# A New Proof on the Growth Factor in Gaussian Elimination for Generalized Higham Matrices 

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#### Abstract

The generalized Higham matrix is a complex symmetric matrix $A=B+i C$, where both $B \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{n \times n}$ are Hermitian positive definite, and $i=\sqrt{-1}$ is the imaginary unit. The growth factor in Gaussian elimination is less than $3 \sqrt{2}$ for this kind of matrices. In this paper, we give a new brief proof on this result by different techniques, which can be understood very easily, and obtain some new findings.


Keywords-CSPD matrix, positive definite, Schur complement, Higham matrix, Gaussian elimination, Growth factor.

## I. Introduction

COMPLEX symmetric matrices arise frequently, specially in algebraic eigenvalue problems see $[2,3]$ and in the computational electrodynamics see [4] etc. The Higham matrix is a complex symmetric matrix $A=B+i C$, where both $B$ and $C$ are real, symmetric and positive definite, which was firstly presented by Higham in [3] (It was called by a CSPD matrix.). In order to research the accuracy and stability of their LU factorizations, the growth factor (denoted by $\rho_{n}(A)$ ) in Gaussian elimination was conjectured in [2] that

$$
\rho_{n}(A) \leq 2
$$

for any Higham matrix $A$. Subsequently, the paper proved the following result

$$
\begin{equation*}
\rho_{n}(A)<3 \tag{1}
\end{equation*}
$$

for such matrix $A$, and so LU factorization without pivoting is perfectly normwise backward stable see [3]. Moreover, they pointed out that if the Higham matrix is extended by allowing $B$ and $C$ to be arbitrary Hermitian positive definite matrices, i.e., $A=B+i C$ is a generalized Higham matrix, then

$$
\begin{equation*}
\rho_{n}(A)<3 \sqrt{2} \tag{2}
\end{equation*}
$$

whose proof was quite lengthy in [1]. In addition, authors in [5] also noted that the above bound (2) remains true when $B$ or $C$ or both are negative (rather than positive) definite.

In this paper, we mainly give a new brief proof of the results (1) and (2) by different techniques, which can be understood more easily than the proof of [1]. Next, for convenience, we use the same notations as in [1].

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## II. Auxiliary results

In this section, we mainly list some results and lemmas which will be essential to prove our results.

Lemma 1 ([6]). Let $A$ be a CSPD matrix, then $A$ is nonsingular, and any principal submatrix of $A$ and any schur complement in $A$ are also CSPD matrices. Obviously, Lemma 1 shows that, being a CSPD matrix is a hereditary property of active submatrices in Gaussian elimination.

Lemma 2 ([6]). The largest element of a CSPD matrix $A$ lies on its main diagonal.

The above property also holds for generalized Higham matrices in the following slightly weakened form.

Lemma 3 ([1]). If $A$ is a generalized Higham matrix, then

$$
\begin{equation*}
\sqrt{2} \max _{l}\left|a_{l l}\right| \geq \max _{l \neq j}\left|a_{l j}\right| \tag{3}
\end{equation*}
$$

Thus, for a CSPD matrix $A$, the growth factor

$$
\begin{equation*}
\rho_{n}(A)=\frac{\max _{i, j, k}\left|a_{i j}^{(k)}\right|}{\max _{i, j}\left|a_{i j}\right|} \tag{4}
\end{equation*}
$$

can be replaced by

$$
\begin{equation*}
\rho_{n}(A)=\frac{\max _{j, k}\left|a_{j j}^{(k)}\right|}{\max _{j}\left|a_{j j}\right|} \tag{5}
\end{equation*}
$$

By Lemma 1 and 2, one can obtain broader bounds for the growth factor of a CSPD matrix $A$.

Lemma 4 ([7]). Let $Z_{1}$ and $Z_{2}$ be $m \times n$ matrices and

$$
\begin{equation*}
H=Z_{1}^{*} Z_{2}+Z_{2}^{*} Z_{1} \tag{6}
\end{equation*}
$$

then

$$
\begin{equation*}
H \leq Z_{1}^{*} Z_{1}+Z_{2}^{*} Z_{2} \tag{7}
\end{equation*}
$$

Lemma 5 ([8]). If $B_{1}$ and $B_{2}$ are $n \times n$ Hermitian positive definite, then inequalities $B_{1} \geq B_{2}$ if and only if $B_{2}^{-1} \geq B_{1}^{-1}$. In addition, according to the Theorem 2.1 in [1] and its proof, we easily obtain the following corollary.

Corollary 1. Let $A=B+i C$, where $B$ and $C$ are Hermitian and positive definite matrices, then $A$ is nonsingular, and $A^{-1}=X+i Y, X$ is a positive (semi)definite matrix when $B$ is positive (semi)definite and $Y$ is a negative (semi)definite matrix when $C$ is positive (semi)definite.

## III. MAIN RESULTS

The following theorem has been proved in [1], but the proof of [1] is lengthy. Based on the ideas in [1] and [7], we next give its a new proof, which can be understood more easily than the proof of [1].

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Theorem 1. Let $A$ be a generalized Higham matrix, then

$$
\begin{equation*}
\frac{\left|a_{j j}^{(k)}\right|}{\left|a_{j j}\right|}<3, \quad j=1,2, \cdots, n, \quad k=1,2, \cdots, n-1 \tag{8}
\end{equation*}
$$

Proof. Similarly to [1], fix the number $k \in\{1,2, \cdots, n-1\}$ and $j$, where $j \geq k+1$. Denote $A_{k}, B_{k}$ and $C_{k}$ by the leading principal order $k$ submatrices in $A, B$ and $C$, respectively. We split the matrix $A_{k j}$, a principal order $(k+1) \times(k+1)$ submatrix in $A$, into

$$
A_{k j}=\left(\begin{array}{cc}
A_{k} & \alpha \\
\beta^{T} & a_{j j}
\end{array}\right)
$$

where

$$
\alpha^{T}=\left(a_{1 j}, a_{2 j}, \cdots, a_{k j}\right)
$$

and

$$
\beta^{T}=\left(a_{j 1}, a_{j 2}, \cdots, a_{j k}\right)
$$

Defining the vectors

$$
b^{T}=\left(b_{1 j}, b_{2 j}, \cdots, b_{k j}\right)
$$

and

$$
c^{T}=\left(c_{1 j}, c_{2 j}, \cdots, c_{k j}\right)
$$

we can rewrite $A_{k j}$ as

$$
A_{k j}=\left(\begin{array}{cc}
B_{k}+i C_{k} & b+i c  \tag{9}\\
b^{*}+i c^{*} & b_{j j}+i c_{j j}
\end{array}\right)
$$

It is easy to see that $a_{j j}^{(k)}$ can be obtained by performing block Gaussian elimination in $A_{k j}$, namely,

$$
a_{j j}^{(k)}=a_{j j}-\beta^{T} A_{k}^{-1} \alpha
$$

Similarly to [1], setting $a_{j j}^{(k)}=\beta+i \gamma, \beta, \gamma \in \mathbb{R}$ and using (9), we have

$$
\begin{equation*}
\beta+i \gamma=b_{j j}+i c_{j j}-\left(b^{*}+i c^{*}\right)\left(B_{k}+i C_{k}\right)^{-1}(b+i c) \tag{10}
\end{equation*}
$$

Next, we use the same method as in [1] to deal with $\left(B_{k}+i C_{k}\right)^{-1}$. By [1], we know that $\left(B_{k}+i C_{k}\right)^{-1}$ can be written as

$$
\begin{equation*}
\left(B_{k}+i C_{k}\right)^{-1}=X_{k}+i Y_{k} \tag{11}
\end{equation*}
$$

where $X_{k}$ is positive definite and $Y_{k}$ is negative definite by Corollary 1. Substituting (11) into (10) yields

$$
\beta+i \gamma=b_{j j}+i c_{j j}-\left(b^{*}+i c^{*}\right)\left(X_{k}+i Y_{k}\right)(b+i c)
$$

we have

$$
\begin{equation*}
\beta=b_{j j}-b^{*} X_{k} b+c^{*} X_{k} c+c^{*} Y_{k} b+b^{*} Y_{k} c \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma=c_{j j}-b^{*} Y_{k} b+c^{*} Y_{k} c-c^{*} X_{k} b-b^{*} X_{k} c \tag{13}
\end{equation*}
$$

Now, we use the other technique, which is different from [1], to obtain the upper bounds on $\beta$ and $\gamma$.

Since $X_{k}$ is a positive definite matrix, $Y_{k}$ is negative definite. It is obvious that $-b^{*} X_{k} b$ in (12) and $c^{*} Y_{k} c$ in (13) are negative semidefinite, so (12) and (13) can rewrite

$$
\begin{equation*}
\beta \leq b_{j j}+c^{*} X_{k} c+c^{*} Y_{k} b+b^{*} Y_{k} c \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma \leq c_{j j}-b^{*} Y_{k} b-c^{*} X_{k} b-b^{*} X_{k} c \tag{15}
\end{equation*}
$$

Now we mainly consider the last two summands on the right hand side for the above two inequalities (14) and (15). First, for (14), we apply the Lemma 4 with

$$
Z_{1}=G b \quad \text { and } \quad Z_{2}=G c
$$

where $G$ is the Hermitian positive definite square root of the matrix $-Y_{k}$, we get

$$
c^{*} Y_{k} b+b^{*} Y_{k} c \leq-b^{*} Y_{k} b-c^{*} Y_{k} c
$$

thus

$$
\begin{equation*}
\beta \leq b_{j j}+c^{*} X_{k} c-b^{*} Y_{k} b-c^{*} Y_{k} c \tag{16}
\end{equation*}
$$

The last summand on the right-hand side of (15) may be proved in the same way. Thus we have the following inequality

$$
-c^{*} X_{k} b-b^{*} X_{k} c \leq b^{*} X_{k} b+c^{*} X_{k} c
$$

So

$$
\begin{equation*}
\gamma \leq c_{j j}-b^{*} Y_{k} b+b^{*} X_{k} b+c^{*} X_{k} c \tag{17}
\end{equation*}
$$

In addition, by [1], we see that

$$
\begin{align*}
& X_{k}=\left(B_{k}+C_{k} B_{k}^{-1} C_{k}\right)^{-1} \leq \frac{1}{2} C_{k}^{-1}  \tag{18}\\
& -Y_{k}=\left(C_{k}+B_{k} C_{k}^{-1} B_{k}\right)^{-1} \leq \frac{1}{2} B_{k}^{-1} . \tag{19}
\end{align*}
$$

Note that $\left(\begin{array}{cc}C_{k} & c \\ c^{*} & c_{j j}\end{array}\right)$ and $\left(\begin{array}{cc}B_{k} & b \\ b^{*} & b_{j j}\end{array}\right)$ are positive definite, by [1], the Schur complement $C_{k j} / C_{k}$ and $B_{k j} / B_{k}$ are also positive definite, i.e.,

$$
c^{*} C_{k}^{-1} c<c_{j j} \text { and }-b^{*} B_{k}^{-1} b<b_{j j}
$$

which implies that

$$
c^{*} X_{k} c<\frac{1}{2} c_{j j}, \quad \text { and } \quad-b^{*} Y_{k} b<\frac{1}{2} b_{j j}
$$

Coming back to (18), from the trivial inequality

$$
B_{k}+C_{k} B_{k}^{-1} C_{k} \geq B_{k}
$$

we can deduce the bound $X_{k} \leq B_{k}^{-1}$ by Lemma 5 . In addition, note that $\left(\begin{array}{cc}B_{k} & b \\ b^{*} & b_{j j}\end{array}\right)$ is positive definite, we have

$$
b^{*} X_{k} b \leq b^{*} B_{k}^{-1} b<b_{j j}
$$

Similarly, (19) implies the bound

$$
-Y_{k}<C_{k}^{-1}
$$

and

$$
-c^{*} Y_{k} c \leq c^{*} C_{k}^{-1} c<c_{j j}
$$

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Summarizing the above results, we conclude that

$$
\begin{aligned}
\beta & \leq b_{j j}+c^{*} X_{k} c-b^{*} Y_{k} b-c^{*} Y_{k} c \\
& <b_{j j}+\frac{1}{2} c_{j j}+\frac{1}{2} b_{j j}+c_{j j} \\
& =\frac{3}{2}\left(b_{j j}+c_{j j}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma & \leq c_{j j}-b^{*} Y_{k} b+b^{*} X_{k} b+c^{*} X_{k} c \\
& <c_{j j}+\frac{1}{2} b_{j j}+b_{j j}+\frac{1}{2} c_{j j} \\
& =\frac{3}{2}\left(b_{j j}+c_{j j}\right) .
\end{aligned}
$$

So both the matrix $\beta$ and matrix $\gamma$ are bounded above by the same matrix $\frac{3}{2}\left(b_{j j}+c_{j j}\right)$.

It follows that

$$
\begin{aligned}
\beta^{2}+\gamma^{2} & <\left[\frac{3}{2}\left(b_{j j}+c_{j j}\right)\right]^{2}+\left[\frac{3}{2}\left(b_{j j}+c_{j j}\right)\right]^{2} \\
& =\frac{9}{2}\left(b_{j j}+c_{j j}\right)^{2} \\
& =\frac{9}{2}\left(b_{j}^{2}+c_{j j}^{2}\right)+9 b_{j j} c_{j j} \\
& \leq \frac{9}{2}\left(b_{j j}^{2}+c_{j j}^{2}\right)+\frac{9}{2}\left(b_{j j}^{2}+c_{j j}^{2}\right) \\
& =9\left(b_{j j}^{2}+c_{j j}^{2}\right) .
\end{aligned}
$$

which is equivalent to (8).
Remark 1. Here, we obtain the same result as the paper [1] by Lemma 4, but our proof may be easily understood. In addition, according to the above analysis, we know that both the matrix $\beta$ and matrix $\gamma$ are bounded by the same matrix $\frac{3}{2}\left(b_{j j}+c_{j j}\right)$, while the paper [1] indicated that

$$
\beta<2 b_{j j}+c_{j j} \text { and } \gamma<b_{j j}+2 c_{j j} .
$$

This seems to be interesting, and we will continue to study them in the future.
Finally, by (5), the following results are obvious.
Corollary 2 ([1]). Let $A$ be a Higham matrix, then

$$
\begin{equation*}
\rho_{n}(A)<3 . \tag{20}
\end{equation*}
$$

Corollary 3 ([1]). Let $A$ be a generalized Higham matrix, then

$$
\begin{equation*}
\rho_{n}(A)<3 \sqrt{2} . \tag{21}
\end{equation*}
$$

## IV. Conclusion

The main result of the paper has been proved in [1], but the proof of [1] is lengthy and is not to be understood easily. Based on the ideas in [1] and [7], we give its a new proof, which can be understood more easily than the proof of [1], and obtain some new findings.

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