $S D \subseteq S D D$.

In [1], [2], the following main results are presented:
Theorem 1.1 ([1]) Let $H \in S D$, then $\rho\left(G_{\omega, \gamma}\right)$ satisfies the following inequality

$$
\begin{align*}
& |\omega-1|+\min _{i}\left\{|\omega| J_{i}+|\omega \gamma| K_{i}\right\} \leq \rho\left(G_{\omega, \gamma}\right) \leq \\
& |\omega-1|+\max _{i}\left\{|\omega| J_{i}+|\omega \gamma| K_{i}\right\} \tag{3}
\end{align*}
$$

where $J_{i}$ and $K_{i}$ are the $i$-row sums of the modulus of the entries of $J$ and $K$, respectively.

Theorem 1.2 ([2]) Let $H \in S D$, then $\rho\left(G_{\omega, \gamma}\right)$ satisfies the following inequality

$$
\begin{align*}
& \min _{i}\left\{|\omega-1|-|\omega|(J+\gamma K)_{i}\right\} \leq \rho\left(G_{\omega, \gamma}\right) \leq \\
& \max _{i}\left\{|\omega-1|+|\omega|(J+\gamma K)_{i}\right\} \tag{4}
\end{align*}
$$

where $(J+\gamma K)_{i}$ denotes the $i$-row sums of the modulus of the entries of matrix $J+\gamma K$.

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In this note, we will continue to study this problem and obtain some new inequalities which improves the corresponding results in [1], [2].
The paper is organized as follows. In Section 2, based on our results [8], [9], we obtain new upper and lower bounds for the spectral radius of $G_{\omega, \gamma}$ when $H \in S D D$, which is better than one of Theorem 1.1 and Theorem 1.2. In Section 3 , we discuss the convergence of the GAOR method for SDD. In Section 4, we present numerical examples to show that our results are better than Theorem 1.1 and 1.2

## II. UpPER AND LOWER BOUNDS FOR $\rho\left(G_{\omega, \gamma}\right)$

Theorem 2.1 Let $H \in S D D$. Then $\rho\left(G_{\omega, \gamma}\right)$ satisfies the following inequality
$\rho\left(G_{\omega, \gamma}\right) \geq \max \left\{0, \min _{i \neq j}\left\{|\omega-1|-|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}\right\}\right.$,
or

$$
\begin{equation*}
\rho\left(G_{\omega, \gamma}\right) \leq \max _{i \neq j}\left\{|\omega-1|+|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}\right\} . \tag{5}
\end{equation*}
$$

Proof. Let $\lambda$ be an arbitrary eigenvalue of iterative matrix $G_{\omega, \gamma}$, then

$$
\begin{equation*}
\operatorname{det}\left(\lambda I-G_{\omega, \gamma}\right)=0 \tag{7}
\end{equation*}
$$

We can show that Eq.(7) holds if and only if

$$
\operatorname{det}((\lambda+\omega-1) I-\omega J-\omega \gamma K)=0
$$

If we take the parameter $\gamma, \omega$ and $\lambda$ in order that

$$
(\lambda+\omega-1) I-\omega J-\omega \gamma K \in S D D,
$$

i.e., for any $i, j \in N, i \neq j$,

$$
\begin{aligned}
& \omega^{2}\left((J+\gamma K)_{i}-(J+\gamma K)_{i i}\right)\left((J+\gamma K)_{j}-(J+\gamma K)_{j j}\right)< \\
& \left|\lambda+\omega-1-\omega(J+\gamma K)_{i i}\right|\left|\lambda+\omega-1-\omega(J+\gamma K)_{j j}\right|
\end{aligned}
$$

then $\lambda$ is not an eigenvalue of $G_{\omega, \gamma}$, where $(J+\gamma K)_{i i}$ denotes the diagonal element of matrix $J+\gamma K$.

Obviously, especially when

$$
\omega^{2}(J+r K)_{i}(J+\gamma K)_{j}<|\lambda+\omega-1|^{2}, \forall i, j \in N, i \neq j,
$$

i.e.,

$$
|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}<|\lambda+\omega-1|
$$

then $\lambda$ can not an eigenvalue of $G_{\omega, \gamma}$. Hence if $\lambda$ is an eigenvalue of $G_{\omega, \gamma}$, we must have

$$
|\lambda+\omega-1| \leq|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}},
$$

especially,

$$
||\lambda|-|\omega-1|| \leq|\omega| \sqrt{(J+\gamma K)_{i}(J+r K)_{j}}
$$

or

$$
\begin{aligned}
& |\omega-1|-|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}} \leq|\lambda| \leq \\
& |\omega-1|+|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}} .
\end{aligned}
$$

i.e.,
$\rho\left(G_{\omega, \gamma}\right) \geq \max \left\{0, \min _{i \neq j}\left\{|\omega-1|-|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}\right\}\right.$,
or
and

$$
\rho\left(G_{\omega, \gamma}\right) \leq \max _{i \neq j}\left\{|\omega-1|+|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}\right\} .
$$

So the assertion holds. The proof is completed.
Remark 2.1 The results (5) and (6) of Theorem 2.1 are better than ones of Theorem 1.1 and Theorem 1.2, since for any $i, j \in N, i \neq j$

$$
\begin{align*}
& \min _{i}\left\{(J+\gamma K)_{i}\right\} \leq \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}} \leq  \tag{8}\\
& \max _{i}\left\{(J+\gamma K)_{i}\right\} .
\end{align*}
$$

In addition, for some values of $\gamma$ and $\omega$, the GAOR method reduces to the well-known methods, i.e.,

1) The GAOR method reduces to the GSOR method when $\omega=\gamma$, thus

$$
\begin{aligned}
& \rho\left(G_{\omega, \omega}\right) \geq \min _{i \neq j}\left\{0,|\omega-1|-|\omega| \sqrt{(J+\omega K)_{i}(J+\omega K)_{j}}\right\}, \\
& \rho\left(G_{\omega, \omega}\right) \leq \max _{i \neq j}\left\{|\omega-1|+|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}\right\} .
\end{aligned}
$$

2) The GAOR method reduces to the generalized Jacobi method when $\gamma=0$, thus

$$
\begin{aligned}
& \rho\left(G_{\omega, 0}\right) \geq \min _{i \neq j}\left\{|\omega-1|-|\omega| \sqrt{J_{i} J_{j}}\right\}, \\
& \rho\left(G_{\omega, 0}\right) \leq \max _{i \neq j}\left\{|\omega-1|+|\omega| \sqrt{J_{i} J_{j}}\right\} .
\end{aligned}
$$

## III. Convergence of the GaOr method

Theorem 3.1 Let $H \in S D D$ and assume that $\gamma$ and $\omega$ satisfy

$$
\max _{i \neq j}(J+\gamma K)_{i}(J+\gamma K)_{j}<1
$$

$$
0<\omega<\frac{2}{1+\sqrt{\max _{i \neq j}(J+\gamma K)_{i}(J+\gamma K)_{j}}}, \forall i, j \in N .
$$

then the GAOR is convergent.
Proof. By Theorem 2.1, we see that $\rho\left(G_{\omega, \gamma}\right)<1$ if

$$
|\omega-1|+|\omega| \sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}<1, \forall i, j \in N, i \neq j .
$$

Hence, $\omega$ must satisfy $0<\omega<2$.
Next, we consider the following two cases:
Case 1: If $0<\omega \leq 1$, i.e.,

$$
(J+\omega K)_{i}(J+\omega K)_{j}<1, \forall i, j \in N
$$

Case 2: If $1<\omega<2$, i.e.,

$$
0<\omega<\frac{2}{1+\sqrt{(J+\gamma K)_{i}(J+\gamma K)_{j}}}, \forall i, j \in N, i \neq j .
$$

which implies

$$
(J+\omega K)_{i}(J+\omega K)_{j}<1, \forall i, j \in N, i \neq j .
$$

## Combining Case 1 with 2, we get

$$
\max _{i \neq j}(J+\gamma K)_{i}(J+\gamma K)_{j}<1
$$

and
$0<\omega<\frac{2}{1+\sqrt{\max _{i \neq j}(J+\gamma K)_{i}(J+\gamma K)_{j}}}, \forall i, j \in N, i \neq j$.
Hence the assertion holds. The proof is completed.
According to the inequality (8), our results are obviously better than the ones in [1], [2]. In addition, some other conclusions in [1], [2] may be also obtained similarly. Here, we can not describe these results in detail.

## IV. Numerical Example

The following two simple examples show that the results of Theorem 2.1 are better than ones of Theorem 1.1 and 1.2.

## Example 4.1 Let

$$
H=\left[\begin{array}{cc|c}
1 & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & 1 & \frac{1}{8} \\
\hline \frac{1}{4} & \frac{1}{3} & 1
\end{array}\right]=\left[\begin{array}{cc}
I-B_{1} & D \\
C & I-B_{2}
\end{array}\right] .
$$

Clearly, $H \in S D$. Meanwhile, $H \in S D D$. For convenient, supposing that $\omega=\gamma=1.0$. By Theorem 2.1, we have $0 \leq \rho\left(l_{\omega, \gamma}\right) \leq 0.5046$, but we have, by Theorem 1.2, $0 \leq \rho\left(l_{\omega, \gamma}\right) \leq 0.8333$. In fact, $\rho\left(l_{\omega, \gamma}\right)=0.2392$. These show that our results are better than ones of Theorem 1.1 and 1.2.

## Example 4.2 Let

$$
H=\left[\begin{array}{cc|c}
1 & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{6} & 1 & \frac{1}{8} \\
\hline \frac{1}{2} & \frac{2}{3} & 1
\end{array}\right]=\left[\begin{array}{cc}
I-B_{1} & D \\
C & I-B_{2}
\end{array}\right] .
$$

Obviously, $H \in S D D$, but $H \notin S D$, therefore Theorem 1.1 and Theorem 1.2 are not valid. For convenient, supposing that $\omega=\gamma=1.0$. By Theorem 2.1, we have

$$
0 \leq \rho\left(l_{\omega, \gamma}\right) \leq \frac{\sqrt{35}}{6}<1,
$$

which shows that our conclusions are valid.

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