

# A nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations

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**Abstract**—In this paper, a nonconforming mixed finite element method is studied for semilinear pseudo-hyperbolic partial integro-differential equations. By use of the interpolation technique instead of the generalized elliptic projection, the optimal error estimates of the corresponding unknown function are given.

**Keywords**—Pseudo-hyperbolic partial integro-differential equations; Nonconforming mixed element method; Semilinear; Error estimates.

## I. INTRODUCTION

**I**N this paper, we consider the following semilinear pseudo-hyperbolic partial integro-differential equations

$$\begin{cases} u_{tt} - \nabla \cdot (a(x,t)\nabla u_t + b(x,t)\nabla u \\ + \int_0^t c(x,t,s)\nabla u(x,s)d\tau) = f(u), (x,t) \in \Omega \times J, \\ u(x,t) = 0, (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \end{cases} \quad (1)$$

where  $\Omega$  is a bounded convex polygonal domain in  $R^2$  with Lipschitz continuous boundary  $\partial\Omega$ ,  $J = (0, T]$  is the time interval with  $0 < T < \infty$ . Assume that the coefficients  $a(x,t)$ ,  $b(x,t)$ ,  $c(x,t)$  are sufficiently smooth and bounded functions, and  $0 < a_0 \leq a(x,t) \leq a_1 < \infty$ ,  $0 < b_0 \leq b(x,t) \leq b_1 < \infty$ ,  $0 < c_0 \leq c(x,t) \leq c_1 < \infty$ ,  $a_t(x,t) \leq |b_0|$  for some positive constants  $a_0, a_1, b_0, b_1, c_0, c_1$ .  $u_0$  and  $u_1$  are given functions,  $f(u)$  and its partial derivatives are sufficiently smooth and nonlinear functions of  $u$ .

The pseudo-hyperbolic equations [1],[2] are a high-order partial differential equations with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction-diffusion and nerve conduction, and other physical phenomena. In [3], a finite element method for pseudo-hyperbolic partial integro-differential equations was studied and The Sobolev-Volterra projection was given. In [4] and [5], the three splitting positive definite mixed finite element schemes were proposed for pseudo-hyperbolic equations, and semidiscrete and fully discrete error estimates were studied. In [6], [8], two  $H^1$ -Galerkin mixed finite element method were proposed for pseudo-hyperbolic equations and pseudo-hyperbolic integro-differential equations. In [9], the

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$H^1$ -Galerkin expanded mixed finite element method is proposed for pseudo-hyperbolic equations. Liu et al. [7] proposed a new splitting  $H^1$ -Galerkin mixed finite element method for pseudo-hyperbolic equation. In [10], a least-squares mixed finite element methods were studied for pseudo-hyperbolic equations.

In recent years, a lot of researchers have studied mixed finite element methods for partial differential equation and made a great contribution to the mixed finite element methods ([11], [12], [13], [14], [15]), With the research and development of the mixed finite element methods, some new mixed finite element method were proposed, such as splitting positive definite mixed finite element method [4], [16], nonconforming mixed finite element method [17], [18], [19],  $H^1$ -Galerkin mixed finite element method [6], [8], [9], [20], and expanded mixed finite element method [21]. Compared to standard mixed finite element methods, the nonconforming mixed finite element method uses the interpolation technique instead of the generalized elliptic projection to obtain the optimal error estimates of the corresponding unknown function. In this paper, we study the rectangle nonconforming mixed finite element method for semilinear pseudo-hyperbolic partial integro-differential equations and obtain the semidiscrete and fully discrete error estimates.

## II. MIXED WEAK FORM AND SEMIDISCRETE SCHEME

To formulate the mixed weak form of (1), let

$$p = -(a\nabla u_t + b\nabla u + \int_0^t c\nabla u d\tau),$$

and set  $d = -\nabla a(x,t)$ ,  $e = -\nabla b(x,t)$ ,  $g = -\nabla c(x,t)$ . Then, problem (1) can be written in the mixed form of the first order system:

$$\begin{cases} u_{tt} + \nabla \cdot p = f(u), (x,t) \in \Omega \times (0, T], \\ p + \nabla (au_t + bu) + \int_0^t \nabla cud\tau \\ + du_t + eu + \int_0^t gud\tau = 0, (x,t) \in \Omega \times (0, T], \\ u(x,t) = 0, (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases} \quad (2)$$

Let  $W = H(\text{div}, \Omega) = \{w \in (L(\Omega))^2; \text{div} w \in L^2(\Omega)\}$ , normed by  $\|\cdot\|_{H(\text{div}, \Omega)}^2 = \|\cdot\|^2 + \|\text{div} \cdot\|^2$  and  $V = L^2(\Omega)$ .

The mixed weak form of (2) is: find  $\{u, p\} : [0, t] \rightarrow V \times W$  such that

$$\begin{cases} (u_{tt}, v) + (\operatorname{div} p, v) - (f(u), v) = 0, v \in V, t \in (0, T], \\ (p, w) - (au_t + bu, \operatorname{div} w) - \left(\int_0^t cud\tau, \operatorname{div} w\right) + (du_t, w) \\ + (eu, w) + \left(\int_0^t gud\tau, w\right) = 0, w \in W, t \in (0, T], \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega. \end{cases} \quad (3)$$

Let  $\Omega$  be a bounded convex polygonal domain in  $R^2$  with boundary  $\partial\Omega$  parallel to the x-axis or y-axis.  $T_h$  be a rectangular mesh of  $\Omega$ ,  $V_h \subset T_h$  is the space of piecewise constant functions,  $W_h \subset T_h$  is the nonconforming piecewise cross-element space, that is

$$V_h = \{u : u|_K \in Q_{0,0}(K), \forall K \in T_h\}, \quad (4)$$

$$W_h = \{p = (p^1, p^2); \forall K \in T_h, \int_e [p^i] ds = 0, e \in \partial K, i = 1, 2\}, \quad (5)$$

where,  $p^1 = \operatorname{span}\{1, x, y, y^2\}$ ,  $p^2 = \operatorname{span}\{1, x, y, x^2\}$ ,  $[p^i]$  denote the jump term of  $p$  on the edge of element,  $[p^i] = p^i, (i = 1, 2)$ , if  $e \subset \partial\Omega$ .

Now we define the interpolation functions of  $u, p$  on the element  $K$ :  $I_h^1 u \in V_h$ :  $I_h^1 u = \frac{1}{|K|} \int_K u dx dy$ ,  $I_h^2 p = (\Pi_n^1 p^1, \Pi_n^2 p^2) \in W_h$ ;  $\int_{l_i} \Pi_n^j p_i^j = \int_{l_i} p_i^j ds, i = 1, 2, 3, 4, j = 1, 2$ , where  $|K|$  and  $l_i$  denote the area of a rectangle and the edge of  $K$ . For any  $p_h \in W_h$ , let us define  $\|p_h\|_h = \sum_K (\|p_h\|^2 + \|\operatorname{div} p_h\|^2)^{\frac{1}{2}}$ , it is straightforward that  $\|\cdot\|_h$  is the norm in  $W_h$ .

The semidiscrete nonconforming mixed finite element scheme for (3) consists in determining  $\{u_h, p_h\} : [0, t] \rightarrow V_h \times W_h$  such that

$$\begin{cases} (u_{htt}, v_h) + (\operatorname{div} p_h, v_h)_h - (f(u_h), v_h) = 0, v_h \in V_h, \\ (p_h, w_h) - (au_{ht} + bu_h, \operatorname{div} w_h)_h \\ - \left(\int_0^t cu_h d\tau, \operatorname{div} w_h\right)_h + (du_{ht}, w_h) + (eu_h, w_h) \\ + \left(\int_0^t gu_h d\tau, w_h\right) = 0, w_h \in W_h, \\ u_h(x, 0) = I_h^1 u_0(x), u_{ht}(x, 0) = I_h^1 u_1(x), \\ p_h(x) = I_h^2 p(x, 0), x \in \Omega, \end{cases} \quad (6)$$

where  $(u, v)_h = \sum_K \int_K uv dx$ .

We shall demonstrate the existence and uniqueness of the solution of (6).

**Theorem 2.1:** There exists a unique discrete solution to the system (6).

*Proof:* In fact, if  $V_h = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{r_1}\}$  and  $W_h = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_{r_2}\}$ , let

$$u_h = \sum_{i=1}^{r_1} h_i(t) \phi_i(x), p_h = \sum_{j=1}^{r_2} g_j(t) \psi_j(x)$$

Then (6) can be written as

$$\begin{cases} (a) \quad A \frac{d^2 H(t)}{dt^2} - RG(t) = B(H(t)), \\ (b) \quad DG(t) - (E - J) \frac{dH(t)}{dt} - (F - L - X)H(t) = 0, \\ (c) \quad H(0) = I_h^1 u_0(0), H'(0) = I_h^1 u_1(0), G(0) = I_h^2 p(0), \end{cases} \quad (7)$$

where

$$\begin{aligned} A &= (\phi_i, \phi_j)_{r_1 \times r_2}, H(t) = (h_1(t), \dots, h_{r_1}(t))', \\ B(H(t)) &= (f(\sum_{k=1}^{r_1} h_k \phi_k), \phi_j)_{r_1 \times r_1} \\ D &= (\psi_i, \psi_j)_{r_2 \times r_2}, E = (\operatorname{div} \psi_j, a \phi_i)_{r_2 \times r_1}, \\ F &= (\operatorname{div} \psi_j, b \phi_i)_{r_2 \times r_1}, G = (g_1(t), \dots, g_{r_2}(t))', \\ J &= (\psi_j, d \phi_i)_{r_2 \times r_1}, L = (\psi_j, e \phi_i)_{r_2 \times r_1}, \\ R &= (\phi_i, \operatorname{div} \psi_j)_{r_1 \times r_2}, X = (\psi_j, \int_0^t c \nabla \phi_i ds)_{r_2 \times r_1}. \end{aligned}$$

Noting that  $A$  and  $D$  are positive definite matrices and combing with (a) and (b) of (7), we get

$$\begin{aligned} A \frac{d^2 H(t)}{dt^2} - RD^{-1}(E - J) \frac{dH(t)}{dt} \\ - RD^{-1}(F - L - X)H(t) = B(H(t)), \end{aligned} \quad (8)$$

which is a differential equation of  $H(t)$ . According to Caratheodory theorem in the theory of ODE, when  $t > 0$ ,  $H(t)$  has the unique solution. Moreover,  $G(t)$  has a unique solution. That is, problem (6) has a unique solution. ■

### III. SOME LEMMAS AND SEMI-DISCRETE ERROR ESTIMATES

Set  $u - u_h = (u - I_h^1 u) + (I_h^1 u - u_h) = \xi + \eta$ ,  $p - p_h = (p - I_h^2 p) + (I_h^2 p - p_h) = \rho + \theta$ . From [17] and [19], we can obtain the following important lemmas.

**Lemma 3.1:** If  $u, u_t \in L^2(\Omega)$ , then  $\forall p_h \in W_h$ , we have

$$(\xi, \operatorname{div} p_h)_h = 0, (\xi_t, \operatorname{div} p_h)_h = 0.$$

**Lemma 3.2:** If  $p \in H(\operatorname{div}; \Omega)$ , then  $\forall v_h \in V_h$ , we have

$$(v_h, \operatorname{div} \rho) = 0, (v_{ht}, \operatorname{div} \rho) = 0,$$

$$(v_{htt}, \operatorname{div} \rho) = 0, (v_{httt}, \operatorname{div} \rho) = 0.$$

**Lemma 3.3:** If  $\forall u, v \in H^1(\Omega)$ ,  $a(x)$  is bounded then

$$(u, I_h^1 v) \leq C \|u\| \cdot \|v\|.$$

**Lemma 3.4:** If  $u \in H^2(\Omega)$ , then  $\forall \psi \in W_h$ , we have

$$\left| \sum_K \int_{\partial K} u \psi \cdot n ds \right| \leq Ch |u|_2 \|\psi\|.$$

**Theorem 3.5:** Let  $(u, p)$  be the solution of (3) and  $(u_h, p_h)$  be that of (6), then for  $u \in H^3(\Omega) \cap H_0^1(\Omega)$ ,  $p \in (H^1(\Omega))^2$ , such that

$$\begin{aligned} \|u - u_h\| \leq Ch [\|u\|_1 + \left(\int_0^t (\|u\|_2^2 + \|u_t\|_2^2 + \|u_{tt}\|_2^2 \right. \\ \left. + |u_{ttt}|_1^2 + |p|_1^2 + |p_t|_1^2) d\tau \right)^{\frac{1}{2}}], \end{aligned} \quad (9)$$

$$\begin{aligned} \|p - p_h\|_h &\leq Ch(|p|_1 + |u|_1 + \|u_{tt}\|_1) \\ &+ Ch\left(\int_0^t \|u\|_2^2 + \|u_t\|_2^2 + \|u_{tt}\|_2^2 \right. \\ &\left. + |u_{tt}|_1^2 + |p|_1^2 + |p_t|_1^2 d\tau\right)^{\frac{1}{2}}. \end{aligned} \quad (10)$$

*Proof:* Set  $\bar{a} = \frac{1}{|K|} \int_K a dx dy$ ,  $\bar{a}_t = \frac{1}{|K|} \int_K a_t dx dy$ ,  $\bar{b} = \frac{1}{|K|} \int_K b dx dy$ ,  $\bar{b}_t = \frac{1}{|K|} \int_K b_t dx dy$ . and combin Lemma 3.1 and Lemma 3.2 together with (2), (6), for  $\forall v_h \in V_h$ ,  $\forall w_h \in W_h$  to have

$$\begin{cases} (\xi_{tt} + \eta_{tt}, v_h) + (\text{div}\theta, v_h)_h = (f(u) - f(u_h), v_h), \\ (\theta, w_h) - (a\eta_t + b\eta, \text{div}w_h)_h = -(\rho, w_h) \\ + (a\xi_t + b\xi, \text{div}w_h)_h - (d(\xi_t + \eta_t), w_h) + (e(\xi + \eta), w_h) \\ + \left(\int_0^t c(\xi + \eta) ds, \text{div}w_h\right)_h + \left(\int_0^t g(\xi + \eta) ds, w_h\right) \\ + \sum_K \int_{\partial\Omega} (au_t + bu + \int_0^t cud\tau) w_h \cdot n ds. \end{cases} \quad (11)$$

Let  $v_h = a\eta_t + I_h^1(b\eta)$  and  $w_h = \theta$  in (11), combine the two equations in(11) and using the lemma 3.1, we get

$$\begin{aligned} &(\eta_{tt}, \eta_t) + (\theta, \theta) \\ &= -(\rho, w_h) + (a\xi_t + b\xi, \text{div}\theta)_h - (\xi_{tt}, a\eta_t) \\ &- (\xi_{tt} + \eta_{tt}, I_h^1(b\eta)) - (d(\xi_t + \eta_t), \theta) + (e(\xi + \eta), \theta) \\ &+ \left(\int_0^t c(\xi + \eta) ds, \text{div}\theta\right)_h + (f(u) - f(u_h), a\eta_t + I_h^1(b\eta)) \\ &+ \left(\int_0^t g(\xi + \eta) ds, \theta\right) + \sum_K \int_{\partial\Omega} (au_t + bu \\ &+ \int_0^t cud\tau)\theta \cdot n ds = \sum_{i=1}^{10} B_i. \end{aligned} \quad (12)$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of  $f(u)$  and Lipschitz continuity, we get

$$\begin{aligned} &B_1 + B_3 + B_6 + B_{10} \\ &\leq C(\|\rho\| + \|\xi\| + \|\eta\|)\|\theta\| + C\|\xi_{tt}\| \cdot \|\eta_t\| \\ &\leq C(\|\eta\|^2 + \|\eta_t\|^2) + Ch^2(|p|_1^2 + |u|_1^2 \\ &+ |u|_2^2 + |u_t|_2^2 + |u_{tt}|_1^2) + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} B_2 &= (a\xi_t + b\xi, \text{div}\theta)_h = ((b - \bar{b})\xi, \text{div}\theta)_h + (\bar{b}\xi, \text{div}\xi)_h \\ &+ ((a - \bar{a})\xi_t, \text{div}\theta)_h + (\bar{a}\xi_t, \text{div}\theta)_h \\ &= ((b - \bar{b})\xi, \text{div}\theta)_h + ((a - \bar{a})\xi_t, \text{div}\theta)_h \\ &\leq Ch\|\xi\| \cdot |\theta|_1 + Ch\|\xi_t\| \cdot |\theta|_1 \\ &\leq Ch^2(|u|_1^2 + |u_t|_1^2) + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} B_4 &= (\xi_{tt} + \eta_{tt}, I_h^1(b\eta)) \\ &\leq C\|\xi_{tt} + \eta_{tt}\| \cdot \|b\eta\| \leq C(\|\eta\|^2 + \|\eta_{tt}\|^2) + Ch|u_{tt}|_1^2, \end{aligned}$$

$$\begin{aligned} B_5 &= (d(\xi_t + \eta_t), \theta) \leq C\|\xi_t + \eta_t\| \cdot \|\theta\| \\ &\leq C\|\eta_t\| + Ch^2|u|_1^2 + \varepsilon\|\theta\|^2, \end{aligned}$$

$$\begin{aligned} B_7 + B_9 &= \left(\int_0^t c\xi d\tau + \int_0^t c\eta d\tau, \text{div}\theta\right)_h + \left(\int_0^t g(\xi + \eta) d\tau, \theta\right) \\ &\leq C\left(\int_0^t \xi\|d\tau + \int_0^t \eta\|d\tau\right) \cdot \|\theta\| \\ &\leq C\left(\int_0^t \|\xi\|d\tau + \int_0^t \|\eta\|d\tau\right) \cdot \|\theta\| \leq Ch^2|u|_1^2 \\ &+ C\|\eta\|^2 + \varepsilon\|\theta\|^2, \end{aligned}$$

$$B_8 = (f(u) - f(u_h), a\eta_t + I_h^1(b\eta)) \leq C(\|\xi\|^2 + \|\eta\|^2 + \|\eta_t\|^2).$$

Then, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|^2 + (\theta, \theta) &\leq C(\|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2) \\ &+ Ch^2(|p|_1^2 + \|u\|_2^2 + \|u_t\|_2^2 + |u_{tt}|_1^2). \end{aligned} \quad (13)$$

Noting that  $\eta(0) = 0$ ,  $\eta_t(0) = 0$ , integrating (13) with respect to  $t$  and using Gronwall's inequality, we have

$$\begin{aligned} \|\eta_t\|^2 + \int_0^t \|\theta\|^2 d\tau &\leq C \int_0^t (\|\eta\|^2 + \|\eta_{tt}\|^2) d\tau \\ &+ Ch^2 \int_0^t (|p|_1^2 + \|u\|_2^2 + \|u_t\|_2^2 + |u_{tt}|_1^2) d\tau. \end{aligned} \quad (14)$$

Noting that  $\eta = \int_0^t \eta_t d\tau$  and using Gronwall's inequality as well as (14), we have

$$\begin{aligned} \|\eta\|^2 &\leq C \int_0^t \|\eta_{tt}\|^2 d\tau + Ch^2 \int_0^t (|p|_1^2 + \|u\|_2^2 \\ &+ \|u_t\|_2^2 + |u_{tt}|_1^2) d\tau. \end{aligned} \quad (15)$$

Differentiating the second equation in (11) with respect to  $t$ , we get

$$\begin{aligned} &(\theta_t, w_h) - (a_t\eta_t + a\eta_{tt} + b\eta_t + b_t\eta, \text{div}w_h)_h \\ &= -(\rho_t, w_h) + (a_t\xi_t + a\xi_{tt} + b\xi_t + b_t\xi, \text{div}w_h)_h \\ &+ (d_t(\xi_t + \eta_t) + d(\xi_{tt} + \eta_{tt}), w_h) + (e_t(\xi + \eta) \\ &+ e(\xi_t + \eta_t), w_h) - (c(\xi + \eta), \text{div}w_h)_h - (g(\xi + \eta), w_h) \\ &- \sum_K \int_{\partial K} (au_{tt} + a_tu_t + bu_t + b_tu + cu) w_h \cdot n ds. \end{aligned} \quad (16)$$

Take  $w_h = \theta$  in (16) and  $v_h = a_t\eta_t + a\eta_{tt} + I_h^1(b\eta_t + b_t\eta)$  in

the first equation of (11) and add the two equations to have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\theta, \theta) + (\eta_{tt}, a\eta_{tt}) \\ &= -(\rho_t, \theta) + (a_t \xi_t + a \xi_{tt} + b \xi_t + b_t \xi, \operatorname{div} \theta)_h + (e_t(\xi + \eta) \\ &+ e(\xi_t + \eta_t), \theta) + (\xi_{tt}, a\eta_{tt}) + (\xi_{tt} + \eta_{tt}, I_h^1(b\eta_t + b_t\eta)) \\ &+ (f(u) - f(u_h), a_t \eta_t + a \eta_{tt} + I_h^1(b\eta_t + b_t\eta)) \\ &- (\xi_{tt} + \eta_{tt}, a_t \eta_t) + (d_t(\xi_t + \eta_t) + d(\xi_{tt} + \eta_{tt}), \theta) \\ &- (c(\xi + \eta), \operatorname{div} \theta)_h - (g(\xi + \eta), \theta) - \sum_K \int_{\partial K} (a u_{tt} \\ &+ a_t u_t + b u_t + b_t u + c u) \theta \cdot n ds = \sum_{i=1}^{11} D_i. \end{aligned} \tag{17}$$

We will obtain the estimates from  $D_1$  to  $D_{11}$

$$D_1 + D_3 \leq C(\|\rho_t\|^2 + \|\theta\|^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\theta\|^2),$$

$$D_2$$

$$\begin{aligned} &= (a_t \xi_t + a \xi_{tt} + b \eta_t + b_t \eta, \operatorname{div} \theta)_h \\ &= ((a_t - \bar{a}_t) \xi_t + (b_t - \bar{b}_t) \xi + (b_t - \bar{b}) \xi_t, \operatorname{div} \theta)_h + (a \xi_{tt}, \operatorname{div} \theta)_h \\ &\leq Ch(\|\xi_{tt}\| + \|\xi_t\| + \|\xi\|) \|\theta\|_1 \leq C(\|\xi\|^2 + \|\xi_t\|^2 + \|\theta\|^2), \end{aligned}$$

$$\begin{aligned} D_4 + D_5 &\leq C(\|\xi_t\| \|\eta_{tt}\| + \|\xi_{tt} + \eta_{tt}\| \cdot (\|\eta\| + \|\eta_t\|)) \\ &\leq C(\|\xi_{tt}\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2), \end{aligned}$$

$$\begin{aligned} D_6 &= (f(u) - f(u_h), a_t \eta_t + a \eta_{tt} + I_h^1(b\eta_t + b_t\eta)) \\ &\leq C(\|\xi\| + \|\eta\|) \cdot (\|\eta_t\| + \|\eta_{tt}\| + \|\eta\|) \\ &\leq C(\|\xi\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2), \end{aligned}$$

$$D_7 \leq C\|\xi_{tt} + \eta_{tt}\| \cdot \|\eta_t\| \leq C(\|\xi_{tt}\|^2 + \|\eta_{tt}\|^2 + \|\eta_t\|^2),$$

$$\begin{aligned} D_8 &\leq C(\|\xi_t\| + \|\xi_{tt}\| + \|\eta_t\| + \|\eta_{tt}\|) \cdot \|\theta\| \\ &\leq C(\|\xi_t\|^2 + \|\xi_{tt}\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2 + \|\theta\|^2), \end{aligned}$$

$$D_9 \leq Ch(\|\xi\| + \|\eta\|) \cdot \|\theta\|_1 \leq C(\|\xi\|^2 + \|\eta\|^2 + \|\theta\|^2),$$

$$D_{10} \leq C(\|\xi + \eta\|) \cdot \|\theta\| \leq C(\|\xi\|^2 + \|\eta\|^2 + \|\theta\|^2),$$

$$\begin{aligned} D_{11} &\leq Ch(|u_{tt}|_2 + |u_t|_2 + |u|_2) \cdot \|\theta\| \\ &\leq Ch^2(|u|_2^2 + |u_t|_2^2 + |u_{tt}|_2^2) + C\|\theta\|^2. \end{aligned}$$

Substitute the above inequalities into (17), integrate with respect to  $t$ , and use Gronwall's inequality to have

$$\begin{aligned} \|\theta\|^2 &\leq C \int_0^t (\|\eta\|^2 + \|\eta_t\|^2) d\tau + Ch^2 \int_0^t (|p|_1^2 + |p_t|_1^2 \\ &+ |u|_2^2 + |u_t|_2^2 + |u_{tt}|_2^2) d\tau. \end{aligned} \tag{18}$$

Noting that  $\eta = \int_0^t \eta_t d\tau$ , we have

$$\begin{aligned} \|\theta\|^2 &\leq C \int_0^t \|\eta_t\|^2 d\tau + Ch^2 \int_0^t (|p|_1^2 + |p_t|_1^2 + \|u\|_2^2 \\ &+ |u_t|_2^2 + |u_{tt}|_2^2) d\tau. \end{aligned} \tag{19}$$

Differentiate the first equation of (11) with respect to  $t$  to get

$$(\xi_{ttt} + \eta_{ttt}, v_h) + (\operatorname{div} \theta, v_h)_h = ([f(u) - f(u_h)]_t, v_h). \tag{20}$$

Choose  $v_h = \eta_{tt} + I_h^1(b_t \eta + b \eta_t)$  in (20) and  $w_n = \theta_t$  in (16) and add the two equations to obtain

$$\begin{aligned} & \frac{1}{2} \|\eta_{tt}\|^2 + (\theta_t, \theta_t) \\ &= -(\rho_t, \theta_t) + (a_t \xi_t + a \xi_{tt} + b \xi_t + b_t \xi, \operatorname{div} \theta_t)_h \\ &+ (d_t \xi_t + d \xi_{tt}, \theta_t) + (d_t \eta_t + d \eta_{tt}, \theta_t) + (e_t(\xi + \eta) \\ &+ e(\xi_t + \eta_t), \theta_t) - (c(\xi + \eta), \operatorname{div} \theta_t)_h - (g(\xi + \eta), \theta_t) \\ &+ ([f(u) - f(u_h)]_t, \eta_{tt} + I_h^1(b_t \eta + b \eta_t)) - (\xi_{ttt}, \eta_{tt}) \\ &- (\xi_{ttt}, I_h^1(b_t \eta + b \eta_t)) + (\eta_{ttt}, I_h^1(b_t \eta + b \eta_t)) \\ &- \sum_K \int_{\partial K} (a u_{tt} + a_t u_t + b u_t + b_t u + c u) \theta_t \cdot n ds \\ &= \sum_{i=1}^{12} E_i, \end{aligned} \tag{21}$$

where

$$E_1 + E_5 \leq C(\|\rho_t\|^2 + \|\xi\|^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2) + \varepsilon \|\theta_t\|^2,$$

$$\begin{aligned} E_2 &= ((a_t - \bar{a}_t) \xi_t + (a - \bar{a}) \xi_{tt} + (b - \bar{b}) \xi_t \\ &+ (b_t - \bar{b}_t) \xi, \operatorname{div} \theta_t)_h \leq Ch(\|\xi\| + \|\xi_t\| + \|\xi_{tt}\|) \|\theta_t\|_1 \\ &\leq C(\|\xi\|^2 + \|\xi_t\|^2 + \|\xi_{tt}\|^2) + \varepsilon \|\theta_t\|^2, \end{aligned}$$

$$\begin{aligned} E_3 + E_4 &\leq C(\|\eta_t\| + \|\eta_{tt}\| + \|\xi_t\| + \|\xi_{tt}\|) \cdot \|\theta_t\| \\ &\leq C(\|\xi_t\|^2 + \|\xi_{tt}\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2) + \varepsilon \|\theta_t\|^2, \end{aligned}$$

$$\begin{aligned} E_6 &= -((c - \bar{c})(\xi + \eta), \operatorname{div} \theta_t) \leq Ch(\|\xi\| + \|\eta\|) \|\theta_t\|_1 \\ &\leq C(\|\xi\|^2 + \|\eta\|^2) + \varepsilon \|\theta_t\|^2 \end{aligned}$$

$$E_7 \leq C\|\xi + \eta\| \cdot \|\theta_t\| \leq C(\|\xi\|^2 + \|\eta\|^2) + \varepsilon \|\theta_t\|^2,$$

$$\begin{aligned} E_8 &= ([f(u) - f(u_h)]_t, \eta_{tt} + I_h^1(b_t \eta + b \eta_t)) \\ &= (f_u(u) u_t - f_u(u_h) u_{ht}, \eta_{tt} + I_h^1(b_t \eta + b \eta_t)) \\ &= (f_u(u) \xi_t + f_u(u_h) \eta_t + (f_u(u) - f_u(u_h)) I_h^1 u_t, \\ &\eta_{tt} + I_h^1(b_t \eta + b \eta_t)) \\ &\leq C(\|\xi\|^2 + \|\xi_t\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2), \end{aligned}$$

$$\begin{aligned} E_9 + E_{10} &\leq C(\|\xi_{ttt}\| \cdot \|\eta_{tt}\| + \|\xi_{ttt}\| \cdot \|b_t \eta + b \eta_t\|) \\ &\leq C(\|\xi_{ttt}\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2), \end{aligned}$$

$$\begin{aligned} E_{11} &= -\frac{d}{dt} (\eta_{tt}, I_h^1(b_t \eta + b \eta_t)) + (\eta_{tt}, I_h^1(b_{tt} \eta + 2b_t \eta_t + b \eta_{tt})) \\ &\leq -\frac{d}{dt} (\eta_{tt}, I_h^1(b_t \eta + b \eta_t)) + C(\|\eta\|^2 + \|\eta_t\|^2 + \|\eta_{tt}\|^2), \end{aligned}$$

$$\begin{aligned} E_{12} &\leq Ch(|u_{tt}|_2 + |u_t|_2 + |u|_2) \cdot \|\theta_t\| \\ &\leq Ch^2(|u|_2^2 + |u_t|_2^2 + |u_{tt}|_2^2) + \varepsilon \|\theta_t\|^2. \end{aligned}$$

Note that  $(\eta_{tt}, I_h^1(b_t \eta + b \eta_t)) \leq C(\|\eta\|^2 + \|\eta_t\|) + \varepsilon \|\eta_t\|^2$ , substitute these inequalities into (21) and integrate with

respect to  $t$  to have

$$\begin{aligned} \|\eta_{tt}\|^2 &\leq C(\|\eta\|^2 + \|\eta_t\|^2) + C \int_0^t (\|\eta\|^2 + \|\eta_t\|^2 \\ &+ \|\eta_{tt}\|^2 + \|\theta\|^2) d\tau + Ch^2 \int_0^t (|p|_1^2 + |p_t|_1^2 \\ &+ \|u\|_2^2 + \|u_t\|_1^2 + \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2) d\tau. \end{aligned} \quad (22)$$

Noting that  $\eta = \int_0^t \eta_t d\tau$ ,  $\eta_t = \int_0^t \eta_{tt} d\tau$ , using Gronwall's inequality combing with (14), (15), (18) and (22), we have

$$\begin{aligned} &\|\eta_{tt}\|^2 + \|\eta_t\|^2 + \|\eta\|^2 + \|\theta\|^2 \\ &\leq Ch^2 \int_0^t (|p|_1^2 + |p_t|_1^2 + \|u\|_2^2 + \|u_t\|_1^2 + \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2) d\tau. \end{aligned} \quad (23)$$

Take  $v_h = \text{div}\theta$  in the first equation of (11) to get

$$\begin{aligned} \|\text{div}\theta\|^2 &= -(\xi_{tt} + \eta_{tt}, \text{div}\theta) + (f(u) - f(u_h), \text{div}\theta) \\ &\leq C(\|\xi\|^2 + \|\eta\|^2 + \|\xi_{tt}\|^2 + \|\eta_{tt}\|^2) + \varepsilon\|\text{div}\theta\|^2. \end{aligned}$$

that is

$$\begin{aligned} \|\text{div}\theta\|^2 &\leq Ch^2(\|u_{tt}\|_1^2 + \|u\|_1^2 + \int_0^t (|p|_1^2 + |p_t|_1^2 \\ &+ \|u\|_2^2 + \|u_t\|_1^2 + \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2) d\tau). \end{aligned} \quad (24)$$

Applying the inequality with (23) and (24), we complete the proof. ■

By the proof of Theorem 3.5 as well as [17], Theorem 3.6 can be obtained.

**Theorem 3.6:** Let  $(u, p)$  be the solution of (3) and  $(u_h, p_h)$  be that of (6), then for any  $u \in H^3(\Omega) \cap H_0^1(\Omega)$ ,  $p \in (H^1(\Omega))^2$ , such that

$$\begin{aligned} \|u_t - u_{ht}\| &\leq Ch[\|u_t\|_1 + (\int_0^t (\|u\|_2^2 + \|u_t\|_2^2 \\ &+ \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2 + |p|_1^2 + |p_t|_1^2) d\tau)^{\frac{1}{2}}], \end{aligned} \quad (25)$$

$$\begin{aligned} \|u_{tt} - u_{h tt}\| &\leq Ch[\|u_{tt}\|_1 + (\int_0^t (\|u\|_2^2 + \|u_t\|_2^2 \\ &+ \|u_{tt}\|_2^2 + \|u_{ttt}\|_1^2 + |p|_1^2 + |p_t|_1^2) d\tau)^{\frac{1}{2}}]. \end{aligned} \quad (26)$$

#### IV. FULLY-DISCRETE ERROR ESTIMATES

Let  $0 = t_0 < t_1 < \dots < t_N = T$  be a given partition of the time interval  $[0, T]$  with step length  $\Delta t = T/N$ , for some positive integer  $N$  and define  $t_n = n\Delta t$ . For a smooth function  $\phi$  on  $[0, T]$ , let

$$\phi_n = \phi(t_n), \phi^{n+1/2} = \frac{1}{2}(\phi_n + \phi_{n+1}), \partial_t \phi^{n+1/2} = \frac{\phi_{n+1} - \phi_n}{\Delta t},$$

$$\partial_t \phi^n = \frac{\phi_{n+1/2} - \phi_{n-1/2}}{2} = \frac{\phi_{n+1} - \phi_{n-1}}{2\Delta t},$$

$$\partial_t^2 \phi^n = \frac{\partial_t \phi^{n+1/2} - \partial_t \phi^{n-1/2}}{\Delta t} = \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{\Delta t^2},$$

$$\phi^{n:1/4} = \frac{1}{2}(\phi^{n+1/2} + \phi^{n-1/2}) = \frac{1}{4}(\phi_{n+1} + 2\phi_n + \phi_{n-1}).$$

To approximate the integral, we introduce the composite trapezoid formulaplgh

$$\Delta_n = \frac{\Delta t}{2} \sum_{i=0}^{n-1} [\phi(t_i) + \phi(t_{i+1})] \simeq \int_0^{t_n} \phi(s) d\tau.$$

and the quadrature error satisfies

$$|\Delta_n - \int_0^{t_n} \phi(s)| \leq \frac{\Delta t^2}{24} \max_{0 \leq t \leq t_n} |\phi(t)|. \quad (27)$$

We define

$$\Delta_{n+1/2} = \frac{1}{2}(\Delta_n + \Delta_{n+1}), \Delta_{n:1/4} = \frac{1}{2}(\Delta_{n+1/2} + \Delta_{n-1/2}).$$

Let  $U^n$  and  $Z^n$  be the approximations of  $u$  and  $p$  at  $t = t_n$ , respectively, which through the following implicit scheme. We determine a sequence of pairs  $\{U^n, Z^n\} \in V_n \times W_h, n = 1, 2, \dots, N$ , satisfying

$$\begin{aligned} (U^0, v_h) &= (u_0, v_h), \\ (Z^0, w_h) &= (p(0), w_h), \forall (v_h, w_h) \in V_h \times W_h, \end{aligned} \quad (28)$$

$$\begin{aligned} &(\frac{2}{\Delta t} \partial_t U^{\frac{1}{2}}, v_h) + (\text{div} Z^{\frac{1}{2}}, v_h)_h \\ &= (f^{\frac{1}{2}}(U), v_h) + (\frac{2}{\Delta t} \partial_t u(0), v_h), \end{aligned} \quad (29)$$

$$\begin{aligned} &(Z^{\frac{1}{2}}, w_h) - (a^{\frac{1}{2}} \partial_t U^{\frac{1}{2}} + b^{\frac{1}{2}} U^{\frac{1}{2}}, \text{div} w_h)_h + (d^{\frac{1}{2}} \partial_t U^{\frac{1}{2}}, w_h) \\ &+ (e^{\frac{1}{2}} U^{\frac{1}{2}}, w_h) + (\Delta_{\frac{1}{2}}(cU), \text{div} w_h)_h + (\Delta_{\frac{1}{2}}(gU), w_h) = 0, \end{aligned} \quad (30)$$

$$(\partial_t^2 U^n, v_h) + (\text{div} Z^{n:1/4}, v_h)_h = (f^{n:1/4}, v_h), \forall v_h \in V_h, \quad (31)$$

$$\begin{aligned} &(Z^{n:1/4}, w_h) - (a^n \partial_t U^n, \text{div} w_h)_h - (b^n U^{n:1/4}, \text{div} w_h)_h \\ &+ (d^n \partial_t U^n, w_h) + (e^n U^{n:1/4}, w_h) - (\Delta_{n:1/4}(cU), \text{div} w_h)_h \\ &+ (\Delta_{n:1/4}(gU), w_h) = 0, \forall w_h \in W_h. \end{aligned} \quad (32)$$

For the fully discrete error estimates, we split the error

$$u(t_n) - U^n = u(t_n) - I_h^1 u(t_n) + I_h^1 u(t_n) - U^n = \xi^n + \eta^n,$$

$$p(t_n) - Z^n = p(t_n) - I_h^2 p(t_n) + I_h^2 p(t_n) - Z^n = \rho^n + \theta^n.$$

Noting that (28)-(32) together with (6), we obtain the error equations for  $\eta^n$  and  $\theta^n$  for  $v_h \in V_h$  and  $w_h \in W_h$

$$\begin{aligned} &(\frac{2}{\Delta t} \partial_t \eta^{\frac{1}{2}}, v_h) + (\text{div} \theta^{\frac{1}{2}}, v_h)_h \\ &= (f^{\frac{1}{2}}(u) - f^{\frac{1}{2}}(U), v_h) - (K_1, v_h) \\ &- (\frac{2}{\Delta t} \partial_t \xi^{\frac{1}{2}}, v_h), \forall v_h \in V_h, n \geq 0. \end{aligned} \quad (33)$$

$$\begin{aligned}
 & (\theta^{\frac{1}{2}}, w_h) + (a^{\frac{1}{2}} \partial_t \eta^{\frac{1}{2}} + b^{\frac{1}{2}} \eta^{\frac{1}{2}}, \operatorname{div} w_h)_h \\
 = & -(\rho^{\frac{1}{2}}, w_h) - (d \partial_t \eta^{\frac{1}{2}}, w_h) - (e \eta^{\frac{1}{2}}, w_h) + (a^{\frac{1}{2}} \partial_t \xi^{\frac{1}{2}} \\
 & + b^{\frac{1}{2}} \xi^{\frac{1}{2}}, \operatorname{div} w_h)_h - (d^{\frac{1}{2}} \partial_t \xi^{\frac{1}{2}}, w_h) - (e^{\frac{1}{2}} \xi^{\frac{1}{2}}, w_h) \\
 & - (\Delta_{\frac{1}{2}}(c\xi), \operatorname{div} w_h)_h - (\Delta_{\frac{1}{2}}(c\eta), \operatorname{div} w_h)_h \\
 & + (K_2, \operatorname{div} w_h) + (K_3, w_h) + (K_4, w_h) \\
 & - (\Delta_{\frac{1}{2}}(g\xi), w_h) - (\Delta_{\frac{1}{2}}(g\eta), w_h) - (\tau_1, w_h) \\
 & - \sum_K \int_{\partial K} (a^{\frac{1}{2}} \frac{u^1 - u^0}{2\Delta t} + b^{\frac{1}{2}} u^{\frac{1}{2}} + \Delta_{\frac{1}{2}}(cu)) w_h \cdot n ds.
 \end{aligned} \tag{34}$$

$$\begin{aligned}
 & (\partial_t^2 \eta^n, v_h) - (\operatorname{div} \theta^{n:1/4}, v_h)_h \\
 = & (\partial_t^2 \xi^n, v_h) + (K_5, v_h) + (f^{n:1/4}(u) - f^{n:1/4}(U), v_h).
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 & (\theta^{n:1/4}, w_h) - (a^n \partial_t \eta^n + b^n \eta^{n:1/4}, \operatorname{div} w_h)_h \\
 = & -(\rho^{n:1/4}, w_h) + (a^n \partial_t \xi^n + b^n \xi^{n:1/4}, \operatorname{div} w_h)_h \\
 & - (d^n (\partial_t \xi^n + \partial_t \eta^n), w_h) - (e^n (\xi^{n:1/4} + \eta^{n:1/4}), w_h) \\
 & + (K_6, \operatorname{div} w_h) + (K_7, w_h) + (K_8, \operatorname{div} w_h) \\
 & - (\Delta_{n:1/4}(c\xi), \operatorname{div} w_h)_h - (\Delta_{n:1/4}(c\eta), \operatorname{div} w_h)_h \\
 & - (\Delta_{n:1/4}(g\xi), w_h) - (\Delta_{n:1/4}(g\eta), w_h) - (\tau_2, w_h) \\
 & - \sum_K \int_{\partial K} (a^n \frac{u^{n+1} - u^{n-1}}{2\Delta t} + b^n u^{n:1/4} \\
 & + \Delta_{n:1/4}(cu)) w_h \cdot n ds.
 \end{aligned} \tag{36}$$

where

$$\begin{aligned}
 K_1 &= u_{tt}^{\frac{1}{2}} + \frac{2}{\Delta t} (u_t(0) - \partial_t u^{\frac{1}{2}}), \quad K_2 = a(u_t - \partial_t u_h^{\frac{1}{2}}), \\
 K_3 &= d(u_t - \partial_t u_h^{\frac{1}{2}}), \quad K_4 = \int_0^{t^{\frac{1}{2}}} cuds - \Delta_{\frac{1}{2}}(cu_h), \\
 K_5 &= u_{tt}^{n:1/4} - \partial_t^2 u_h^n, \quad K_6 = a(u_t^{n:1/4} - \partial_t u_h), \\
 K_7 &= d(u_t^{n:1/4} - \partial_t u_h), \quad K_8 = \int_0^{t_{n:1/4}} cud\tau - \Delta_{n:1/4}(cu), \\
 \tau_1 &= \int_0^{t^{\frac{1}{2}}} gud\tau - \Delta_{\frac{1}{2}}(gu_h), \quad \tau_2 = \int_0^{t_{n:1/4}} gud\tau - \Delta_{n:1/4}(gu_h), \\
 \int_0^{t^{\frac{1}{2}}} \phi(s) d\tau &= \frac{1}{2} (\int_0^{t_n} \phi(s) d\tau + \int_0^{t_{n+1}} \phi(s) d\tau), \\
 \int_0^{t_{n:1/4}} \phi(s) d\tau &= \frac{1}{2} (\int_0^{t_{n+\frac{1}{2}}} \phi(s) d\tau + \int_0^{t_{n-\frac{1}{2}}} \phi(s) d\tau).
 \end{aligned}$$

**Theorem 4.1:** Let  $(u^n, p^n)$  be the solution of (6) and  $(U^n, Z^n)$  be that of (28)-(32). If  $u \in H^4(\Omega) \cap H_0^1(\Omega)$ ,  $p \in (H^1(\Omega))^2$ , then for  $0 \leq t \leq T$ , there exist the constant  $C > 0$ , independent of  $h$  and  $t$ , such that

$$\begin{aligned}
 \|u^n - U^n\| &\leq Ch(|u_1| + |u_2|) + Ch(\Delta t)^2(|p_1| + |u_1| + |u_2|) \\
 &+ C(\Delta t)^{\frac{7}{2}} (\int_0^T \|u_{ttt}\| dt + \int_0^T \|u_{tttt}\| dt) \\
 &+ C(\Delta t)^3 (\|\frac{\partial^3 u}{\partial t^3}\| + \|\frac{\partial^4 u}{\partial t^4}\| + \|u(0)\|),
 \end{aligned} \tag{37}$$

$$\begin{aligned}
 & \|p^{n:1/4} - Z^{n:1/4}\|_h \\
 \leq & Ch(|u_1| + |p_1| + |u_2|) + Ch(\Delta t)^2(|p_1| + |u_1| + |u_2|) \\
 &+ C(\Delta t)^3 (\|\frac{\partial^3 u}{\partial t^3}\| + \|\frac{\partial^4 u}{\partial t^4}\| + \|u(0)\|) \\
 &+ C(\Delta t)^{\frac{7}{2}} (\int_0^T \|u_{ttt}\| dt + \int_0^T \|u_{tttt}\| dt).
 \end{aligned} \tag{38}$$

**Proof:** Set  $v_h = a \partial_t \eta^n + I_h^1(b^n \eta^n)$  in (35) and  $w_h = \theta^{n:1/4}$  in (36) to get

$$\begin{aligned}
 & (\partial_t^2 \eta^n, a \partial_t \eta^n) + (\theta^{n:1/4}, \theta^{n:1/4}) \\
 = & -(\rho^{n:1/4}, \theta^{n:1/4}) + (a^n \partial_t \xi^n + b^n \xi^n, \operatorname{div} \theta^{n:1/4})_h \\
 & - (\partial_t^2 \xi^n + \partial_t^2 \eta^n, I_h^1(b^n \eta^n)) - (d^n (\partial_t \xi^n + \partial_t \eta^n), \theta^{n:1/4}) \\
 & - (e^n (\xi^{n:1/4} + \eta^{n:1/4}), \theta^{n:1/4}) - (\partial_t^2 \xi^n, a^n \partial_t \eta^n) \\
 & + (b^n \eta^n, \operatorname{div} \theta^{n:1/4})_h - (I_n^1(b^n \eta^n), \operatorname{div} \theta^{n:1/4})_h \\
 & + (K_5, a^n \partial_t \eta^n + I_h^1(b^n \eta^n)) + (K_6, \operatorname{div} \theta^{n:1/4})_h \\
 & + (K_7, \theta^{n:1/4}) + (K_8, \operatorname{div} \theta^{n:1/4})_h \\
 & + (f^{n:1/4}(u) - f^{n:1/4}(U), a^n \partial_t \eta^n + I_n^1(b^n \eta^n)) \\
 & + (\Delta_{n:1/4}(c\xi), \operatorname{div} \theta^{n:1/4})_h - (\Delta_{n:1/4}(c\eta), \operatorname{div} \theta^{n:1/4})_h \\
 & - (\Delta_{n:1/4}(g\xi), \theta^{n:1/4}) - (\Delta_{n:1/4}(g\eta), \theta^{n:1/4}) \\
 & - (\tau_2, \theta^{n:1/4}) - \sum_K \int_{\partial K} (a^n \frac{u^{n+1} - u^{n-1}}{2\Delta t} \\
 & + b^n u^n + \Delta_{n:1/4}(cu)) \theta^{n:1/4} \cdot n ds = \sum_{i=1}^{19} G_i.
 \end{aligned} \tag{39}$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of  $f(u)$  and Lipschitz continuity, we get

$$\begin{aligned}
 G_1 &\leq C \|\rho^{n:1/4}\| \|\theta^{n:1/4}\| \leq C \|\rho^{n:1/4}\|^2 + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_2 &\leq C (\|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\xi^{n:1/4}\|^2) + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_3 &\leq C (\|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \\
 &\|\partial_t \eta^{n-\frac{1}{2}}\|^2 + \|\eta^n\|^2), \\
 G_4 &\leq C (\|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \\
 &\|\partial_t \eta^{n-\frac{1}{2}}\|^2) + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_5 &\leq C (\|\xi^{n:1/4}\|^2 + \|\eta^{n:1/4}\|^2) + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_6 &\leq C (\|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \\
 &\|\partial_t \eta^{n-\frac{1}{2}}\|^2), \\
 G_7 &\leq C \|\eta^{n:1/4}\|^2 + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_8 &\leq C \|\eta^n\|^2 + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_9 &\leq C (\|K_5\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \|\partial_t \eta^{n-\frac{1}{2}}\|^2 + \|\eta^n\|^2), \\
 G_{10} + G_{11} &\leq C (\|K_6\|^2 + \|K_7\|^2) + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_{12} &\leq C \|K_8\|^2 + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_{13} &\leq C (\|\xi^{n:1/4}\|^2 + \|\eta^{n:1/4}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \|\partial_t \eta^{n-\frac{1}{2}}\|^2 + \\
 &\|\eta^n\|^2), \\
 G_{14} + G_{15} + G_{16} + G_{17} &\leq C (\|\Delta_{n:1/4}(\xi)\|^2 + \|\Delta_{n:1/4}(\eta)\|^2) + \\
 &\varepsilon \|\theta^{n:1/4}\|^2, \\
 G_{18} &\leq C \|\tau_2\|^2 + \varepsilon \|\theta^{n:1/4}\|^2, \\
 G_{19} &\leq \frac{C}{\Delta t} h |u^n|_2 \|\theta^{n:1/4}\| \leq Ch^2 |u^n|_2^2 + \varepsilon \|\theta^{n:1/4}\|^2.
 \end{aligned}$$

Note that

$$(\partial_t^2 \eta^n, a^n \partial_t \eta^n) = \frac{a^n}{2\Delta t} (\|\partial_t \eta^{n+\frac{1}{2}}\|^2 - \|\partial_t \eta^{n-\frac{1}{2}}\|^2).$$

Then we obtain

$$\begin{aligned} & \frac{1}{2\Delta t} (\|\partial_t \eta^{n+\frac{1}{2}}\|^2 - \|\partial_t \eta^{n-\frac{1}{2}}\|^2) \\ & \leq C(\|\rho^{n;1/4}\|^2 + \|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 \\ & \quad + \|\xi^{n;1/4}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \|\partial_t \eta^{n-\frac{1}{2}}\|^2) \\ & \quad + \|\eta^n\|^2 + \|\eta^{n;1/4}\|^2 + \|K_5\|^2 + \|K_6\|^2 \\ & \quad + \|K_7\|^2 + \|K_8\|^2 + \|\Delta_{n;1/4}(\xi)\|^2 \\ & \quad + \|\Delta_{n;1/4}(\eta)\|^2 + \|\tau_2\|^2 + h^2|u^n|_2^2). \end{aligned} \tag{40}$$

Multiplying by  $2\Delta t$  and then summing from  $n = 1, 2, 3, \dots, N$ , we obtain

$$\begin{aligned} & \|\partial_t \eta^{N+\frac{1}{2}}\|^2 - \|\partial_t \eta^{\frac{1}{2}}\|^2 \\ & \leq Ch^2|u^n|_2^2 + C\Delta t \sum_{n=1}^N (\|\rho^{n;1/4}\|^2 + \|\partial_t \xi^{n+\frac{1}{2}}\|^2 \\ & \quad + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\xi^{n;1/4}\|^2 + \|\partial_t \eta^{n+\frac{1}{2}}\|^2 + \|\partial_t \eta^{n-\frac{1}{2}}\|^2 \\ & \quad + \|\eta^n\|^2 + \|\eta^{n;1/4}\|^2 + \|K_5\|^2 + \|K_6\|^2 + \|K_7\|^2 \\ & \quad + \|K_8\|^2 + \|\Delta_{n;1/4}(\xi)\|^2 + \|\Delta_{n;1/4}(\eta)\|^2 + \|\tau_2\|^2). \end{aligned} \tag{41}$$

Using the Granwall's lemma, we get

$$\begin{aligned} & \|\partial_t \eta^{N+\frac{1}{2}}\|^2 \\ & \leq \|\partial_t \eta^{\frac{1}{2}}\|^2 + Ch^2|u^n|_2^2 + C\Delta t \sum_{n=1}^N (\|\rho^{n;1/4}\|^2 \\ & \quad + \|\partial_t \xi^{n+\frac{1}{2}}\|^2 + \|\partial_t \xi^{n-\frac{1}{2}}\|^2 + \|\xi^{n;1/4}\|^2 \\ & \quad + \|\eta^n\|^2 + \|\eta^{n;1/4}\|^2 + \|K_5\|^2 + \|K_6\|^2 + \|K_7\|^2 \\ & \quad + \|K_8\|^2 + \|\Delta_{n;1/4}(\xi)\|^2 + \|\Delta_{n;1/4}(\eta)\|^2 + \|\tau_2\|^2). \end{aligned} \tag{42}$$

Noting that  $\frac{1}{\Delta t} (\|\eta^{n+1}\| - \|\eta^n\|) \leq \|\partial_t \eta^{n+\frac{1}{2}}\|$ , using the Granwall's lemma and then summing from  $n = 1, 2, 3, \dots, N$ , we obtain

$$\begin{aligned} & \|\eta^{N+1}\| \\ & \leq C(\|\eta^0\| + \|\eta^1\|) + Ch|u^n|_2 + C(\Delta t)^{\frac{3}{2}} \sum_{n=1}^N (\|\rho^{n;1/4}\| \\ & \quad + \|\partial_t \xi^{n+\frac{1}{2}}\| + \|\partial_t \xi^{n-\frac{1}{2}}\| + \|\xi^{n;1/4}\|^2 + \|K_5\| + \|K_6\| \\ & \quad + \|K_7\| + \|K_8\| + \|\Delta_{n;1/4}(\xi)\| + \|\tau_2\|). \end{aligned} \tag{43}$$

To estimate (33) and (34), setting  $v_h = a^{\frac{1}{2}} \partial_t \eta^{\frac{1}{2}} + I_h^1(b^{\frac{1}{2}} \eta^{\frac{1}{2}})$

and  $w_h = \theta^{\frac{1}{2}}$ , we get

$$\begin{aligned} & \left(\frac{2}{\Delta t} \partial_t \eta^{\frac{1}{2}}, a^{\frac{1}{2}} \eta^{\frac{1}{2}}\right) + (\theta^{\frac{1}{2}}, \theta^{\frac{1}{2}}) \\ & = -(\rho^{\frac{1}{2}}, \theta^{\frac{1}{2}}) + (a^{\frac{1}{2}} \partial_t \xi^{\frac{1}{2}} + b^{\frac{1}{2}} \xi^{\frac{1}{2}}, \text{div} \theta^{\frac{1}{2}})_h \\ & \quad - (d^{\frac{1}{2}} (\partial_t \xi^{\frac{1}{2}} + \partial_t \eta^{\frac{1}{2}}), \theta^{\frac{1}{2}}) - (e^{\frac{1}{2}} (\xi^{\frac{1}{2}} + \eta^{\frac{1}{2}}), \theta^{\frac{1}{2}}) \\ & \quad - \left(\frac{2}{\Delta t} (\partial_t \xi^{\frac{1}{2}} + \partial_t \eta^{\frac{1}{2}}), I_h^1(b^{\frac{1}{2}} \eta^{\frac{1}{2}})\right) - \left(\frac{2}{\Delta t} \partial_t \xi^{\frac{1}{2}}, a^{\frac{1}{2}} \partial_t \eta^{\frac{1}{2}}\right) \\ & \quad + (K_1, a^{\frac{1}{2}} \partial_t \eta^{\frac{1}{2}} + I_h^1(b^{\frac{1}{2}} \eta^{\frac{1}{2}})) + (K_2, \text{div} \theta^{\frac{1}{2}})_h + (K_3, \theta^{\frac{1}{2}}) \\ & \quad + (K_4, \theta^{\frac{1}{2}}) - (\Delta_{\frac{1}{2}}(c\xi) + \Delta_{\frac{1}{2}}(c\eta), \text{div} \theta^{\frac{1}{2}})_h \\ & \quad + (\Delta_{\frac{1}{2}}(g\xi) + \Delta_{\frac{1}{2}}(g\eta), \theta^{\frac{1}{2}}) - (\tau_1, \theta^{\frac{1}{2}}) \\ & \quad - (f^{\frac{1}{2}}(u) - f^{\frac{1}{2}}(U), a^{\frac{1}{2}} \partial_t \eta^{\frac{1}{2}} + I_h^1(b^{\frac{1}{2}} \eta^{\frac{1}{2}})) \\ & \quad - \sum_K \int_{\partial K} \left(a^{\frac{1}{2}} \frac{u^1 - u^0}{2\Delta t} + b^{\frac{1}{2}} u^{\frac{1}{2}} + \Delta_{\frac{1}{2}}(cu)\right) \theta^{\frac{1}{2}} \cdot n ds \\ & = \sum_{i=1}^{15} E_i. \end{aligned} \tag{44}$$

Using Cauchy-Schwarz inequality and Young inequality, we obtain

$$\|\eta_0\| + \|\eta_1\| \leq Ch(\Delta t)^2 (\|\rho^{\frac{1}{2}}\| + \|\xi^{\frac{1}{2}}\| + \|K_1\| + \|K_2\| + \|K_3\| + \|K_4\| + \|\tau_1\|).$$

To estimate the right-hand terms, we note that

$$\|K_1\| \leq C\Delta t (\|\frac{\partial^3 u}{\partial t^3}\| + \|\frac{\partial^4 u}{\partial t^4}\|),$$

$$\|K_2\| \leq C(\Delta t)^{\frac{3}{2}} \int_{t_0}^{t_1} \|u_{ttt}\| ds,$$

$$\|K_3\| \leq C(\Delta t)^{\frac{3}{2}} \int_{t_0}^{t_1} \|u_{ttt}\| ds,$$

$$\|K_4\| = \|\int_0^{t^{\frac{1}{2}}} cuds - \Delta_{\frac{1}{2}}(cu_h)\| \leq C\Delta t \|u\|,$$

$$\|\tau_1\| \leq C\Delta t \|u\|.$$

Using the above estimates, we get

$$\|\eta_0\| + \|\eta_1\| \leq Ch(\Delta t)^2 (|p|_1 + |u|_1 + |u|_2) + C(\Delta t)^3 (\|\frac{\partial^3 u}{\partial t^3}\| + \|\frac{\partial^4 u}{\partial t^4}\| + \|u\|) + C(\Delta t)^{\frac{7}{2}} \int_{t_0}^{t_1} \|u_{ttt}\| ds.$$

Note that

$$\|K_5\| = \|u_{tt}^{n;1/4} - \partial_t^2 u_h^n\| \leq C(\Delta t)^{\frac{3}{2}} \int_0^T \|u_{tttt}\| dt,$$

$$\|K_6\| + \|K_7\| \leq C(\Delta t)^{\frac{3}{2}} \int_0^T \|u_{ttt}\| ds,$$

$$\|K_8\| = \|\int_0^{t_{n;1/4}} cuds - \Delta_{n;1/4}(cu_h)\| \leq C\Delta t \|u\|,$$

$$\|\tau_2\| = \|\int_0^{t_{n;1/4}} guds - \Delta_{n;1/4}(gu_h)\| \leq C\Delta t \|u\|.$$

Using (43) and the above estimates, we complete the proof of (37).

Further, set  $v_h = \text{div}\theta^{n;1/4}$  in (35) and  $w_h = \theta^{n;1/4}$  in (36) to get

$$\begin{aligned} & (\text{div}\theta^{n;1/4}, \text{div}\theta^{n;1/4}) + (\theta^{n;1/4}, \theta^{n;1/4}) \\ &= -(\rho^{n;1/4}, \theta^{n;1/4}) + (a^n(\partial_t\eta^n + \partial_t\xi^n), \text{div}\theta^{n;1/4})_h \\ &+ (b^n(\xi^{n;1/4} + \eta^{n;1/4}), \text{div}\theta^{n;1/4})_h \\ &- (d^n(\partial_t\xi^n + \partial_t\eta^n), \theta^{n;1/4}) - (e^n(\xi^{n;1/4} + \theta^{n;1/4}), \theta^{n;1/4}) \\ &+ (\partial_t^2\xi^n + \partial_t^2\eta^n, \text{div}\theta^{n;1/4})_h + (K_5, \text{div}\theta^{n;1/4}) \\ &+ (K_6, \text{div}\theta^{n;1/4})_h + (K_7, \theta^{n;1/4}) + (K_8, \text{div}\theta^{n;1/4})_h \\ &- (\tau_2, \theta^{n;1/4}) - (\Delta_{n;1/4}(c\xi) + \Delta_{n;1/4}(c\eta), \text{div}\theta^{n;1/4}) \\ &+ (\Delta_{n;1/4}(g\xi) + \Delta_{n;1/4}(g\eta), \theta^{n;1/4}) \\ &+ (f^{n;1/4}(u) - f^{n;1/4}(U), \text{div}\theta^{n;1/4}) \\ &- \sum_K \int_{\partial K} (a^n \frac{u^{n+1} - u^{n-1}}{2\Delta t} + b^n u^{n;1/4} \\ &+ \Delta_{n;1/4}(cu))\theta^{n;1/4} \cdot nds. \end{aligned}$$

Using Hölder's inequality and Young's inequality combining with Lemma 3.1-3.4, inverse inequality, average value technique in [18], and the boundedness of  $f(u)$  and Lipschitz continuity, the estimate of the right-hand terms are obtained

$$\begin{aligned} & \|\theta^{n;1/4}\| + \|\text{div}\theta^{n;1/4}\| \\ &\leq Ch|u^n|_2 + C(\|\rho^{n;1/4}\| + \|\xi^{n;1/4}\| + \|\xi^n\| \\ &+ \|\eta^{n;1/4}\| + \|\eta^n\| + \|K_5\| + \|K_6\| + \|K_7\| \\ &+ \|K_8\| - \|\tau_2\| + \|\Delta_{n;1/4}(\xi)\| \\ &+ \|\Delta_{n;1/4}(\eta)\| + \|f^{n;1/4}(u) - f^{n;1/4}(U)\|). \end{aligned}$$

Summing from  $n = 1, 2, 3, \dots, N$ , we obtain

$$\begin{aligned} & \|\theta^{n;1/4}\|_h = \sum_{n=1}^N (\|\theta^{n;1/4}\| + \|\text{div}\theta^{n;1/4}\|) \\ &\leq Ch|u^n|_2 + C \sum_{n=1}^N (\|\rho^{n;1/4}\| + \|\xi^{n;1/4}\| + \|\xi^n\| \\ &+ \|\eta^{n;1/4}\| + \|\eta^n\| + \|K_5\| + \|K_6\| + \|K_7\| \\ &+ \|K_8\| - \|\tau_2\| + \|\Delta_{n;1/4}(\xi)\| + \|\Delta_{n;1/4}(\eta)\| \\ &+ \|f^{n;1/4}(u) - f^{n;1/4}(U)\|). \end{aligned}$$

Using the similar proof, we get

$$\begin{aligned} & \|\theta^{n;1/4}\|_h \leq Ch(|u|_1 + |p|_1 + |u|_2) + Ch(\Delta t)^2(|p|_1 \\ &+ |u|_1) + C(\Delta t)^3(\|\frac{\partial^3 u}{\partial t^3}\| + \|\frac{\partial^4 u}{\partial t^4}\| + \|u(0)\|) \\ &+ C(\Delta t)^{\frac{7}{2}}(\int_0^T \|u_{ttt}\|dt + \int_0^T \|u_{tttt}\|dt). \end{aligned}$$

Finally, we apply the triangle inequality to complete the proof. ■

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#### REFERENCES

- [1] J Nagumo, S Arimoto, S Yoshizawa. An active pulse transmission line simulating nerve axon, Proc. IRE., 1962, 50: 91-102.
- [2] C V Pao. A mixed initial boundary value problem arising in neurophysiology, J. Math. Anal. Appl., 1975, 52: 105-119.
- [3] X Cui. Sobolev-Volterra projection and numerical analysis of finite element methods for integro-differential equations, Acta Mathematicae Applicatae Sinica, 2001, 24(3): 441-454.
- [4] Y Liu, H Li, W Gao, S He. A new splitting positive definite mixed element method for pseudo-hyperbolic equations, Mathematica Applicata, 2011, 24(1): 104-111.
- [5] Y Liu, H Li, J F Wang, S He. Splitting positive definite mixed element methods for pseudo-hyperbolic equations, Numer. Methods Partial Differential Equations, DOI 10.1002/num.20650(2010).
- [6] Y Liu, H Li.  $H^1$ -Galerkin mixed finite element methods for pseudo-hyperbolic equations, Appl. Math. Comput., 2009, 212: 446-457.
- [7] Y Liu, J F Wang, H Li, W Gao, S He. A new splitting  $H^1$ -Galerkin mixed method for pseudo-hyperbolic equations, International Journal of Engineering and Natural Sciences, 2011, 5(2): 58-63.
- [8] Y Liu, H Li, S He. Error estimates of  $H^1$ -Galerkin mixed finite element methods for pseudo-hyperbolic partial integro-differential equation [J]. Numerical Mathematics A Journal of Chinese Universities, 2010, 32(1): 1-20.(in Chinese)
- [9] Y Liu, H Li. A new mixed finite element method for pseudo-hyperbolic equation, Mathematica Applicata, 2010, 23(1): 150-157.
- [10] H Guo, H X Rui. Least-squares Galerkin procedures for pseudo-hyperbolic equations, Appl. Math. Comput., 2007, 189: 425-439.
- [11] J Jr Douglas, R Ewing, M Wheeler. A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media, RAIRO Anal. Numér., 1983, 17: 249-265.
- [12] F Brezzi, J Jr Douglas, L Marini. Two families of mixed finite elements for second order elliptic problems, Numer. Math., 1985, 47: 217-235.
- [13] F Brezzi, J Jr Douglas, R Durán, M Fortin. Mixed finite elements for second order elliptic problems in three variables, Numer. Math., 1987, 51: 237-250.
- [14] Z D Luo. Theory Bases and Applications of Mixed Finite Element Methods, Science Press, Beijing, 2006.(in Chinese)
- [15] Y P Chen, Y Q Huang. The superconvergence of mixed finite element methods for nonlinear hyperbolic equations, Communications in Non-linear Science and Numerical Simulation, 1998, 3(3): 155-158.
- [16] D P Yang. A splitting positive definite mixed element method for miscible displacement of compressible flow in porous media, Numerical Methods for Partial Differential Equations, 2001, 17: 229-249.
- [17] G Ma, D Y Shi. The nonconforming mixed finite element method for generalized nerve conduction type equation, Mathematics In Practice And Theory, 2010, 40(4): 217-223.(in Chinese)
- [18] D Y Shi, H B Guan. A kind of full-discrete nonconforming finite element method for the parabolic variational inequality, Acta Mathematicae Applicatae Sinica, 2008, 31(1): 90-96.
- [19] D Y Shi, J C Ren. Nonconforming mixed finite element method for the stationary conduction-convection problem, Inter. J. Numer. Anal. Model., 2009, 6(2): 293-310.
- [20] A K Pani, R K Sinha, A K Otta. An  $H^1$ -Galerkin mixed method for second order hyperbolic equations, Inter. J. Numer. Anal. Model., 2004, 1(2): 111-129.
- [21] Z X Chen. Expanded mixed finite element methods for linear second order elliptic problems I, RAIRO Modél. Math. Anal. Numér., 1998, 32(4): 479-499.