A New Shock Model for Systems Subject to Random Threshold Failure

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Abstract—This paper generalizes Yeh Lam's shock model for renewal shock arrivals and random threshold. Several interesting statistical measures are explicitly obtained. A few special cases and an optimal replacement problem are also discussed.

Keywords— $\delta - shock$ model, optimal replacement, random threshold, shocks.

I. INTRODUCTION

ALL real world systems are deteriorating in nature and the progressive system degradation is often reflected in higher production cost, lower product quality and missed targets. Thus the study of optimal replacement and repair strategies of deteriorating systems have widely attracted the attention of several operations researchers in the recent past. Among the many approaches to modeling deteriorating systems, shock models have found favor with reliability analysts because of their tractability and wide applicability to diverse areas (stochastic clearing systems, drug administration in chemotherapy [1] and fatigue failure). Shocks are events which cause perturbation to the system, leading to its deterioration and consequent failure. The effect of these shocks on the system is measured by a process called wear process or damage process. The wear process denoted by $\{D(t): t \geq 0\}$, represents the deterioration level or the cumulative damage level at time t. The shocks arrive at random instants of time, and are described by the associated counting process $\{N(t): t \geq 0\}$. The system failure is viewed as the first passage problem of $D(t)$ past a threshold, fixed or random. The shock that leads to the threshold crossing is known as the lethal shock. Interesting variations of the first passage problem have been studied by Shanthikumar and Sumita [2], Stadje [1] and Yeh Lam and Zhang [3]. Yeh Lam and Zhang [3] in a refreshing departure introduced a new class of shock models and called them $\delta - shock$ models. While the earlier shock models concentrated solely on the magnitude of the damage caused by the shocks, Yeh Lam and Zhang's model paid attention to the frequency of the shocks. Thus in a $\delta - shock$ model, a shock is a deadly shock if the time elapsed from the previous shock to this shock is less than a prespecified value δ hereinafter referred to as the threshold value, and the system fails at the time of the occurrence of the deadly shock. This approach is more practical because the cumulative damage process is abstract and many a times not physically observable. In addition, systems may not withstand successive shocks at short intervals. For instance, elastic materials will stretch on the application of a shock and will take time to recover. Any further shock before the recovery is complete will make the material break. The threshold time δ is the recovery time. In view of the relevance of this class of models in real systems, it seems worthwhile to make a comprehensive analysis of such a modeling approach. This paper attempts such an analysis of $\delta - shock$ models in which the shock counting process is generalized to a renewal process and the threshold times are considered as random variables. Apart from deriving explicitly various statistical characteristics of the model we analyze an optimal replacement problem of such a system.

II. NOTATION USED

Z:Random variable denoting the time between two successive shocks.

$f(.), F(.), \overline{F}(.)$:Probability density, cumulative distribution and survivor functions of Z.

D: Random variable denoting the threshold value.

$g(.), G(.), \overline{G}(.)$: Probability density, cumulative distribution and survivor functions of D.

W: Random variable denoting time between two successive failures.

$k(t), K(t), \overline{K}(t)$: Probability density, cumulative distribution and survivor functions of W.

$N(t)$:Counting variable denoting the number of failures in $(0, t)$.

$M(t) = E\{N(t)\}$.

$L_f(s)$: Laplace Transform of the function $f(t)$.

c: Repair cost rate.

r:Reward rate when the system is operating.

R:Replacement cost.

Y_n: Repair time after the nth failure $\{Y_n; n = 1, 2, ...\}$; Y_n's form an increasing geometric process with rate b with expectation of $E(Y_1) = \mu$.

W_n:Operating time of the system after the $(n-1)$th failure, called the length of the nth subcycle.

D_n : Random threshold in the nth subcycle.

$G_n(.)$: Distribution function of the threshold time D_n in the n th subcycle.

U : Random replacement time with $E(U) = \tau$.

III. THE MODEL

Our model is governed by the following assumptions:

A1. A new system is put on operation at time $t = 0$. The system on failure is repaired and successive repairs are assumed to take negligible amount of time.

A2. The system is subject to shocks. The interval between shocks Z , are assumed to be independently and identically distributed with distribution function $F(.)$.

A3. A shock is classified as a nonlethal shock if the time elapsed from the previous shock to this shock is greater than the threshold D . A shock is lethal if it occurs within D . A lethal shock results in system failure leading to its repair.

A4. Threshold time D is a random variable with distribution function $G(.)$.

A5. The shock arrival times and the threshold time are independent of each other.

- The term "shock" is used in a broad sense, denoting any perturbation to the system caused by environment or inherent factors, leading to a degeneration of the system. If shocks are due to environmental factors like high temperature, voltage fluctuations, humidity and wrong handling, then shocks due to each of such factors will arrive according to a renewal process. Thus the shock arrival process can be seen to be the superposition of independent renewal processes. Thus a Poisson process will provide an adequate approximation [3]. On the other hand, if the shocks are due to internal causes, then the renewal process is a reasonable approximation. For instance, shocks could be viewed as the failure of a component in a multi-component system.

- The random threshold D can be viewed as a built in mechanism in the system, which counters the after effects of a shock restoring the system to its original state. Thus any shock which arrives before the termination of D can prove to be fatal. For instance if the system is a "2 out of n" system and a component failure is identified as a shock, then the system continues to function, despite the shocks. While the failed component is repaired, if the next shock (component failure) arrives before the failed component is repaired, then the system fails.

IV. THE STATISTICAL CHARACTERISTICS

We first obtain the probability density function of W , the time between two successive failures. From the stated assumptions, the survivor function $\overline{K}(t)$ of the time between failures satisfies the integral equation

$$\overline{K}(t) = \overline{F}(t) + \int_0^t f(\tau)G(\tau)\overline{K}(t - \tau)d\tau \qquad (1)$$

Equation (1) may be derived as follows. The event $\{W > t\}$ can be decomposed into two mutually exclusive events as given below.

1) The first shock itself occurs only after t , the probability of which is $\overline{F}(t)$.

2) The other possible event is a conjunction of the following three events.

   a. The first shock occurs at some instant $\tau \in (0, t]$, the corresponding density being $f(\tau)$.

   b. The threshold time starting from $t = 0$ is over by time τ , the probability of which is $G(\tau)$ and

   c. In the remaining interval $(\tau, t]$ of length $(t - \tau)$ there is no failure, the probability of which is $\overline{K}(t - \tau)$. Integrating over all possible $\tau \in (0, t]$ we obtain the second term of equation (1). Simple differentiation of (1) yields the probability density $k(t)$ of the random variable W as

$$k(t) = f(t)\overline{G}(t) + \int_0^t f(\tau)G(\tau)k(t - \tau)dt \qquad (2)$$

Finally application of Laplace transforms and rearrangement of terms yields

$$L_k(s) = \frac{L_{f\overline{G}}(s)}{1 - L_{fG}(s)} \qquad (3)$$

We will now provide an alternate derivation of (3) which will be useful in our further analysis. Towards this end, it is to be noted that during W, a random number of shocks occur of which the last one is lethal while the earlier ones are not. From the definition of a lethal shock, we are led to the fact that W, the time between two successive failures comprises of the sum of a random number of intervals each of which is greater than D and one lone interval whose length is less than D. We first define a sequence of independently and identically distributed random variables X_i's which are distributed as Z but conditional on Z>D. We also define a random variable Y_N distributed as Z but conditional on Z≤D. Y_N is assumed to be independent of the sequence X_i's. We observe that W can be represented as the sum of a random number of random variables, so that

$$W = \sum_{i=1}^{N-1} X_i + Y_N \qquad (4)$$

The number of terms $N-1$ in the summation is a random variable representing the number of nonlethal shocks experienced by the system during one cycle. From our assumptions, it is immediate that N has the geometric distribution given by

$$P[N = n] = qp^n, n = 0,1,2,...$$

where $q = P[Z \le D]$ and $p = 1 - q$. $\qquad (5)$

Define the conditional distributions of X_i and Y_N as

$$\alpha(x) = P[x < Z < x+dx/Z > D] = \frac{f(x)G(x)}{P(Z > D)} \qquad (6)$$

and $\beta(x) = P[x < Z < x+dx/Z \le D] = \dfrac{f(x)\overline{G}(x)}{P(Z \le D)} \qquad (7)$

Now,

$$k(t) = P[t \le W \le t+dt] = \sum_{n=0}^{\infty} P[t \le W \le t+dt/N = n]P(N=n)$$

$$= \sum_{n=0}^{\infty} [\alpha^{(n)} * \beta(t)]P(Z \le D)[P(Z > D)]^n \qquad (8)$$

where $\alpha^{(n)} * \beta(t)$ is the convolution of the n fold convolution of $\alpha(t)$ and $\beta(t)$.

Taking the Laplace Transform on both sides of (8) yields,

$$L_k(s) = \frac{L_{f\overline{G}}(s)}{1 - L_{fG}(s)} \qquad (9)$$

The moments of W for any shock arrival distribution $f(t)$ are simply obtained by differentiating $L_k(s)$ successively and setting $s = 0$. By observing that

$$\frac{d^n L_{fG}(s)}{ds^n}\bigg|_{s=0} = (-1)^n E[Z^n / Z > D]P(Z > D) \text{ and}$$

$$\frac{d^n L_{f\overline{G}}(s)}{ds^n}\bigg|_{s=0} = (-1)^n E[Z^n / Z \le D]P(Z \le D) \quad n = 0,1,2,...$$

we obtain the mean and variance of W, after simplification as

$$E[W] = \frac{E(Z)}{P(Z \le D)} \qquad (10)$$

$$Var[W] = \frac{E(Z^2)}{P(Z \le D)} + \frac{2E(Z)E(Z / Z > D)P(Z > D) - E^2(Z)}{P(Z \le D)^2} \qquad (11)$$

We next proceed to obtain the statistical characteristics of the failure counting process $N(t)$. To this end, we appeal to elementary results of renewal theory. We first define the generating function of $N(t)$ as

$$V(u,t) = E[u^{N(t)}] = \sum_{0}^{\infty} u^n p[N(t) = n] \qquad (12)$$

$$= \sum_{0}^{\infty} u^n [K_n(t) - K_{n+1}(t)] \qquad (13)$$

Where $K_n(t)$ is the n fold convolution of $K(t)$ with itself. Taking the Laplace Transform of (13), we obtain

$$V(u,s) = \frac{1}{s} + \frac{(u-1)L_k(s)}{s[1 - uL_k(s)]} \qquad (14)$$

Differentiation of $V(u,s)$ with respect to u and setting $u = 1$, yields the Laplace Transform of the mean function $M(t) = E[N(t)]$. Thus

$$L_M(s) = \frac{L_k(s)}{s[1 - L_k(s)]} \qquad (15)$$

$L_M(s)$ could be inverted for specific forms of $f(.)$ and $g(.)$. To obtain $Var[N(t)]$, instead of differentiating (4.14) twice, we use the formula,

$$Var[N(t)] = M(t) + 2\int_{0}^{t} M(t-x)dM(x) - [M(t)]^2 \quad (16)$$

V. SPECIFIC MODELS AND DISCUSSIONS

Case 1: We first consider the case when the systems are subjected to the same kind of shocks but consider different systems whose thresholds for recovery are different. Specifically we let the shocks arrive according to a Poisson counting process so that $f(t) = \lambda e^{-\lambda t}, t > 0$. If the threshold density is chosen to be exponential so that $g(t) = \mu e^{-\mu t}, t > 0$ then we observe that $E(Z) = \dfrac{1}{\lambda}$ and

$$P(Z \le D) = \int_{0}^{\infty} P(Z \le x | D = x)g(x)dx = \frac{\lambda}{\lambda + \mu}.$$

Thus $E(W)$ is easily seem to be $\dfrac{\lambda + \mu}{\lambda^2}$. In a similar manner in the case of constant and uniform threshold densities specified by $g(t) = \delta(t-d)$ and $g(t) = \dfrac{1}{b}$, $0 < t < b$, the respective mean failure times are easily seen to be $\dfrac{1}{\lambda(1 - e^{-\lambda d})}$ and $\dfrac{b}{\lambda b + e^{-\lambda b} - 1}$. In Figure 1 we plot the mean failure times for the three cases discussed above. To bring out the dependence of the mean failure time on the threshold distribution, we have chosen the mean of each of threshold distribution to be the same, and plotted $E(W)$. The plotted graph clearly exhibits the sensitivity of the mean failure times due to the threshold distributions for smaller mean values.
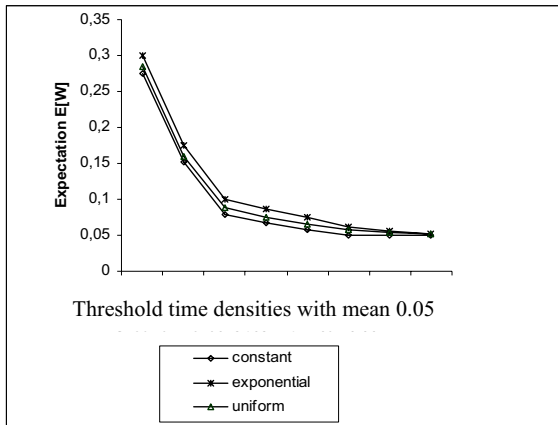
Fig. 1 Mean failure times case 1

Case 2: When the system is subjected to the same kind of shock each time, the threshold time of the system is likely to remain a constant, a case discussed by Yeh Lam [3],[4]. Under such a scenario, we consider a few models for different shock arrival distributions.

We choose the threshold times to be a constant d so that

$$G(t) = \begin{cases} 0 & 0 \le t < d \\ 1 & t \ge d \end{cases} \qquad (17)$$

First, we assume the shock arrivals are according to an exponential density with mean $\frac{1}{\lambda}$. The relevant statistical characteristics can be derived as

$$k(t) = \lambda e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left\{ (t - nd)^n U_{nd} - \left(t - \overline{n+1}d\right)^n U_{(n+1)d} \right\} \quad (18)$$

where U_α is the Heaviside unit step function defined as

$$U_a(t) = \begin{cases} 1 & t \ge \alpha \\ 0 & t < \alpha \end{cases}$$

$$E[W] = \frac{1}{\lambda\left(1 - e^{-\lambda d}\right)} \qquad (19)$$

$$Var(W) = \frac{2\left(1 - e^{-\lambda d}\right) + 2\lambda e^{-\lambda d}\left(\lambda d e^{-\lambda d} - e^{-\lambda d}\right)}{\lambda^2\left(1 - e^{-\lambda d}\right)^2} \qquad (20)$$

$$M(t) = E[N(t)] = \lambda t - \lambda e^{-\lambda d}(t - d), \qquad t > d \quad (21)$$

$$Var[N(t)] = \lambda t\left(1 - e^{-\lambda d}\right) + \lambda d e^{-\lambda d} - \left(\lambda d e^{-\lambda d}\right)^2 \qquad (22)$$

Observations:

We first observe that the expected time between failures given in (19) is the same as that obtained by Yeh Lam and Yuan Lin Zhang [3]. Consider the limiting case as $d \to \infty$. This implies that with the threshold value being large, every shock leads to system failure. In this case, we see that $k(t) = \lambda e^{-\lambda t}$, (which is the shock arrival density)

and $M(t) = \lambda t$. Also for a given λ, the expected time between failures is a function of the threshold time d, say $g(d) = \dfrac{1}{\lambda\left(1 - e^{-\lambda d}\right)}$. We note that $g(d)$ is a monotonically decreasing function of d with $g(d) = \dfrac{1}{\lambda}$ as an asymptote. On the other hand as $d \to 0$, the system cannot fail, which is confirmed by the governing equations with $E[W] \to \infty$. Further, for a fixed d, $E[W]$ is again a decreasing function of λ. Finally we note that the expected number of failures is λt, dampened by a factor $\lambda e^{-\lambda d}(t - d)$. This factor arises out of the interaction of shock arrival process and threshold process.

We consider four more shock arrival distributions namely:

(a) $f(t) = \dfrac{e^{-bt}(bt)^{a-1} b}{\Gamma(a)}, t > 0, a, b > 0$

(b) $f(t) = \dfrac{1}{b - a}, a < t < b$

(c) $f(t) = \theta_\eta t^{\eta-1} e^{-\theta t^\eta}, t > 0$

(d) $f(t)$ is degenerate at $t = \ell$.

The mean failure time using (11) are specifically given by

(a) $\dfrac{\Gamma(a+1)}{b\gamma(a, bd)}$ where $\gamma(a, b)$ is the complete gamma function

(b) $\dfrac{b+a}{2} * \dfrac{b-a}{d-a}$

(c) $\dfrac{\Gamma\left(1 + \dfrac{1}{\eta}\right)}{\left(1 - e^{-\theta\delta\eta}\right)^{\frac{1}{\eta}}}$

(d) $\begin{cases} \ell, & \ell \le d \\ \infty, & \ell > d \end{cases}$

The other statistical characteristics could be obtained in a straight forward manner. It is to be noted that (a) and (c) have been obtained by Yeh Lam but using lengthy probability arguments.

To conclude this section, we present in Figure 2, the mean time to failure for various shock arrival distributions. As before to bring out the degree of dependence of the mean failure time on the shock arrival density, we have chosen the mean of each of the shock arrival distribution to be the same. This ensures that for large d, $E(W)$ remains the same while the sensitivity gets pronounced for smaller values of d.

We observe that for a given threshold value the exponential density gives the least mean failure time. Also, as the order of the gamma density increases, the mean failure times increase.
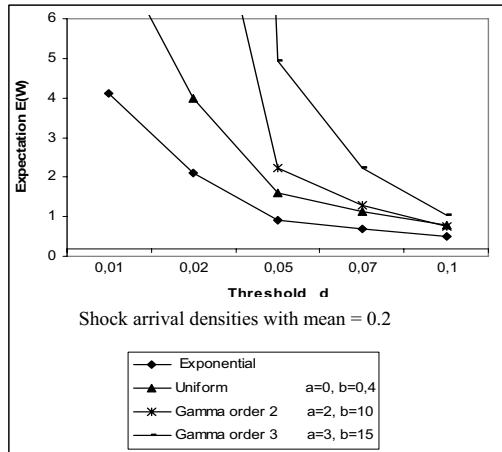
Fig. 2 Mean of Failure Times case 2

VI. OPTIMAL REPLACEMENT MODEL

With our success in obtaining the various statistical characteristics of the $\delta - shock$ model under a more general setup, we analyze an optimal replacement model first introduced by Yeh lam [3]. The replacement model is described by the following assumptions:

A1. A new system is put on operation at $t = 0$. The system is repaired on failure and replaced after N such failures by a new and identical system. The successive times between failures are called subcycles so that replacement is done at the end of the end of the Nth of the subcycle. The replacement time is a random variable U write $E(U) = \tau$.

A2. The system is subject to shocks with a shock coinciding with the installation of the system after repairs. The time between shocks, Z are independently and identically distributed with the distribution function $F(.)$.

A3. If the system has been repaired n times, then in the $(n+1)$th subcycle the threshold time D_{n+1} is a random variable with the distribution function $G_{n+1}(.)$. This means that during the operating time of the system after the (n+1)th repair, the system fails whenever the time between two successive shocks is less than D_{n+1} for the first time. We assume that the system is deteriorating so that $\{G_n(.)\}$ is stochastically decreasing.

A4. The successive repair times Y_n of the system after failures form a geometric process with rate b so that $E[Y_n] = \frac{\mu}{b^{n-1}}$ where $E[Y_1] = \mu$. During the repair time, the arriving shocks have no effect on the system.

A5. The shock arrival times, threshold times and repair times are mutually independent variable times.

To compute the long run average cost we first observe that the system renews after every replacement, so that the time till the N th failure since its installation forms the length of each cycle. Thus the successive cycles together with the cost incurred in each cycle will form a renewal reward process. By applying the standard result in renewal reward process, the long run average cost per unit time is given by C(N)= (Expected cost incurred in a cycle)/ (Expected length of a cycle)

Let W_n be the operating time of the system following the $(n-1)$th repair in a cycle and let Y_n be the repair time after the nth failure in a cycle. We see that

$$C(N) = \frac{E\left(c \sum_{n=1}^{N-1} Y_n - r \sum_{n=1}^{N} W_n + R \right)}{E\left(\sum_{n=1}^{N} W_n + \sum_{n=1}^{N-1} Y_n + U \right)} \qquad (23)$$

Let $\lambda_n = E(W_n)$. λ_n's could be obtained using the formula (11) by replacing D with D_n in the n th subcycle. Thus

$$C(N) = \frac{c\mu \sum_{n=1}^{N-1} \frac{1}{b^{n-1}} - r \sum_{n=1}^{N} \lambda_n + R}{\sum_{n=1}^{N} \lambda_n + \mu \sum_{n=1}^{N-1} \frac{1}{b^{n-1}} + \tau} \qquad (24)$$

Lam Yeh showed that the optimal replacement policy N^* for the deteriorating system whose long run average cost C(N) is (24) can be determined by $N^* = \min\{N \mid B(N) \geq 1\}$ where

$$B(N) = \frac{(c+r)\mu \left(\sum_{n=1}^{N} \lambda_n - \lambda_{N+1} \sum_{n=1}^{N-1} b^{N-n} + \tau \right)}{(R + r\tau)(\lambda_{N+1} b^{N-1} + \mu)}.$$ Now we present

certain numerical illustrations to demonstrate the model and methodology and obtain explicitly the optimal replacement time and cost. Specifically we assume that the time between shocks is exponentially distributed with mean $\frac{1}{\lambda}$. To demonstrate the effect of threshold times on the optimal N* we consider two different distributions for $G_n(.)$. They are

$$G_n(t) = \begin{cases} 0, & 0 < t < d_n \\ 1, & t \geq d_n \end{cases} \qquad (25)$$

and $\quad G_n(t) = 1 - \mu_n e^{-\mu_n} \qquad (26)$

Let us choose the parameter values to be $c = 6, r = 10, R = 6000, \tau = 50, \lambda = 20, d_n = \alpha^{n-1}\delta$, $\alpha = 1.05, \delta = 1$; $\mu_n = \frac{\mu}{\alpha^{n-1}}$ where $\mu = 1, b = 0.95$. The choice of d_n, μ_n where motivated by the fact that we have the same mean threshold for the two distributions that we consider. The optimal N^* and the associated costs are presented in Table I.

TABLE I
THRESHOLD DENSITIES

Constant		Exponential	
N^*	$C(N)^*$	N^*	$C(N)^*$
1	4.127875	1	14.07407
2	-2.26592	2	3.578641
3	-4.51336	3	-0.27856
4	-5.65205	4	-2.26962
5	-6.3344	5	-3.47612
6	-6.78457	6	-4.34709
7	-7.10031	7	-4.90558
8	-7.33103	8	-5.31643
9	-7.5044	9	-5.62738
10	-7.6371	10	-5.86737
11	-7.73977	11	-6.05497
12	-7.81953	12	-6.20261
13	-7.88129	13	-6.31893
14	-7.92855	14	-6.41012
15	-7.96388	15	-6.48071
16	-7.98919	**16**	**-6.53411**
17	-8.00593	17	-6.26741
18	-8.01519	18	-6.30628
19	**-8.01786**	19	-6.33284
20	-8.0146	20	-6.39043

VII. CONCLUSION

This paper considers an important class of shock models for deteriorating systems focusing on the frequency of shocks. As remarked earlier, this class of models which analyzes deteriorating systems has wide applicability in diverse areas of which we will mention a few. The classes of warranty models known as renewing warranty policies have been of interest [5]. In such policies, if the item fails during the warranty period W (fixed or random), the manufacturer agrees to replace it by a new item, which carries a further warranty period of W. This goes on until there is no failure during a warranty period. By identifying the instants of shocks with the failure of the item and the threshold times as the warranty period and changing the definition of the lethal shock (item failure in this case) as the occurrence of a shock past the threshold period, we can study renewing warranty cost analysis. An interesting special case of our model arises when the shock occurrences are deterministic. This will lead us to fatigue failure models with cyclic loading, which we encounter in fatigue analysis. The developed model can be used to study reliability of one unit system with a single spare and one repair facility. Identifying shocks with failure times of the system or spare and the threshold times with repair times, the system failure times of our model can be identified with system breakdown. Finally another interesting and important application that we envisage is in the area of single neurons. Neurons which are the carriers of communication signals, receive two kinds of inputs Excitatory Post Synaptic Potential (EPSP) and Inhibitory Post Synaptic Potential (IPSP) from neighboring neurons. These two inputs are assumed to arrive according to two independent renewal processes. A signal or a spike is registered by the selective interaction of these two processes which is of fundamental importance in the study of neurons.

In our model the sequence of events of the shock arrival process is mapped onto the sequence formed by the failure process using the rule determined by the threshold value (selective interaction). Also, given the threshold value D, the sequence $\{Z_i\}$ of shock arrivals and the sequences $\{t_i\}$ of failure times are related by a one to one correspondence and may be reconstructed from each other. Finally our model registers the first shock which arrives during the threshold time as the lethal shock. However one can think of the memory extending to more than one such shock to be called lethal.

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