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A Nonlinear Parabolic Partial Differential Equation Model for Image Enhancement

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Abstract**—**We present a robust nonlinear parabolic partial differential equation (PDE)-based denoising scheme in this article. Our approach is based on a second-order anisotropic diffusion model that is described first. Then, a consistent and explicit numerical approximation algorithm is constructed for this continuous model by using the finite-difference method. Finally, our restoration experiments and method comparison, which prove the effectiveness of this proposed technique, are discussed in this paper.

Keywords**—**Image denoising and restoration, nonlinear PDE model, anisotropic diffusion, numerical approximation scheme, finite differences.

I. Introduction

NONLINEAR PDE based approaches are increasingly used in many image processing and analysis domains, such as image denoising and restoration [1]. Since feature-preserving image restoration is still a challenging task in image processing, these nonlinear PDE-based filters represent the proper solution. The classic 2D image filters, like Average, Gaussian or Median, reduce the noise, but also blur the boundaries and have no localization property [2].

Many nonlinear diffusion-based noise removal techniques have been elaborated since the influential model of Perona and Malik was proposed [3]. Since it is common to derive a PDE from a variational scheme, a lot of variational restoration algorithms have been developed in the last 25 years [4]. The most influential one is the TV denoising developed in 1992 [4], [5]. These variational and nonlinear PDE schemes overcome the blurring effect but usually generate the unintended staircase effect [6]. We developed numerous PDE and variational models that alleviate the undesired effects, in our previous papers in this domain [7]-[10]. In this article, we consider a second-order parabolic PDE-based image restoration technique that overcomes successfully the staircase and blurring effects and outperforms many state-of-the-art diffusion-based methods [11].

The presented mathematical model is described in the second section. Then, the finite-difference based numerical discretization scheme is presented in the third section of this paper.

Our successfully denoising experiments and method comparison results are detailed in the fourth section. Then, the conclusions of this article are drawn in the fifth section.

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II. A Second-Order Parabolic PDE Model

Here we propose an anisotropic diffusion model that achieves an efficient noise removal while preserving successfully the edges and other essential image details. Our PDE-based restoration method is based on the following nonlinear parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = div\left(\xi_u\left(\|\nabla u\|\right)\nabla u\right) - \lambda\left(u - u_0\right) \\ u\left(0, x, y\right) = u_0 \\ u\left(t, x, y\right) = 0, \quad \forall t \geq 0, \ \left(x, y\right) \in \partial\Omega \end{cases} \tag{1}$$

where u_0 is the initial noisy image, $\left(x, y\right) \in \Omega$, $\Omega \subset R^2$, $\lambda \in \left(0,1\right)$ and $\partial\Omega$ represents the frontier of the Ω domain.

We consider the following diffusivity, or edge-stopping function, $\xi_u : [0, \infty) \to [0, \infty)$, for this image enhancement approach:

$$\xi_u\left(s\right) = \frac{\alpha}{\left(\dfrac{s}{\eta_u}\right)^k + \eta_u \left|\ln\left(\dfrac{s}{\eta_u}\right)\right|} \tag{2}$$

where the conductance parameter depends on some statistics of the evolving image *u* at time *t*. Therefore, we consider the following statistics-based form for this parameter:

$$\eta_u = \beta \cdot \mu\left(\|\nabla u\|\right) + \gamma \cdot \left|median\ \left(u\right)\right|, \tag{3}$$

where, $\beta \in \left(2,3\right)$, $\gamma \in \left(0,1\right)$, μ represents the averaging operator and median(u) returns the median of the current image.

The considered edge-stopping function ξ_u is properly selected, satisfying the conditions required by a good denoising [3], [7]. So, it is always positive and also monotonically decreasing, since we get $\xi_u(s_1) \geq \xi_u(s_2)$ for $\forall\, s_1 \leq s_2$. Also, we have $\lim_{s \to \infty} \xi_u(s) = 0$.

The restored image is determined as the solution of the PDE given by (1). One can prove the existence and uniqueness of a weak solution for this differential model. Thus, the proposed

PDE model admits a weak solution if the flux function, computed as $s \cdot \xi_u(s)$, is monotonically increasing. In order for this to happen, its derivative must be positive, which means that $\psi_u(s) + s\frac{\partial \psi_u(s)}{\partial s} \geq 0$. This is generally true for $\forall\, s \geq 0$. This solution of the PDE-based model will be numerically approximated in the third section.

III. Consistent Numerical Approximation Scheme

The nonlinear parabolic PDE model is approximated by constructing a numerical discretization scheme using the finite-difference method [11]. Therefore, we consider a space grid size of h and a time step Δt. The space and time coordinates are quantized as:

$$x = ih, y = jh, t = n\Delta t, \forall i \in \{0,..,I\}, j \in \{0,.,J\}, n \in \{0,..,N\} \quad (4)$$

We may take the values $h = 1$ and $\Delta t = 1$ for our discretization. Then, we re-write the equation of the PDE in the following form:

$$\frac{\partial u}{\partial t} = \xi_u(\|\nabla u\|)\Delta u + \nabla(\xi_u(\|\nabla u\|)) \cdot \nabla u - \lambda(u - u_0) \quad (5)$$

First component of this sum is approximated by using the discretized Laplacian [11], as:

$$C_1^n(i,j) = \xi_u(\|\nabla u^n(i,j)\|) \\ (u^n(i+1,j) + u^n(i-1,j) + u^n(i,j+1) + u^n(i,j-1) - 4u^n(i,j)) \quad (6)$$

The second component from (5) will be computed as following:

$$\nabla(\xi_u(\|\nabla u\|)) \cdot \nabla u = \left(\frac{\partial}{\partial x}\xi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right), \frac{\partial}{\partial y}\xi_u\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\right)\right)\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad (7)$$

which leads to

$$\nabla(\xi_u(\|\nabla u\|)) \cdot \nabla u = \frac{\partial \xi_u}{\partial s}(\|\nabla u\|) \\ \frac{\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}} \quad (8)$$

Then, we perform more approximations and from (8) we obtain:

$$\nabla(\xi_u(\|\nabla u\|)) \cdot \nabla u \approx \frac{\partial \xi_u}{\partial s}\left(\sqrt{(u_x)^2 + (u_y)^2}\right) u_{xy}(u_x + u_y) \quad (9)$$

where the derivatives u_x, u_y, u_{xy} are approximated by using the finite differences [11]. So, we obtain the following discretization of the second component:

$$C_2^n(i,j) = \frac{\partial \xi_u}{\partial s}\left(\sqrt{\left(\frac{\partial u^n(i,j)}{\partial x}\right)^2 + \left(\frac{\partial u^n(i,j)}{\partial y}\right)^2}\right) \\ \frac{\partial^2 u^n(i,j)}{\partial x \partial y}\left(\frac{\partial u^n(i,j)}{\partial x} + \frac{\partial u^n(i,j)}{\partial y}\right) \quad (10)$$

Where

$$\begin{cases} \frac{\partial u^n(i,j)}{\partial x} + \frac{\partial u^n(i,j)}{\partial y} = \frac{u^n(i+1,j) - u^n(i-1,j) + u^n(i,j+1) - u^n(i,j-1)}{2} \\ \frac{\partial^2 u^n(i,j)}{\partial x \partial y} = \frac{u^n(i+1,j+1) - u^n(i+1,j-1) - u^n(i-1,j+1) + u^n(i-1,j-1)}{4} \end{cases} \quad (11)$$

Therefore, the PDE equation from (5) is discretized as following:

$$u^{n+1}(i,j) = u^n(i,j) + C_1^n(i,j) + C_2^n(i,j) - \lambda(u^n(i,j) - u^0(i,j)) \quad (12)$$

where $i \in \{0,..., I\}, j \in \{0,..., J\}, n \in \{0,..., N\}$.

The discretization (12) represents an explicit numerical approximation scheme that is consistent to the continuous PDE model provided by (1)-(3). This iterative restoration scheme begins the denoising process with a degraded $[I \times J]$ image and applies repeatedly the operation given by (13) on u^n, for each n from 0 to N. Also, our numerical approximation algorithm converges fast to the solution of the anisotropic diffusion-based model, since the number of iterations, N, is quite low. That solution represents the enhanced image, u^{N+1}.

IV. Experiments and Method Comparison

The nonlinear PDE-based technique described here has been successfully experimented on numerous degraded digital images. Our approach reduces considerably the Gaussian noise and the image blurring, while preserving the boundaries and other essential details of the image. It also overcomes the staircase (blocky) effect [6].

We have determined empirically the following set of parameters for the PDE-based scheme, which provides the optimal results:

$$\lambda = 0.4,\ \gamma = 0.8,\ \beta = 2.1,\ \alpha = 1.3, N = 15 \quad (13)$$

The performance of this denoising scheme has been assessed by using Peak Signal to Noise Ratio (PSNR) and Norm of the Error Image (NE) performance measures [12]. Our method provides higher PSNR values (lower NE values) than nonlinear PDE-based and variational techniques, like Perona-Malik models [3] and TV denoising [5], and also the classic 2D [3 x 3] filters, such as the Averaging, Gaussian 2D, Median and 2D Wiener [2].

Method comparison results can be observed in Table I and Fig. 1. One can see the PSNR values achieved by several image restoration techniques, and one can remark that the present proposed method obtains the highest value.

Fig. 1 Denoising results achieved by several filtering methods

In Fig. 1, the restoration results generated by several filtering approaches on the $[512 \times 512]$ *Barbara* image are displayed. The original image is displayed in a), the image corrupted by a Gaussian noise with $\mu = 0.08$ and *variance* = 0.05 is displayed in b), the denoising of our PDE-based technique in c), the restoration obtained by some [3 x 3] 2D filters in d) to f), the Perona-Malik restoration in g), and the TV denoising in h).

TABLE I
PSNR VALUES OF VARIOUS FILTERS

Denoising method	**PSNR**
This PDE Model	27.61(dB)
Average	25.13(dB)
Gaussian	25.07(dB)
Wiener 2D	26.48(dB)
Median	26.90(dB)
Perona-Malik	27.23(dB)
Tv Denoising	27.11(dB)

V. CONCLUSION

A nonlinear second-order parabolic PDE model for image denoising has been described in this work. Our technique provides an edge-preserving image noise removal and avoids the undesired effects, like blurring, staircasing and speckle noise.

The presented model, with its novel edge-stopping function and conductance parameter, represents the main contribution of this article. The consistent explicit numerical approximation scheme developed here represents also an important contribution of our article.

The described successfully experiments and method comparison prove the effectiveness of the proposed technique, which outperforms many PDE-based schemes and classic filtering approaches. Our future research in this image processing domain will focus on elaborating new effective image denoising models, based on higher order PDEs or novel edge-stopping functions.

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