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A new approach to the approximate solutions of Hamilton-Jacobi equations

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***Abstract*—** We propose a new approach on how to obtain the approximate solutions of Hamilton-Jacobi (HJ) equations. The process of the approximation consists of two steps. The first step is to transform the standard HJ equations into the virtual time based HJ equations (VT-HJ) by introducing a new idea of 'virtual-time'. The second step is to construct the approximate solutions of the HJ equations through a computationally iterative procedure based on the VT-HJ equations. It should be noted that the approximate feedback solutions evolve by themselves as the virtual-time goes by. Finally, we demonstrate the effectiveness of our approximation approach by means of simulations with linear and nonlinear control problems.

***Keywords*—** Nonlinear Control, Optimal Control, Hamilton-Jacobi Equation, Virtual-Time.

I. INTRODUCTION

FOR optimal feedback controller design of nonlinear systems, we generally meet Hamilton-Jacobi (HJ) equations to be dealt with. Once a solution of HJ equations is obtained, it is relatively easy to construct a feedback controller for the optimal control problem. It means that HJ equations play a key role in the design process of optimal control problems. However, it is well known that in the case of nonlinear optimal control problems HJ equations are almost impossible to be solved analytically. Therefore, a great deal of research on approximate solutions of HJ equations has been reported so far. For example, there are the Taylor expansion approach [1], the successive Galerkin approach [2], [3], the viscosity solution approach, the genetic programming approach [4], the Neural Network approach [5], and others.

In this paper, we propose a new approach for obtaining the approximate solutions of HJ equations, by introducing a new type of HJ equations coming from the idea of 'virtual time'. The major advantage of our approach with HJ equations using the virtual time is that it is computationally simple and there is no need to search for implicit functions contained in the HJ equation as done in [2], [3].

The outline of the paper is organized as follows. In Section 2, the problem formulation for optimal control problems is given. In Section 3, we introduce the idea of the virtual time for HJ equations, and propose the VT-HJ equations using the virtual time, and describe how to obtain the solutions of VT-HJ equations. Note that the solutions of VT-HJ equations evolve as time goes by. In Section 4, several simulation examples with linear and nonlinear optimal control problems are shown to illustrate the effectiveness of the proposed approach.

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II. PROBLEM FORMULATION

In this section, we formulate the optimal control problems, and give a brief description on the standard HJ equations.

A. Optimal control problem

Consider the nonlinear time-invariant systems described by ordinary differential equations that are affine in control.

$$\dot{x} = f(x) + g(x)u\ ,\ f(0) = 0\ ,\ g(0) = 0 \tag{1}$$

Here, $x \in \mathrm{R}^n$, $u \in \mathrm{R}^m$, $f : \mathrm{R}^n \mapsto \mathrm{R}^n$, $g : \mathrm{R}^n \mapsto \mathrm{R}^{n\times m}$, and f, g are sufficiently smooth.

Then, the optimal control problem is formulated as follows. Given the prescribed performance index

$$J = \int_0^\infty l(x) + u^T R u\, dt \tag{2}$$

find a control function that satisfies the system equation (1) and minimizes the performance index (2). Here the state penalty function $l(x)$ is sufficiently smooth and positive-definite, the control penalty matrix $R \in \mathrm{R}^{m\times m}$ is positive-definite, and $(\bullet)^T$ means the transpose of vectors and matrices.

B. HJ equation

We design a feedback controller $u(x)$ for the above-mentioned optimal control problem. Under the assumption of differentiability on solutions, the standard theory of optimal control tells us that the design problem is reduced to finding a solution V^* of the following HJ equation

$$\frac{\partial V^*}{\partial x} f - \frac{1}{4}\frac{\partial V^*}{\partial x} g R^{-1} g^T \frac{\partial V^*}{\partial x}^T + l = 0 \tag{3}$$

with the initial state condition

$$V^*(0) = 0\ . \tag{4}$$

By using the solution V^*, we can construct the optimal feedback controller as follows.

$$u^*(x) = -\frac{1}{2} R^{-1} g^T \frac{\partial V^*}{\partial x}^T \tag{5}$$

Here, $\partial V^*/\partial x$ is defined as

$$\frac{\partial V^*}{\partial x} \equiv \left[\frac{\partial V^*}{\partial x_1}\ \frac{\partial V^*}{\partial x_2} \cdots \frac{\partial V^*}{\partial x_n}\right] \tag{6}$$

Remark 2.1

When the solutions of HJ equations are not differentiable, the appropriate theory should be adopted, i.e. the viscosity solution approach, the generalized Jacobian approach, and others. This is a subject of the further research.

III. VT-HJ Equation and Algorithm

We introduce the virtual time based HJ equation, which plays a key role in the proposed approach.

Definition(VT-HJ equation)

Consider the optimal control problem with system equation (1) and the performance index (2). Then, we define the VT-HJ equation as

$$\frac{\partial W}{\partial \tau} = \frac{\partial W}{\partial x} f - \frac{1}{4}\frac{\partial W}{\partial x} g R^{-1} g^T \frac{\partial W}{\partial x}^T + l \qquad (7)$$

and denote the solution of (7) by $W(x,\tau)$, where $W(x,\tau)$: $\mathrm{R}^n \times \mathrm{R} \mapsto \mathrm{R}$ is continuous and sufficiently smooth with $W(0,\tau) = 0$.

Convergence

We give some comments on the relationship between $W(x,\tau)$ and $V^*(x)$. When it comes to the time-invariant system with an infinite horizon, the HJ equation associated with it is given in the expression of (3), while the VT-HJ equation is given in the expression of (7). A large difference between these two equations seems to exist, because the VT-HJ equation (7) includes the time-derivative term while the HJ equation (3) doses not. However, the equations (3) and (7) can be expected to be identical to each other, if the limit function

$$\lim_{\tau\to\infty} \frac{\partial W(x,\tau)}{\partial \tau} \qquad (8)$$

exists uniformly in x and the value of the limit function is zero.

Here, we give a brief discussion on the possibility that $W(x,\tau)$ converges to $V^*(x)$ as $\tau \to \infty$ in Banach space. Setting $\overline{W}(x,\tau)$ by $\overline{W}(x,\tau) = W(x,\tau) - V^*(x)$, we rewrite the VT-HJ equation (7) into the following.

$$\frac{\partial \overline{W}}{\partial \tau} = \frac{\partial(\overline{W} + V^*)}{\partial x} f - \frac{1}{4}\frac{\partial(\overline{W} + V^*)}{\partial x} g R^{-1} g^T \frac{\partial(\overline{W} + V^*)}{\partial x}^T + l \qquad (7')$$

Focusing on this equation(7') instead of (7), we investigate the possibility that the zero solution of (7') is asymptotically stable in Banach space. For this reason, we introduce a Lyapunov function candidate $U(\overline{W}(\bullet,\tau))$ as follows,

$$U(\overline{W}(\bullet,\tau)) = \iint\cdots\int \overline{W}^2(x,\tau) dx_1 dx_2 \cdots dx_n$$

and apply the Lyapunov stability theorem of abstract nonlinear dynamical systems in Banach space [7]. Under the assumption that for each τ there is a $\delta(\tau) > 0$ such as

$$\text{ess. } \sup_x \left\{ \overline{W}(x,\tau) \times \overline{H}\left(x, \frac{\partial \overline{W}}{\partial x}\right) \right\} \le -\delta(\tau)$$

whenever $\overline{W}(x,\tau) \neq 0$,

we can expect to obtain the convergence property. Here, $\overline{H}$ represents the right hand side of (7').

However, it should be noted that the above-mentioned assumption might be not suitable for practical applications. Moreover, even with such an assumption, the rigorous proof of the convergence property in Banach space is considered to be extremely difficult. Therefore, we demonstrate the usefulness of the VT-HJ based approach, not from a theoretical point of view, but from a computational point of view. We propose a new approach for obtaining the computational solutions of HJ equations through the VT-HJ equations. We will provide some simulations to illustrate the convergence property of our proposed approach.

Basic Algorithm

Step 1. Select an appropriate initial function $W(x,0)$.

Step 2. Solve the VT-HJ equation (7), using the virtual time.

Step 3. If the absolute value of $\left.\frac{\partial W(x,\tau)}{\partial \tau}\right|_{\tau=\tau^*}$ is sufficiently small uniformly in x, determine $W(x,\tau^*)$ as the approximate solution of HJ equation (3). If not, return to Step 2.

Remark 3.1

The VT-HJ equations can be solved through the existing differential equation solvers, such as the Euler method, the Runge-Kutta method, and so on. For example, the case with the Euler method is described as follows. First, we determine the integration step $\Delta\tau$, and then calculate the one-step forward solution using the following relation.

$$W(x,\tau+\Delta\tau) = W(x,\tau) + \frac{\partial W}{\partial \tau}\Delta\tau \qquad (9)$$

Remark 3.2

Note that the calculation process is beyond the standard usage of Euler method in the sense that the function $W(x,\tau)$ evolves as the virtual time passes by. In other words, the calculation processes of the Euler method and the Runge-Kutta method are done in terms of functions.

IV. Simulation

Three design examples are given to illustrate the effectiveness of the proposed approach in a comparison with other existing methods. We choose the LQ problem in the first example, so as to show that $H\left(x, \frac{\partial W}{\partial x}\right)$ is updated in a quadratic expression, where $H\left(x, \frac{\partial W}{\partial x}\right)$ represents the right hand side of the VT-HJ equation (7). The resultant controller is expected to converge to the solution LQ theory implies. The

second example is given for the case of nonlinear optimal control problems. In this case, the number of the terms in the series expansion for $W(x,\tau)$ dramatically increases as $\tau \to \infty$, so that the calculation process with the Euler method can not be carried out because of the blast in the number of series terms. One way to avoid such blast is to delete the higher-order terms by means of the Taylor series expansion method. With this method, we can obtain a local nonlinear feedback controller. For a semi-global nonlinear feedback controller, we take the third example. See [8] for more details.

A. Example (*LQ problem*)

For the case of LQ problems, we describe how to solve the optimal LQ problems, based on Basic Algorithm. In this case, the algorithm (say, Algorithm 1) turns out to be identical to the standard method of numerically obtaining the solution of the Riccati differential equation.

Algorithm 1

Step 1. Select an appropriate integration step $\Delta\tau$.

Step 2. Set $W(x,\tau)=0$ at $\tau=0$.

Step 3. Calculate H_τ by

$$H_\tau = H\left(x, \frac{\partial W(x,\tau)}{\partial x}\right).$$

Step 4. Calculate $W(x,\tau+\Delta\tau)$ by the Euler method.

$$W(x,\tau+\Delta\tau) = W(x,\tau) + H_\tau \Delta\tau$$

Step 5. Set $\tau = \tau + \Delta\tau$, and go to Step 2.

We now consider the state equation and performance index as follows.

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u \tag{10}$$

$$J = \int_0^\infty x^T x + u^T u \; dt \tag{11}$$

It is easy to obtain the analytical solution of the associated HJ equation.

$$\begin{aligned} V^* &= x_1^2 - 2x_1x_2 + \left(\sqrt{3}+2\right)x_2^2 \\ &\cong x_1^2 - 2x_1x_2 + 3.732x_2^2 \end{aligned} \tag{12}$$

Keeping this solution in mind, we apply Algorithm 1 to the optimal LQ problem using the Runge-Kutta method, instead of the Euler method. Given $\Delta\tau = 0.1$, $W(x,\tau)$ numerically converges to

$$W(x,\tau^*) = 1.000x_1^2 - 2.000x_1x_2 + 3.732x_2^2$$

uniformly in x. This function is obtained at $\tau^* = 6$, which is almost equal to the analytical solution (12).

B. Example 2 (*Local controller*)

Here is Algorithm 2, combined with the Taylor series expansion method.

Algorithm 2

Step 1. Select an appropriate integration step $\Delta\tau$.

Step 2. Set $W(x,\tau)=0$ at $\tau=0$.

Step 3. Calculate H_τ by

$$H_\tau = H\left(x, \frac{\partial W(x,\tau)}{\partial x}\right)$$

Step 4. Form the Taylor series expansion of H_τ about the origin, and delete the higher-order terms of the series expansion.

Step 5. Calculate $W(x,\tau+\Delta\tau)$ via the Euler method.

$$W(x,\tau+\Delta\tau) = W(x,\tau) + H_\tau \Delta\tau$$

Step 6. Set $\tau = \tau + \Delta\tau$, and go to Step 2.

For a comparison with the results of [6], we consider the following nonlinear optimal control problem.

$$\dot{x} = \begin{bmatrix} -x_1^2 + x_2 \\ -x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{13}$$

$$J = \int_0^\infty \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right) dt \tag{14}$$

A local nonlinear controller V_{taylor} is given in [6] as follows.

$$\begin{aligned} V_{taylor} &= x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 - 1.593x_1^3 \\ &\quad - 2x_1^2x_2 - 0.889x_1x_2^2 - 0.148x_2^3 \end{aligned} \tag{15}$$

Keeping the solution (15) in mind, we apply Algorithm 2 to the nonlinear optimal control problem using the Runge-Kutta method. Given $\Delta\tau = 0.1$, $W(x,\tau)$ numerically converges to

$$\begin{aligned} W(x,\tau^*) &= 1.000x_1^2 + 1.000x_1x_2 + 0.500x_2^2 - 1.593x_1^3 \\ &\quad - 2.000x_1^2x_2 - 0.889x_1x_2^2 - 0.148x_2^3 \end{aligned} \tag{16}$$

uniformly in x. The function (16) is obtained at $\tau^* = 8$, which is equal to the solution (15).

C. Example 3 (*Semi-global controller*)

It should be noticed that the controller (16) is a local one and does work only in a local region. In this section, we restrict ourselves to a prescribed compact set $\Omega \subset R^n$, not to a local set. Then, it is important to note that we have the same kind of the blast problem as seen in Example 2, and also important to note that we can not use the Taylor series expansion because we are focusing on a semi-global controller in the prescribed compact set Ω. In order to avoid such a blast problem, we apply another approximation scheme, such as the Galerkin approximation method or least square method. By means of these approximation schemes, we could construct the semi-global controller without any blast problem, based on the solution of VT-HJ equations. Details are given in the following algorithm.

Algorithm 3

Step 1. Select an appropriate integration step $\Delta\tau$, and determine a compact set Ω and a set of basis functions $\Phi = \{\phi_i\}_{i=1}^{N}$.

Step 2. Select an appropriate initial function $W(x,\tau)$ at $\tau = 0$.

Step 3. Based on the set of the basis functions Φ, approximate the following function

$$W(x,\tau) + H\left(x, \frac{\partial W(x,\tau)}{\partial x}\right)\Delta\tau$$

over the region of Ω, denoted by $W(x,\tau+\Delta\tau)$.

Step 4. Set $\tau = \tau + \Delta\tau$, and go to Step 2.

In a comparison with the successive Galerkin approximation in [3], we deal with the following nonlinear optimal control problem.

$$\dot{x} = \begin{bmatrix} -x_1^3 - x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{17}$$

$$J = \int_0^\infty x_1^2 + x_2^2 + u^2 \, dt \tag{18}$$

A semi-global controller u_{GHJB} is given in [3], over the region of $\Omega = [-1,1]\times[-1,1]$, with the set of the basis functions being $\{ x_1^2, x_1x_2, x_2^2, x_1^4, x_1^3x_2, \cdots, x_2^4, x_1^6, \cdots x_2^6 \}$.

$$\begin{aligned} u_{GHJB} = {} & 0.1643x_1 - 2.5822x_2 - 0.9661x_1^3 \\ & + 1.3757x_1^2x_2 - 0.8441x_1x_2^2 + 0.3010x_2^3 \\ & + 0.4071x_1^5 - 0.7337x_1^4x_2 + 0.6204x_1^3x_2^2 \\ & - 0.3463x_1^2x_2^3 + 0.0995x_1x_2^4 - 0.0574x_2^5 \end{aligned} \tag{19}$$

Keeping this solution in mind, we apply Algorithm 3 to the optimal control problem using the Runge-Kutta method, instead of the Euler method. Given $\Delta\tau = 0.05$, $W(x,\tau)$ numerically converges, resulting in

$$\begin{aligned} u_{\tau^*=10} = {} & 0.1643x_1 - 2.5822x_2 - 0.9661x_1^3 \\ & + 1.3757x_1^2x_2 - 0.8441x_1x_2^2 + 0.3010x_2^3 \\ & + 0.4071x_1^5 - 0.7337x_1^4x_2 + 0.6204x_1^3x_2^2 \\ & - 0.3463x_1^2x_2^3 + 0.0995x_1x_2^4 - 0.0574x_2^5 \end{aligned} \tag{20}$$

This function (20) is obtained at $\tau^* = 10$, which is equal to the solution (19). Eventually, the semi-global controller is obtained.

V. Conclusion

We proposed a new approach for obtaining the approximate solutions of HJ equations. The approximation process consists of two steps. Firstly, introducing a concept of virtual-time, we transformed the HJ equations into the VT-HJ equations. Secondly, we numerically solved the VT-HJ equations by means of existing differential equation solvers.

We demonstrated the effectiveness of our approximation approach through numerical simulations with linear and nonlinear control problems.

References

[1] D.L. Lukes, "Optimal regulation of nonlinear dynamical systems," *SIAM J. Control*, 7, 1969, pp. 75-100

[2] R. Beard, "Improving the closed-loop performance of nonlinear systems," PhD Thesis, Rensselaer Polytechnic Institute, Troy, New York, 1995

[3] R. Beard, G. Saridis and J. Wen, "Galerkin approximation of the generalized Hamilton-Jacobi-Bellman equation," *Automatica*, Vol.33, No.12, 1997, pp. 2159-2177.

[4] J. Imae, Y. Kikuchi, N. Ohtsuki, T. Kobayashi and G. Zhai, "Design of nonlinear control systems by means of differential genetic programming," *Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004, pp. 2734-2739.

[5] A.K. Murad and F.L. Lewis, "Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach," *Automatica*, Vol. 41, No. 5, 2005, pp. 779-791.

[6] S.T. Glad, "Robustness of nonlinear state feedback: a survey," *Automatica*, Vol. 23, No.4, 1987, pp. 425-435.

[7] G.Q.Xu, and S. P. Yung, "Lyapunov stability of abstract nonlinear dynamic system in Banach space," *IMA Journal of Mathematical Control and Information*, 20, 2003, pp.105-127.

[8] J. Imae, K. Shinagawa, A. Ueda, T. Kobayashi and G. Zhai, "An approach to Hamilton-Jacobi-Bellman equations using virtual time," *Transaction of the Japan Society for Simulation Technology*, Vol. 3, No. 1, 2011, pp.11-17, to be published.