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A New Proof on the Growth Factor in Gaussian Elimination for Generalized Higham Matrices

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Abstract—The generalized Higham matrix is a complex symmetric matrix $A = B + iC$, where both $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times n}$ are Hermitian positive definite, and $i = \sqrt{-1}$ is the imaginary unit. The growth factor in Gaussian elimination is less than $3\sqrt{2}$ for this kind of matrices. In this paper, we give a new brief proof on this result by different techniques, which can be understood very easily, and obtain some new findings.

Keywords—CSPD matrix, positive definite, Schur complement, Higham matrix, Gaussian elimination, Growth factor.

I. Introduction

COMPLEX symmetric matrices arise frequently, specially in algebraic eigenvalue problems see [2,3] and in the computational electrodynamics see [4] etc. The Higham matrix is a complex symmetric matrix $A = B + iC$, where both B and C are real, symmetric and positive definite, which was firstly presented by Higham in [3] (It was called by a CSPD matrix.). In order to research the accuracy and stability of their LU factorizations, the growth factor (denoted by $\rho_n(A)$) in Gaussian elimination was conjectured in [2] that

$$\rho_n(A) \le 2$$

for any Higham matrix A. Subsequently, the paper proved the following result

$$\rho_n(A) < 3, \tag{1}$$

for such matrix A, and so LU factorization without pivoting is perfectly normwise backward stable see [3]. Moreover, they pointed out that if the Higham matrix is extended by allowing B and C to be arbitrary Hermitian positive definite matrices, i.e., $A = B + iC$ is a generalized Higham matrix, then

$$\rho_n(A) < 3\sqrt{2}, \tag{2}$$

whose proof was quite lengthy in [1]. In addition, authors in [5] also noted that the above bound (2) remains true when B or C or both are negative (rather than positive) definite.

In this paper, we mainly give a new brief proof of the results (1) and (2) by different techniques, which can be understood more easily than the proof of [1]. Next, for convenience, we use the same notations as in [1].

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II. Auxiliary results

In this section, we mainly list some results and lemmas which will be essential to prove our results.

Lemma 1 ([6]). Let A be a CSPD matrix, then A is nonsingular, and any principal submatrix of A and any schur complement in A are also CSPD matrices. Obviously, Lemma 1 shows that, being a CSPD matrix is a hereditary property of active submatrices in Gaussian elimination.

Lemma 2 ([6]). The largest element of a CSPD matrix A lies on its main diagonal.

The above property also holds for generalized Higham matrices in the following slightly weakened form.

Lemma 3 ([1]). If A is a generalized Higham matrix, then

$$\sqrt{2}\max_{l}|a_{ll}| \ge \max_{l\ne j}|a_{lj}|, \tag{3}$$

Thus, for a CSPD matrix A, the growth factor

$$\rho_n(A) = \frac{\max_{i,j,k}|a_{ij}^{(k)}|}{\max_{i,j}|a_{ij}|} \tag{4}$$

can be replaced by

$$\rho_n(A) = \frac{\max_{j,k}|a_{jj}^{(k)}|}{\max_{j}|a_{jj}|}. \tag{5}$$

By Lemma 1 and 2, one can obtain broader bounds for the growth factor of a CSPD matrix A.

Lemma 4 ([7]). Let Z_1 and Z_2 be $m \times n$ matrices and

$$H = Z_1^* Z_2 + Z_2^* Z_1, \tag{6}$$

then

$$H \le Z_1^* Z_1 + Z_2^* Z_2. \tag{7}$$

Lemma 5 ([8]). If B_1 and B_2 are $n \times n$ Hermitian positive definite, then inequalities $B_1 \ge B_2$ if and only if $B_2^{-1} \ge B_1^{-1}$.

In addition, according to the Theorem 2.1 in [1] and its proof, we easily obtain the following corollary.

Corollary 1. Let $A = B+iC$, where B and C are Hermitian and positive definite matrices, then A is nonsingular, and $A^{-1} = X + iY$, X is a positive (semi)definite matrix when B is positive (semi)definite and Y is a negative (semi)definite matrix when C is positive (semi)definite.

III. Main results

The following theorem has been proved in [1], but the proof of [1] is lengthy. Based on the ideas in [1] and [7], we next give its a new proof, which can be understood more easily than the proof of [1].

Theorem 1. Let A be a generalized Higham matrix, then

$$\frac{|a_{jj}^{(k)}|}{|a_{jj}|} < 3, \quad j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, n-1. \quad (8)$$

Proof. Similarly to [1], fix the number $k \in \{1, 2, \cdots, n-1\}$ and j, where $j \geq k+1$. Denote A_k, B_k and C_k by the leading principal order k submatrices in A, B and C, respectively. We split the matrix A_{kj}, a principal order $(k+1) \times (k+1)$ submatrix in A, into

$$A_{kj} = \begin{pmatrix} A_k & \alpha \\ \beta^T & a_{jj} \end{pmatrix},$$

where

$$\alpha^T = \left(\; a_{1j}, a_{2j}, \cdots, a_{kj} \; \right),$$

and

$$\beta^T = \left(\; a_{j1}, a_{j2}, \cdots, a_{jk} \; \right).$$

Defining the vectors

$$b^T = \left(\; b_{1j}, b_{2j}, \cdots, b_{kj} \; \right)$$

and

$$c^T = \left(\; c_{1j}, c_{2j}, \cdots, c_{kj} \; \right),$$

we can rewrite A_{kj} as

$$A_{kj} = \begin{pmatrix} B_k + iC_k & b + ic \\ b^* + ic^* & b_{jj} + ic_{jj} \end{pmatrix}. \quad (9)$$

It is easy to see that $a_{jj}^{(k)}$ can be obtained by performing block Gaussian elimination in A_{kj}, namely,

$$a_{jj}^{(k)} = a_{jj} - \beta^T A_k^{-1} \alpha.$$

Similarly to [1], setting $a_{jj}^{(k)} = \beta + i\gamma, \; \beta, \; \gamma \in \mathbb{R}$ and using (9), we have

$$\beta + i\gamma = b_{jj} + ic_{jj} - (b^* + ic^*)(B_k + iC_k)^{-1}(b + ic). \quad (10)$$

Next, we use the same method as in [1] to deal with $(B_k + iC_k)^{-1}$. By [1], we know that $(B_k + iC_k)^{-1}$ can be written as

$$(B_k + iC_k)^{-1} = X_k + iY_k, \quad (11)$$

where X_k is positive definite and Y_k is negative definite by Corollary 1. Substituting (11) into (10) yields

$$\beta + i\gamma = b_{jj} + ic_{jj} - (b^* + ic^*)(X_k + iY_k)(b + ic),$$

we have

$$\beta = b_{jj} - b^* X_k b + c^* X_k c + c^* Y_k b + b^* Y_k c, \quad (12)$$

and

$$\gamma = c_{jj} - b^* Y_k b + c^* Y_k c - c^* X_k b - b^* X_k c. \quad (13)$$

Now, we use the other technique, which is different from [1], to obtain the upper bounds on β and γ.

Since X_k is a positive definite matrix, Y_k is negative definite. It is obvious that $-b^* X_k b$ in (12) and $c^* Y_k c$ in (13) are negative semidefinite, so (12) and (13) can rewrite

$$\beta \leq b_{jj} + c^* X_k c + c^* Y_k b + b^* Y_k c, \quad (14)$$

and

$$\gamma \leq c_{jj} - b^* Y_k b - c^* X_k b - b^* X_k c. \quad (15)$$

Now we mainly consider the last two summands on the right hand side for the above two inequalities (14) and (15). First, for (14), we apply the Lemma 4 with

$$Z_1 = Gb \;\; \text{and} \;\; Z_2 = Gc,$$

where G is the Hermitian positive definite square root of the matrix $-Y_k$, we get

$$c^* Y_k b + b^* Y_k c \leq -b^* Y_k b - c^* Y_k c,$$

thus

$$\beta \leq b_{jj} + c^* X_k c - b^* Y_k b - c^* Y_k c. \quad (16)$$

The last summand on the right-hand side of (15) may be proved in the same way. Thus we have the following inequality

$$-c^* X_k b - b^* X_k c \leq b^* X_k b + c^* X_k c.$$

So

$$\gamma \leq c_{jj} - b^* Y_k b + b^* X_k b + c^* X_k c. \quad (17)$$

In addition, by [1], we see that

$$X_k = (B_k + C_k B_k^{-1} C_k)^{-1} \leq \frac{1}{2} C_k^{-1} \quad (18)$$

and

$$-Y_k = (C_k + B_k C_k^{-1} B_k)^{-1} \leq \frac{1}{2} B_k^{-1}. \quad (19)$$

Note that $\begin{pmatrix} C_k & c \\ c^* & c_{jj} \end{pmatrix}$ and $\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix}$ are positive definite, by [1], the Schur complement C_{kj}/C_k and B_{kj}/B_k are also positive definite, i.e.,

$$c^* {C_k}^{-1} c < c_{jj} \;\; \text{and} \;\; - b^* {B_k}^{-1} b < b_{jj},$$

which implies that

$$c^* X_k c < \frac{1}{2} c_{jj}, \;\; \text{and} \;\; - b^* Y_k b < \frac{1}{2} b_{jj},$$

Coming back to (18), from the trivial inequality

$$B_k + C_k B_k^{-1} C_k \geq B_k,$$

we can deduce the bound $X_k \leq B_k^{-1}$ by Lemma 5. In addition, note that $\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix}$ is positive definite, we have

$$b^* X_k b \leq b^* B_k^{-1} b < b_{jj}.$$

Similarly, (19) implies the bound

$$-Y_k < C_k^{-1}$$

and

$$-c^* Y_k c \leq c^* C_k^{-1} c < c_{jj}.$$

Summarizing the above results, we conclude that

$$\begin{aligned}\beta &\leq b_{jj} + c^* X_k c - b^* Y_k b - c^* Y_k c \\ &< b_{jj} + \tfrac{1}{2} c_{jj} + \tfrac{1}{2} b_{jj} + c_{jj} \\ &= \tfrac{3}{2}(b_{jj} + c_{jj})\end{aligned}$$

and

$$\begin{aligned}\gamma &\leq c_{jj} - b^* Y_k b + b^* X_k b + c^* X_k c \\ &< c_{jj} + \tfrac{1}{2} b_{jj} + b_{jj} + \tfrac{1}{2} c_{jj} \\ &= \tfrac{3}{2}(b_{jj} + c_{jj}).\end{aligned}$$

So both the matrix β and matrix γ are bounded above by the same matrix $\frac{3}{2}(b_{jj} + c_{jj})$.

It follows that

$$\begin{aligned}\beta^2 + \gamma^2 &< [\tfrac{3}{2}(b_{jj} + c_{jj})]^2 + [\tfrac{3}{2}(b_{jj} + c_{jj})]^2 \\ &= \tfrac{9}{2}(b_{jj} + c_{jj})^2 \\ &= \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2) + 9 b_{jj} c_{jj} \\ &\leq \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2) + \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2) \\ &= 9(b_{jj}^2 + c_{jj}^2).\end{aligned}$$

which is equivalent to (8). ■

Remark 1. Here, we obtain the same result as the paper [1] by Lemma 4, but our proof may be easily understood. In addition, according to the above analysis, we know that both the matrix β and matrix γ are bounded by the same matrix $\frac{3}{2}(b_{jj} + c_{jj})$, while the paper [1] indicated that

$$\beta < 2b_{jj} + c_{jj} \quad \text{and} \quad \gamma < b_{jj} + 2c_{jj}.$$

This seems to be interesting, and we will continue to study them in the future.

Finally, by (5), the following results are obvious.

Corollary 2 ([1]). Let A be a Higham matrix, then

$$\rho_n(A) < 3. \tag{20}$$

Corollary 3 ([1]). Let A be a generalized Higham matrix, then

$$\rho_n(A) < 3\sqrt{2}. \tag{21}$$

IV. Conclusion

The main result of the paper has been proved in [1], but the proof of [1] is lengthy and is not to be understood easily. Based on the ideas in [1] and [7], we give its a new proof, which can be understood more easily than the proof of [1], and obtain some new findings.

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References

[1] A. George, Kh. D. Ikramov, A. B. Kucherov, On the growth factor in Gaussian elimination for generalized Higham matrices, *Numer. Lin. Alg. Appl.*, 9 (2002) 107–114 .

[2] N. J. Higham, Factorizing complex symmetric matrices with positive real and imaginary parts, *Math. Comput.*, 67 (1998) 1591–1599.

[3] N. J. Higham, Accuracy and stablility of numerical algorithms (2nd), *Society for Industrial and Applied Mathematics*, Philadelphia, PA, USA, 2002.

[4] U. V. Rienen. Numerical Methods in Computational Electrodynamics: *Linear Systems in Practical Applications. Number 12 in Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2001.

[5] A. George, K. D. Ikramov, On the growth factor in Gaussian elimination for matrices with sharp angular field of values, *Calcolo*, 41 (2004) 27–36.

[6] Kh. D. Ikramov, A. B. Kucherov, Bounding the growth factor in Gaussian elimination for Buckley's class of complex symmetric matrices, *Numer. Lin. Alg. Appl.*, 7 (2000) 269–274.

[7] Kh. D. Ikramov, Determinantal inequalities for accretive-dissipative matrices, *J. Math. Sci.*, 121 (2004) 2458–2464.

[8] R. A. Horn, C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

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