

# A New Analytic Solution for the Heat Conduction with Time-Dependent Heat Transfer Coefficient

Te Wen Tu, Sen Yung Lee

**Abstract**—An alternative approach is proposed to develop the analytic solution for one dimensional heat conduction with one mixed type boundary condition and general time-dependent heat transfer coefficient. In this study, the physic meaning of the solution procedure is revealed. It is shown that the shifting function takes the physic meaning of the reciprocal of Biot function in the initial time. Numerical results show the accuracy of this study. Comparing with those given in the existing literature, the difference is less than 0.3%.

**Keywords**—Analytic solution, heat transfer coefficient, shifting function method, time-dependent boundary condition.

## I. INTRODUCTION

HEAT transfer with variable heat conduction coefficient and heat transfer coefficient can be important in many engineering applications. The studies on the problems of heat transfer with space and temperature dependent heat conduction coefficient in literature are tremendous [1], [2].

For the heat conduction with mixed type boundary condition and time-dependent heat transfer coefficient, the problem cannot be solved by the method of separation of variable, hence, various approximated and numerical methods were taken [3], [4]. Kozlov [5] used the Laplace transformation method to study the problems with Biot function in a rational combination of sines, cosines, polynomials, and exponentials. Even though it is possible to obtain an exact series solution of the specified transformed system, there always is great difficulty while taking the inverse Laplace transform of the transformed solution in complex domain. Recently, various inverse schemes [6] for determining the time-dependent heat transfer coefficient were developed.

From the existing literature, due to the complexity and difficulty of the solution, it can be found that the study on the heat conduction with mixed type boundary condition and time-dependent heat transfer coefficient is insufficient. After several decades of the development of analytical-numerical and numerical methods for the solution of non-stationary problems of the diffusion of heat and substance with a time-varying heat transfer coefficient, an analytical solution for a plate with an arbitrary law of the variation of this coefficient was obtained by Lee and his colleagues [7]. They

proposed a simple and accurate analytic form solution for wide class of problems. The work has been considered to be of great importance in the literature [8], [9].

This work proposes a different type of shifting function to develop the analytic closed form solution for one dimensional heat conduction with one mixed type boundary condition and general time-dependent heat transfer coefficient. The solution method is an extension of Lee and his colleagues' works [10], [11], by choosing the shifting function which owns its physical meaning.

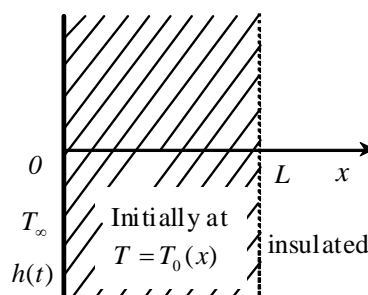


Fig. 1 One-dimensional heat transfer system of a slab with a time-dependent heat convection coefficient

## II. MATHEMATICAL MODELING

Consider the heat conduction in a slab with mixed type boundary condition at one end as shown in Fig. 1. The governing differential equation of the system is:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad 0 < x < L, \quad t > 0. \quad (1)$$

The boundary conditions are

$$-k \frac{\partial T}{\partial x} = h(t)(T_\infty - T), \quad \text{at } x = 0, \quad (2)$$

$$\frac{\partial T}{\partial x} = 0, \quad \text{at } x = L, \quad (3)$$

and the initial condition is

$$T(x, 0) = T_0(x), \quad \text{when } t = 0. \quad (4)$$

Here,  $T$  is the temperature,  $x$  is the spatial-domain variable,  $\alpha$  is the thermal diffusivity,  $t$  is the time,  $L$  is the

Te Wen Tu is Associated Professor with the Department of Mechanical Engineering, Air Force Institute of Technology, Kaohsiung, Taiwan, Republic of China.

Sen Yung Lee is Distinguished Professor with the Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan, Republic of China (Corresponding author, Tel: +886-6-2757575 ext.62150; E-mail: sylee@mail.ncku.edu.tw).

half thickness of the slab,  $k$  is the thermal conductivity,  $h(t)$  is the time-dependent heat transfer coefficient,  $T_\infty$  is an environment temperature constant, and  $T_0$  is the initial temperature. In terms of the following dimensionless quantities:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}, X = \frac{x}{L}, \tau = \frac{\alpha t}{L^2}, Bi(\tau) = \frac{h(t)L}{k}, \theta_0 = \frac{T_0 - T_\infty}{T_\infty}, \quad (5)$$

the associated dimensionless governing differential equation of the system is

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}, \text{ in } 0 < X < 1, \tau > 0, \quad (6)$$

where  $\tau$  and  $Bi(\tau)$  denote Fourier number and Biot function, respectively.

The boundary conditions are

$$\frac{\partial \theta}{\partial X} = Bi(\tau)\theta, \text{ at } X = 0, \quad (7)$$

$$\frac{\partial \theta}{\partial X} = 0, \text{ at } X = 1, \quad (8)$$

and the initial condition is

$$\theta = \theta_0(X), \text{ when } \tau = 0. \quad (9)$$

To simplify the analysis and increase the accuracy, one splits  $Bi(\tau)$  into a constant  $\delta$  and a variable  $F(\tau)$  as

$$Bi(\tau) = \delta + F(\tau) \quad (10)$$

where

$$\delta = Bi(0), \quad (11)$$

$$F(\tau) = Bi(\tau) - Bi(0). \quad (12)$$

It is obvious that  $F(0) = 0$ , and the boundary condition at  $X = 0$ , (8) can be rewritten as

$$\frac{\partial \theta}{\partial X} - \delta \theta = F(\tau)\theta. \quad (13)$$

### III. THE SHIFTING FUNCTION METHOD

#### A. Change of Variable

To find the solution for the partial differential equation with a time dependent heat convection coefficient at one boundary, one extends the shifting function method developed by Lee and Lin [10] and Lee et al. [11] by taking

$$\theta(X, \tau) = v(X, \tau) + f(\tau)g(X), \quad (14)$$

where  $v(X, \tau)$  is the transformed function,  $f(\tau)$  is taken as

$$f(\tau) = F(\tau)\theta(0, \tau), \quad (15)$$

and  $g(X)$  is a shifting function to be specified.

Substituting (14) into (6), (7), (13), and (9), one has the following partial differential equation

$$\frac{\partial^2 v(X, \tau)}{\partial X^2} + f(\tau) \frac{d^2 g(X)}{dX^2} = \frac{\partial v(X, \tau)}{\partial \tau} + g(X) \frac{df(\tau)}{d\tau},$$

in  $0 < X < 1, \tau > 0,$  (16)

along with the boundary conditions and initial condition

$$\frac{\partial v(0, \tau)}{\partial X} + f(\tau) \frac{dg(0)}{dX} - \delta[v(0, \tau) + f(\tau)g(0)] = f(\tau), \quad (17)$$

$$\frac{\partial v(1, \tau)}{\partial X} + f(\tau) \frac{dg(1)}{dX} = 0, \quad (18)$$

$$v(X, 0) + f(0)g(X) = \theta_0(X). \quad (19)$$

It should be noted that (16)-(19) only contain two variables  $v(X, \tau)$  and  $\theta(1, \tau)$ .

#### B. Shifting Function and Its Physical Meaning

To simplify the problem, one specifies a particular shifting function  $g(X)$ . One lets the shifting function  $g(X)$  in (16) - (18) satisfy the differential equation

$$\frac{d^2 g(X)}{dX^2} = 0, \quad (20)$$

and the following boundary conditions

$$\frac{dg(1)}{dX} = 0, \quad (21)$$

$$\frac{dg(0)}{dX} - \delta g(0) = 1. \quad (22)$$

Therefore, the shifting function  $g(X)$  is determined as

$$g(X) = -\frac{1}{\delta}, \quad (23)$$

and takes the physic meaning of the reciprocal of Biot function at time  $\tau = 0$ .

Substitution the function  $g(X)$  in (14), it becomes

$$\theta(X, \tau) = v(X, \tau) - \frac{F(\tau)\theta(0, \tau)}{\delta} \quad (24)$$

Setting  $X = 0$  in the equation above, one obtains

$$\theta(0, \tau) = v(0, \tau) - \frac{F(\tau)\theta(0, \tau)}{\delta} \quad (25)$$

After rearranging these terms, we have

$$\theta(0, \tau) = \frac{\delta}{Bi(\tau)} v(0, \tau) \quad (26)$$

with (23) and (26), the function variables in governing differential equation (16) is reduced from two to one and expressed in terms of the function variable  $v(X, \tau)$ .

$$\frac{\partial^2 v(X, \tau)}{\partial X^2} = \frac{\partial v(X, \tau)}{\partial \tau} - \left\{ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} v(0, \tau) + \frac{F(\tau)}{Bi(\tau)} \dot{v}(0, \tau) \right\} \quad (27)$$

The associated boundary conditions of transformed function  $v(X, \tau)$ , (17) and (18) become the homogeneous type as follows:

$$\frac{\partial v(0, \tau)}{\partial X} - \delta v(0, \tau) = 0 \quad (28)$$

$$\frac{\partial v(1, \tau)}{\partial X} = 0 \quad (29)$$

Since  $f(0) = F(0)\theta(0, 0)$  and  $F(0) = 0$ , therefore, the associated initial condition, (19) becomes

$$v(X, 0) = \theta_0(X) \quad (30)$$

C. Series Expansion

To find the solution for the partial differential equation (27) with boundary conditions (28)-(29) and initial condition (30), one uses the method of series expansion with try functions

$$\phi_n(X) = \frac{\delta}{\lambda_n} \sin(\lambda_n X) + \cos(\lambda_n X), \quad n = 1, 2, 3, \dots \quad (31)$$

satisfying the boundary conditions (28)-(29). Here the characteristic values  $\lambda_n (n = 1, 2, 3, \dots)$  are the roots of the transcendental equation

$$\tan \lambda_n = \frac{\delta}{\lambda_n} \quad (32)$$

The inner products of the try functions are

$$\int_0^1 \phi_m(X) \phi_n(X) dX = \begin{cases} 0 & \text{for } m \neq n \\ N_n & \text{for } m = n \end{cases} \quad (33)$$

where

$$N_n = \frac{1}{2} + \frac{\sin 2\lambda_n}{4\lambda_n} + \frac{\delta}{2\lambda_n^2} (1 - \cos 2\lambda_n) + \frac{\delta^2}{4\lambda_n^3} (2\lambda_n - \sin 2\lambda_n) \quad (34)$$

One can assume the transformed function  $v(X, \tau)$  takes the series form of

$$v(X, \tau) = \sum_{n=1}^{\infty} \phi_n(X) q_n(\tau) \quad (35)$$

Substituting solution from (35) into differential equation (27), it leads to

$$\sum_{n=1}^{\infty} \left\{ \phi_n(X) \dot{q}_n(\tau) - \left[ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} \phi_n(0) q_n(\tau) + \frac{F(\tau)}{Bi(\tau)} \phi_n(0) \dot{q}_n(\tau) \right] - \phi_n''(X) q_n(\tau) \right\} = 0 \quad (36)$$

After taking the inner product with arbitrary try function  $\phi_m(X)$  and integrating  $X$  from 0 to 1, the resulting differential equation becomes

$$\sum_{n=1}^{\infty} \left\{ \int_0^1 \phi_m(X) \phi_n(X) dX \dot{q}_n(\tau) - \left[ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} q_n(\tau) + \frac{F(\tau)}{Bi(\tau)} \dot{q}_n(\tau) \right] \phi_n(0) \int_0^1 \phi_m(X) dX - \int_0^1 \phi_m''(X) \phi_n(X) dX q_n(\tau) \right\} = 0 \quad (37)$$

(Situation I): Consider the  $m = n$  case, (37) reduces to

$$\sum_{n=1}^{\infty} \left\{ N_n \dot{q}_n(\tau) - \left[ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} q_n(\tau) + \frac{F(\tau)}{Bi(\tau)} \dot{q}_n(\tau) \right] N_n \beta_n + \lambda_n^2 N_n q_n(\tau) \right\} = 0 \quad (38)$$

where

$$\beta_n = \frac{\phi_n(0) \int_0^1 \phi_n(X) dX}{N_n} = \frac{\delta}{\lambda_n^2 N_n} \quad (39)$$

From (38), since  $q_n(\tau)$  are independent each other, therefore, one can let

$$N_n \dot{q}_n(\tau) - \left[ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} q_n(\tau) + \frac{F(\tau)}{Bi(\tau)} \dot{q}_n(\tau) \right] N_n \beta_n + \lambda_n^2 N_n q_n(\tau) = 0 \quad (40)$$

or

$$\dot{q}_n(\tau) \left[ 1 - \beta_n \frac{F(\tau)}{Bi(\tau)} \right] + q_n(\tau) \left[ \lambda_n^2 - \beta_n \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} \right] = 0 \quad (41)$$

(Situation II): Consider the case  $m \neq n$ , hence, (37) becomes

$$\sum_{n=1}^{\infty} \left\{ - \left[ \frac{\delta \dot{F}(\tau)}{Bi^2(\tau)} q_n(\tau) + \frac{F(\tau)}{Bi(\tau)} \dot{q}_n(\tau) \right] \left( \frac{\delta}{\lambda_m^2} \right) \right\} = 0 \quad (42)$$

After dropping all constant terms, one obtains

$$\frac{\delta \dot{F}(\tau)}{Bi(\tau)} q_n(\tau) + F(\tau) \dot{q}_n(\tau) = 0 \quad (43)$$

and finds that its solution is the trivial solution

$$q_n(\tau) = 0 \quad (44)$$

As a result, the solution for  $q_n(\tau)$  in (41) is

$$q_n(\tau) = q_n(0) e^{\int_0^{\tau} \frac{\lambda_n^2 (Bi(\tau))^2 - \beta_n \delta \dot{F}(\tau)}{[\beta_n F(\tau) - Bi(\tau)] Bi(\tau)} d\tau} \quad (45)$$

$$= q_n(0) \left( \frac{-Bi(\tau)}{\beta_n F(\tau) - Bi(\tau)} \right) \cdot e^{\frac{\lambda_n^2}{\beta_n - 1} \tau} \cdot e^{\frac{\lambda_n^2 \beta_n \delta}{\beta_n - 1} \int_0^{\tau} \frac{d\tau}{[(\beta_n - 1)F(\tau) - \delta]}}$$

Finally, using the series expansion, the associated initial condition becomes

$$q_n(0) = \frac{\int_0^1 \theta_0(X) \phi_n(X) dX}{N_n} \quad (46)$$

If  $\theta_0(X)$  denotes a constant temperature  $\theta_0$ , we obtain

$$q_n(0) = \frac{\theta_0 \delta}{\lambda_n^2 N_n} \quad (47)$$

After substituting the transformed function  $v(X, \tau)$ , (35), (31), and (45) as well as the shifting function  $g(X)$ , (23), back to (14), one has the analytic solution

$$\theta(X, \tau) = \sum_{n=1}^{\infty} q_n(\tau) \left\{ \phi_n(X) - \frac{F(\tau)}{Bi(\tau)} \right\} \quad (48)$$

**D. Constant Heat Transfer Coefficient**

When the heat transfer coefficient  $h$  is a constant, the Biot function is a constant  $\delta$  and  $F(\tau) = 0$ . The infinite series solution, (48), is reduced to

$$\theta(X, \tau) = \sum_{n=1}^{\infty} q_n(0) e^{-\lambda_n^2 \tau} \left( \frac{\delta}{\lambda_n} \sin \lambda_n X + \cos \lambda_n X \right) \quad (49)$$

The solution is exactly the same as that obtained via the method of separation of variable [1].

**IV. VERIFICATION AND EXAMPLES**

To illustrate the previous analysis and the accuracy of the solution, one examines the following two cases.

**A. Case1:**  $\theta_0(X)$  is a constant.

Consider the heat conduction in a slab with initial constant temperature  $\theta_0$ . Assume the general form of the Biot function is

$$Bi(\tau) = a - b e^{-s\tau} \cos \omega \tau \quad (50)$$

where  $a, b$  are constants;  $s, \omega$  are control variables.

According to (11) - (12), we obtain

$$F(\tau) = b(1 - e^{-s\tau} \cos \omega \tau) \quad (51)$$

$$\delta = a - b \quad (52)$$

and

$$\dot{F}(\tau) = b e^{-s\tau} (s \cos \omega \tau + \omega \sin \omega \tau) \quad (53)$$

Therefore, from (48), the temperature distribution is

$$\theta(X, \tau) = \sum_{n=1}^{\infty} q_n(\tau) \left\{ \cos \lambda_n X - \frac{b(1 - e^{-s\tau} \cos \omega \tau)}{a - b e^{-s\tau} \cos \omega \tau} \cos \lambda_n X \right\} \quad (54)$$

where

$$q_n(\tau) = \frac{\theta_0 \sin \lambda_n}{\lambda_n N_n} \left( \frac{-(a - b e^{-s\tau} \cos \omega \tau)}{\beta_n b(1 - e^{-s\tau} \cos \omega \tau) - (a - b e^{-s\tau} \cos \omega \tau)} \right) e^{\frac{\lambda_n^2}{\beta_n - 1} \tau} e^{\frac{\lambda_n^2 \beta_n \delta}{\beta_n - 1} \int_0^{\tau} \frac{d\tau}{[(\beta_n - 1)b(1 - e^{-s\tau} \cos \omega \tau) - \delta]}} \quad (55)$$

For comparison, we show the results of Chen et al. [7] as follows

$$\theta(X, \tau) = \sum_{n=1}^{\infty} q_n(\tau) \left\{ \cos \lambda_n X - b(1 - e^{-s\tau} \cos \omega \tau) \left( \frac{X^2 - 1}{2} \right) \cos \lambda_n \right\} \quad (56)$$

$$q_n(\tau) = \frac{\theta_0 \sin \lambda_n}{\lambda_n N_n} \left( \frac{1}{\beta_n b(1 - e^{-s\tau} \cos \omega \tau) - 1} \right) e^{\frac{\beta_n}{\gamma_n} \tau} e^{\frac{(\lambda_n^2 - \beta_n)}{\gamma_n} \int_0^{\tau} \frac{d\tau}{[\beta_n b(1 - e^{-s\tau} \cos \omega \tau) - 1]}} \quad (57)$$

$$\gamma_n = \frac{\cos \lambda_n}{N_n} \int_0^1 \left( \frac{X^2 - 1}{2} \right) \cos \lambda_n X dX = \frac{\cos \lambda_n}{\lambda_n^3 N_n} (\lambda_n \cos \lambda_n - \sin \lambda_n) \quad (58)$$

when  $a = 1.2, b = 1, s = 1, \omega = 0$ , and  $\theta_0 = -0.664$ , the case is exactly the same as the one discussed by Ivanov and Salomatov [3] and Postol'nik [4], who calculated the heating of an infinite plate. In Figs. 2, 3, the temperatures of the slab at  $X = 1$ , and various time points evaluated via the present

analysis are compared with those in the existing literature [7]. It shows that the results are very consistent. Fig. 4 shows the numerical difference between present analysis and [7], the maximum difference is less than 0.3%.

In Fig. 5, as times go, the temperature variation along the slab is illustrated. It can be found that the temperature variation of the slab of present analysis coincides with [7].

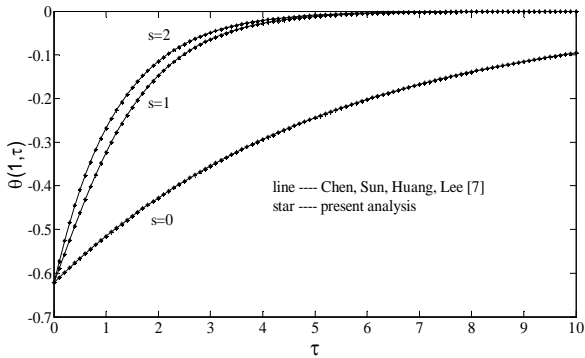


Fig. 2 Influence of  $s$  parameter on the temperature variation of the slab at  $X = 1$  ( $\omega = 0$ )

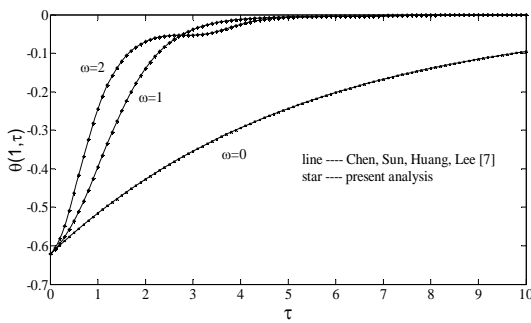


Fig. 3 Temperature variation of the slab with various  $\omega$  at  $X = 1$  ( $s = 0$ )

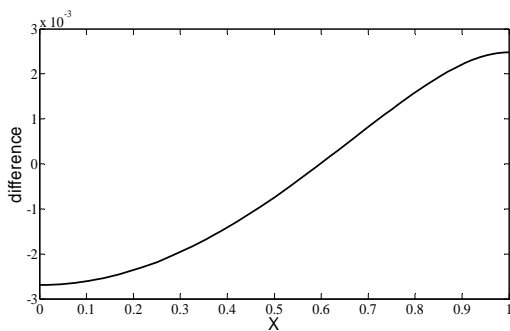


Fig. 4 Difference between present analysis and [7] with various  $X$  ( $\tau = 0.5, s = 5, \omega = 5$ )

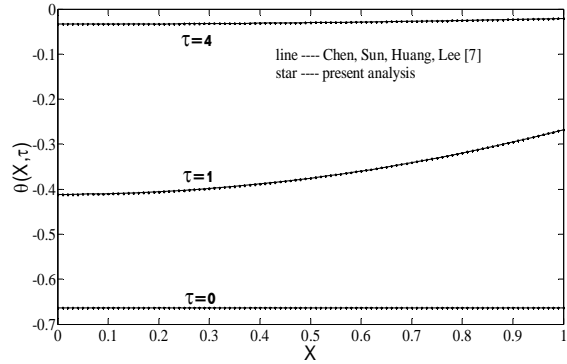


Fig. 5 Temperature variation along the slab at  $\tau = 0, 1, 4$  ( $a = 1.2, b = 1, s = 2, \omega = 0, \theta_0 = -0.664$ )

**B. Case2 :  $\theta_0(X)$  is not a constant.**

Consider the heat conduction in a slab with initial temperature  $\theta_0(X) = \cos(\frac{\pi}{2} X)$ . The other constants are the same as Case 1, and  $q_n(0)$  is determined as

$$q_n(0) = \frac{2\pi \cos \lambda_n}{\pi^2 - 4\lambda_n^2} \tag{59}$$

The numerical result is shown in Fig. 6. It is obvious when the Fourier constant  $\tau$  is equal to 4, the temperatures along the boundary surfaces approach to the environment temperature zero.

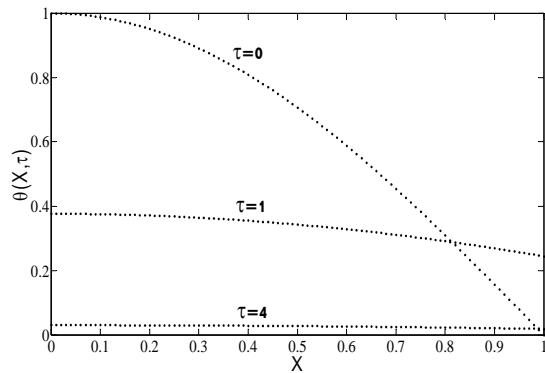


Fig. 6 Temperature variation along the slab at  $\tau = 0, 1, 4$  [ $a = 1.2, b = 1, s = 2, \omega = 0, \theta_0(X) = \cos(\frac{\pi X}{2})$ ]

**V. CONCLUSION**

In this paper, an alternative approach is proposed to develop the analytic solution for the one dimensional heat conduction with one mixed type boundary condition and general time-dependent heat transfer coefficient. In the present study, the physic meaning of the solution procedures are revealed. It is shown that the solution is also simple accurate and fast

convergent. Numerical results are shown to be consistent with those in literature. The proposed solution method can also be extended to the studies of the problems with boundary conditions of different kinds and those of multiple dimensions.

#### ACKNOWLEDGMENT

This research work was supported by the National Science Council of Taiwan, the Republic of China, under grant: NSC 101-2221-E-006-031.

#### REFERENCES

- [1] Özisik, M. N., *Boundary Value Problems of Heat Conduction*, International Textbook Company, Pennsylvania, 1968, pp. 43-69.
- [2] Borukhov, V.T., Tsurko, V.A., and Zayats, G.M., "The functional identification approach for numerical reconstruction of the temperature-dependent thermal conductivity coefficient", *International Journal of Heat and Mass Transfer*, 2009, vol. 52, no. 1-2, pp. 232-238.
- [3] Ivanov, V.V. and Salomatov, V.V., "On the calculation of the temperature field in solids with variable heat-transfer coefficients", *Journal of Engineering Physics and Thermophysics*, 1965, vol. 9, no. 1, pp. 83-85.
- [4] Postol'nik, Yu.S., "One-dimensional convective heating with a time-dependent heat-transfer coefficient", *Journal of Engineering Physics and Thermophysics*, 1970, vol. 18, no. 2, pp. 316-322.
- [5] Kozlov, V.N., "Solution of heat-conduction problem with variable heat exchange coefficient", *Journal of Engineering Physics and Thermophysics*, 1970, vol. 18, no. 1, pp. 133-138.
- [6] Chen, H.T. and Wu, X.Y., "Investigation of heat transfer coefficient in two dimension altransient inverse heat conduction problems using the hybrid inversescheme", *International Journal for Numerical Methods in Engineering*, 2008, vol. 73, no.1, pp. 107-122.
- [7] Han Taw Chen, Shao Lun Sun, Hui Chen Huang, and Sen Yung Lee, "Analytic Closed Solution for the Heat Conduction with Time Dependent Heat Convection Coefficient at One Boundary", 2010, *CMES*, vol. 59, no. 2, pp. 107-126 .
- [8] Yatskiv, O. I., Shvets, R. M., and Bobyk, B. Ya., "Thermostressed State of a Cylinder with Thin Near-Surface Layer Having Time-Dependent Thermophysical Properties", *J. of Mathematical Sciences*, 2012, vol. 187, No. 5, 647-666.
- [9] Caffagni, A., Angeli, D., Barozzi, G.S., and Polidoro, S., "A Revised Approach for One-Dimensional Time-Dependent Heat Conduction in a Slab", *ASME J. of Heat Transfer*, 2013, vol. 135, pp. 31301-1 ~31301-8.
- [10] Lee, S.Y. and Lin, S.M., "Dynamic analysis of nonuniform beams with time-dependent elastic boundary conditions", *ASME Journal of Applied Mechanics*, 1996, vol. 63, no. 2, pp. 474-478.
- [11] Lee, S.Y., Lu, S.Y., Liu, Y.T., and Huang, H.C., "Exact Large Deflection Solutions for Timoshenko Beams with Nonlinear Boundary Conditions", *CMES*, 2008, vol. 33, no. 3, pp. 293-312.



**Sen Yung Lee** is a Distinguished Professor of Department of Mechanical Engineering, National Cheng Kung University, Taiwan, R.O.C.. He received his Ph.D. from State University of New York at Buffalo, USA in 1984. His research interests are solid mechanics, heat conduction, and creative design with more than 80 patents.