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A New Analytical Approach for Free Vibration of Membrane from Wave Standpoint

Mansour Nikkhah-Bahrami, Masih Loghmani, and Mostafa Pooyanfar

Abstract**—**In this paper, an analytical approach for free vibration analysis of rectangular and circular membranes is presented. The method is based on wave approach. From wave standpoint vibration propagate, reflect and transmit in a structure. Firstly, the propagation and reflection matrices for rectangular and circular membranes are derived. Then, these matrices are combined to provide a concise and systematic approach to free vibration analysis of membranes. Subsequently, the eigenvalue problem for free vibration of membrane is formulated and the equation of membrane natural frequencies is constructed. Finally, the effectiveness of the approach is shown by comparison of the results with existing classical solution.

Keywords**—**Rectangular and circular membranes, propagation matrix, reflection matrix, vibration analysis.

I. Introduction

IN classical method for vibration analysis of membrane we apply the boundary conditions to the general solution of the differential equation of motion. It yields to find the natural frequencies [1].

Vibrations can be described as a linear combination of the modes of a structure. An alternative is to describe vibrations as propagating waves travelling in the structure. It was found that the two descriptions often give enlightening complementary perspectives. Wave propagation, transmission and reflection in solids have been studied by a number of researchers [2-6]. By these characteristics Mei solved Longitudinal Vibrations of bars [7]. Mei, Karpenko, Moody and Allen found an analytical approach to free and forced vibrations of axially loaded cracked Timoshenko beams [8].

In this paper we present an analytical solution for free vibration of rectangular and circular membranes by wave propagation method which is organized as follows: In the next section, the equation of motion for membrane is presented and expressions for the propagation of waves in rectangular and circular membranes derived. In Section III, propagation matrix and reflection matrix are derived. We consider the Vibration analysis using wave approach in Section IV. Finally effectiveness of the approach is shown and we describe other applications of this approach.

M. Nikkhah-Bahrami is with Department of Mechanical Engineering, University of Tehran, Tehran, Iran (Corresponding author; phone: +98 21 8800 5677; fax: +98 21 8801 3029; e-mail: mbahrami@ut.ac.ir).

M. Loghmani is with Department of Mechanical Engineering, University of Tehran, Tehran, Iran (e-mail: massloghmany@yahoo.com).

M. Pooyanfar is with Department of Mechanical Engineering, University of Tehran, Tehran, Iran (e-mail: mostafapoyan@yahoo.com).

II. Equation of Motion and Wave Propagation

The equation of motion for membrane is defined as [1]:

$$T\nabla^2 w = \rho \ddot{w} \tag{1}$$

Where $\nabla^2 = \nabla.\nabla$ is known as the Laplacian operator, T is the tension of membrane, w is the transverse displacement and ρ is mass density.

A. Rectangular Membrane

Laplacian operator in rectangular coordinate has the expression

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

Assuming time harmonic motion and using separation of variables, the solution to (1) may be written as:

$$w(x,y,t) = w(x)w(y)\exp(i\omega t) \tag{3}$$

We can rewrite (3) as:

$$w(x,y,t) = x_0 y_0 \exp(-ik_1 x)\exp(-ik_2 y)\exp(i\omega t) \tag{4}$$

Where x_0 and y_0 are arbitrary constant, ω is the frequency, k_1 and k_2 are the wave numbers. Substituting this expression into (1), we have:

$$\left[T(k_1^2 + k_2^2) - \rho\omega^2\right] x_0 y_0 = 0 \tag{5}$$

Or

$$(k_1^2 + k_2^2) = \omega\sqrt{\frac{\rho}{T}} \tag{6}$$

With the time dependence $\exp(i\omega t)$ suppressed, by considering (4) $w(x)$ becomes:

$$w(x) = C_1 e^{-ik_1 x} + C_2 e^{ik_1 x} \tag{7}$$

In which C 's are the wave amplitudes of $w(x)$ and we can write a similar expression for $w(y)$.

B. Circular Membrane

Laplacian operator in circular coordinate has the expression

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{8}$$

Assuming time harmonic motion and using separation of variables, the solution to (1) may be written as:

$$w(r,\theta,t) = w(r)w(\theta)\exp(i\omega t) \tag{9}$$

We can rewrite (9) as:

$$w(r,\theta,t) = AJ_m(k_1 r)\exp(-ik_2\theta)\exp(i\omega t) \tag{10}$$

Where A is an arbitrary constant, ω is the frequency, k_1 and k_2 are the wave numbers. By substituting this expression into (1), and some mathematical operation we have:

$$(k_1^2 + m^2) = \omega\sqrt{\frac{\rho}{T}}, \qquad m = k_2 \tag{11}$$

This is similar (6).
With the time dependence $\exp(i\omega t)$ suppressed, by considering (10) $w(r)$ becomes:

$$w(r) = C_1 J_{k_2}(-k_1 r) + C_2 J_{k_2}(k_1 r) \tag{12}$$

In which C 's are the wave amplitudes of $w(r)$.

III. Propagation and Reflection Wave Matrices

A. Rectangular Membrane

1. Propagation

From wave standpoint, vibrations propagate in an object and reflect at boundaries. We describe them in matrix form, so-called propagation and reflection matrices. Consider two points A and B on a flexurally vibrating membrane along the X-direction at a distance x apart; denoting the positive- and negative-going wave vectors at points A and B as $\mathbf{a}^+$, $\mathbf{a}^-$, $\mathbf{b}^+$ and $\mathbf{b}^-$, respectively. Wave vectors $\mathbf{a}$ and $\mathbf{b}$ are related by [3]:

$$\mathbf{b}^+ = \mathbf{f}(x)\mathbf{a}^+, \qquad \mathbf{a}^- = \mathbf{f}(x)\mathbf{b}^- \tag{13}$$

where from (7),

$$\begin{aligned} \mathbf{a}^+ &= \left\{C_1 e^{-ik_1 x_a}\right\}, \quad \mathbf{a}^- = \left\{C_2 e^{ik_1 x_a}\right\}, \\ \mathbf{b}^+ &= \left\{C_1 e^{-ik_1 x_b}\right\}, \quad \mathbf{b}^+ = \left\{C_2 e^{ik_1 x_b}\right\} \end{aligned} \tag{14}$$

And

$$\mathbf{f}(x) = \left[e^{-ik_1 x}\right] \tag{15}$$

$\mathbf{f}(x)$ is known as the propagation matrix for the distance x.

2. Reflection at Boundaries

The boundary conditions of a rectangular membrane are:

$$\begin{aligned} w(x,y) &= 0, \qquad x = 0, l \\ w(x,y) &= 0, \qquad y = 0, d \end{aligned} \tag{16}$$

A boundary is shown in Fig. 1. The incident wave $\mathbf{a}^+$ give rise to reflected wave $\mathbf{a}^-$, which are related by:

$$\mathbf{a}^- = \mathbf{r}\mathbf{a}^+ \tag{17}$$

The reflection matrix $\mathbf{r}$ can be determined by considering boundary conditions. If the boundary is at $x = 0$ then the equilibrium condition become:

$$\mathbf{a}^- = -\mathbf{a}^+ \tag{18}$$

So the reflection matrix $\mathbf{r}$ is:

$$\mathbf{r} = -1 \tag{19}$$

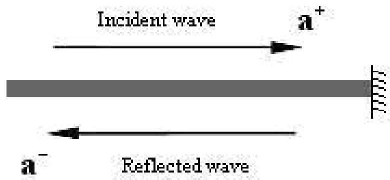

Fig. 1 Side view of a boundary

B. Circular Membrane

1. Propagation

Now consider two points A and B on a flexurally vibrating membrane along the r-direction at a distance r apart; denoting the positive- and negative-going wave vectors at points A and B as $\mathbf{a}^+$, $\mathbf{a}^-$, $\mathbf{b}^+$ and $\mathbf{b}^-$, respectively. Wave vectors $\mathbf{a}$ and $\mathbf{b}$ are related by [3]:

$$\mathbf{b}^+ = \mathbf{f}(r)\mathbf{a}^+, \qquad \mathbf{a}^- = \mathbf{f}(r)\mathbf{b}^- \tag{20}$$

Where from (12)

$$\begin{aligned} \mathbf{a}^+ &= \left\{C_1 J_{k_2}(-k_1 r)\right\}, \quad \mathbf{a}^- = \left\{C_2 J_{k_2}(k_1 r)\right\}, \\ \mathbf{b}^+ &= \left\{C_1 J_{k_2}(-k_1 r)\right\}, \quad \mathbf{b}^- = \left\{C_2 J_{k_2}(k_1 r)\right\} \end{aligned} \tag{21}$$

And

$$f(r) = \frac{J_{k_2}(k_1(r_A + r))}{J_{k_2}(k_1 r_A)} \tag{22}$$

$\mathbf{f}(r)$ is known as the propagation matrix for the distance r and r_A is an arbitrary distance from center of membrane.

2. Reflection at Boundary

The boundary condition of a circular membrane is:

$$w(R,\theta) = 0 \tag{23}$$

In which R is the radius of membrane.

A boundary is shown in Fig. 1. The incident wave $\mathbf{a}^+$ give rise to reflected wave $\mathbf{a}^-$, which are related by:

$$\mathbf{a}^- = \mathbf{r}\mathbf{a}^+ \tag{24}$$

As we anticipated the reflection matrix by considering equilibrium condition becomes such as rectangular case so:

$$\mathbf{r} = -1 \tag{25}$$

IV. Vibration Analysis Using Wave Approach

In this section, the derived propagation and reflection matrices are combined to provide a concise and systematic approach for vibration analysis of both rectangular and circular membranes.

A. Rectangular Membrane

In Fig. 2 a rectangular membrane shown along x axis is depicted with its two boundaries at A and B. Incident and reflected waves at the boundaries, A and B, are denoted by $\mathbf{a}^{\mp}, \mathbf{b}^{\pm}$ respectively. The relationship between the incident and the reflected waves at the boundaries are described as:

$$\mathbf{a}^+ = \mathbf{r}\mathbf{a}^-, \quad \mathbf{b}^- = \mathbf{r}\mathbf{b}^+ \tag{26}$$

In which $\mathbf{r}$ is the reflection matrix that is equal for both boundaries.

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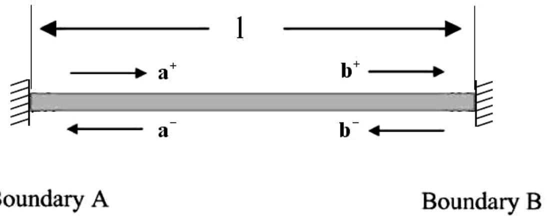

Fig. 2 Side view of rectangular membrane, on boundaries A and B

The propagation relations are**:**

$$\mathbf{b}^+ = \mathbf{f}(l)\mathbf{a}^+ \ , \quad \mathbf{a}^- = \mathbf{f}(l)\mathbf{b}^- \tag{27}$$

Where $\mathbf{f}(l)$ is the propagation matrix between A and B. By rewriting (26), (27) in a matrix form, we have**:**

$$\begin{bmatrix} -1 & \mathbf{r} & 0 & 0 \\ \mathbf{f}(l) & 0 & -1 & 0 \\ 0 & -1 & 0 & \mathbf{f}(l) \\ 0 & 0 & \mathbf{r} & -1 \end{bmatrix} \begin{Bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \\ \mathbf{b}^+ \\ \mathbf{b}^- \end{Bmatrix} = 0 \tag{28}$$

For a non-trivial solution, it follows that:

$$\begin{vmatrix} -1 & \mathbf{r} & 0 & 0 \\ \mathbf{f}(l) & 0 & -1 & 0 \\ 0 & -1 & 0 & \mathbf{f}(l) \\ 0 & 0 & \mathbf{r} & -1 \end{vmatrix} = 0 \tag{29}$$

This leads us to:

$$\begin{aligned} &(1-e^{-2ik_1 l})(1-e^{-2k_1 l}) = 0 \\ &\Rightarrow k_1 = \frac{n\pi}{l}, \quad n = 1, 2, ... \end{aligned} \tag{30}$$

By writing similar expression for the other side of the membrane and assuming its length is d we have:

$$k_2 = \frac{m\pi}{d}, \quad m = 1, 2, ... \tag{31}$$

Substituting (30) and (31) in (6) yields

$$\omega = \omega_{nm} = \pi \sqrt{\left[\left(\frac{n}{l}\right)^2 + \left(\frac{m}{d}\right)^2\right] \frac{T}{\rho}}, \qquad m, n = 1, 2, ... \tag{32}$$

Equation (32) is in full agreement with the classical solution [1].

B. Circular Membrane

In Fig. 3 a circular membrane shown along one of its diameter is depicted with its boundary which we can see it at A and B. Incident and reflected waves at the boundaries, A, B, are denoted by $\mathbf{a}^{\mp}, \mathbf{b}^{\pm}$ respectively. The waves at left and right side of center are denoted by $\mathbf{c}^{\mp}, \mathbf{d}^{\pm}$ relationship between the waves vectors are described as:

$$\begin{aligned} &\mathbf{a}^+ = \mathbf{f}(R)\mathbf{d}^+, \quad \mathbf{a}^- = \mathbf{r}\mathbf{a}^+, \quad \mathbf{d}^- = \mathbf{f}(R)\mathbf{a}^-, \quad \mathbf{c}^+ = \mathbf{d}^-, \\ &\mathbf{b}^+ = \mathbf{f}(R)\mathbf{c}^+, \quad \mathbf{b}^- = \mathbf{r}\mathbf{b}^+, \quad \mathbf{c}^- = \mathbf{f}(R)\mathbf{b}^-, \quad \mathbf{c}^- = \mathbf{d}^+ \end{aligned} \tag{33}$$

In which $\mathbf{r}$ is the reflection matrix for its boundary and $\mathbf{f}(R)$ is the propagation matrix between center of membrane and its boundary.

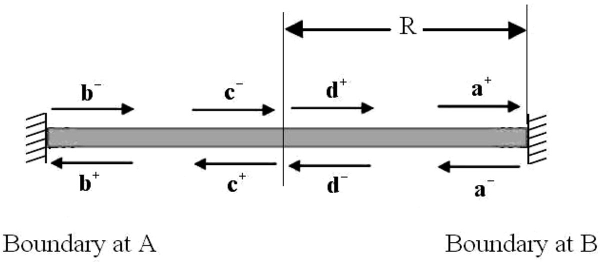

Fig. 3 Side view of circular membrane, on its boundary

After some mathematical operation (33) reduce to

$$\begin{aligned} &\mathbf{a}^+ = \left(\mathbf{f}(R)\right)^2 \mathbf{b}^-, \qquad \mathbf{a}^- = \mathbf{r}\mathbf{a}^+, \\ &\mathbf{b}^+ = \left(\mathbf{f}(R)\right)^2 \mathbf{a}^-, \qquad \mathbf{b}^- = \mathbf{r}\mathbf{b}^+ \end{aligned} \tag{34}$$

By rewriting (34) in a matrix form equation we have

$$\begin{bmatrix} -1 & \mathbf{r} & 0 & 0 \\ -1 & 0 & 0 & \left(\mathbf{f}(R)\right)^2 \\ 0 & \left(\mathbf{f}(R)\right)^2 & -1 & 0 \\ 0 & 0 & \mathbf{r} & -1 \end{bmatrix} \begin{Bmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \\ \mathbf{b}^+ \\ \mathbf{b}^- \end{Bmatrix} = 0 \tag{35}$$

For a non-trivial solution, it follows that:

$$\begin{vmatrix} -1 & \mathbf{r} & 0 & 0 \\ -1 & 0 & 0 & \left(\mathbf{f}(R)\right)^2 \\ 0 & \left(\mathbf{f}(R)\right)^2 & -1 & 0 \\ 0 & 0 & \mathbf{r} & -1 \end{vmatrix} = 0 \tag{36}$$

This leads us to:

$$J_{k_2}(k_1 R) = 0 \tag{37}$$

For very large argument, we have relation:

$$\lim_{z\to\infty} J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{2m+1}{4}\pi\right) \tag{38}$$

So that the frequency equation (37), leads us to the conclusion that, for very large k_1, the natural frequencies can be approximate by:

$$\omega_{mn} = \left(\frac{m}{2} + n - \frac{1}{4}\right)\frac{\pi}{R}\sqrt{\frac{T}{\rho}}, \qquad m, n = 1, 2, ... \tag{39}$$

This is in full agreement with the classical solution [1].

V. Comparison and Discussion

In this paper, we presented an analytical solution for find the natural frequencies of free vibration of rectangular and circular membranes. Classical method [1] leads us to apply boundary conditions to equation of motion, but in this method without solving the equation of motion we can calculate the natural frequencies.

Moreover, we can apply this method to other boundary conditions such as membranes which rest on elastic boundary. Wave approach has high performance to acoustical application and the systems which has interaction between membrane and fluid or other exciting media.

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