International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:3, No:7, 2009

A New Algorithm for Determining the Leading Coefficient of in the Parabolic Equation

Shiping Zhou and Minggen Cui

Abstract—This paper investigates the inverse problem of determining the unknown time-dependent leading coefficient in the parabolic equation using the usual conditions of the direct problem and an additional condition. An algorithm is developed for solving numerically the inverse problem using the technique of space decomposition in a reproducing kernel space. The leading coefficients can be solved by a lower triangular linear system. Numerical experiments are presented to show the efficiency of the proposed methods.

Keywords—parabolic equations, coefficient inverse problem, reproducing kernel.

I. Introduction

IN this paper, we consider the numerical solution of the inverse problem of determining the leading coefficient $a(t)$ satisfying the equation

$$u'_t = a(t)u''_x + f(x,t) \qquad (x,t) \in [\,0,1\,] \times [\,0,T\,], \tag{1}$$

the initial condition

$$u(x,0) = h(x) \qquad x \in [\,0,1\,], \tag{2}$$

the boundary conditions

$$u(0,t) = 0 \qquad u'_x(1,t) = 0 \qquad t \in [\,0,T\,], \tag{3}$$

and the additional condition

$$u'_x(0,t) = g(t) \qquad t \in [\,0,T\,]. \tag{4}$$

Coefficient inverse problems arise in many applied areas. Unlike direct problems where the state of an object under investigation is unknown, for inverse problems, in addition to the state, certain so-called causal characteristics, including boundary conditions, initial conditions, coefficients of equations, and geometric characteristics of domains, are also unknown. In investigating inverse coefficient problems of parabolic equations, much attention is given to problems with unknown leading coefficient, which can also depend on one or two variables [1][2]. Conditions for existence and uniqueness of a solution were established in [3] in the case when the unknown coefficients are functions of the space variables. Conditions for existence and uniqueness of a solution to the inverse problem were established in [4] for a one-dimensional heat equation with unknown time-dependent leading coefficients. In [5] and [6] the authors investigate the problem of simultaneous determination of the time-dependent leading, lower coefficients, and the free term in a one-dimensional parabolic equation and establish existence of a solution over some time interval.

Shiping Zhou is with Dept. of Math., Harbin Institute of Technology, Weihai, Shandong, P. R. China 264209 and School of Computer Science and Technology, Yantai University, Yantai, Shandong, P.R.China. 264005. e-mail: shipingzh@hotmail.com.
Minggen Cui is with Dept. of Math., Harbin Institute of Technology, Weihai, Shandong, P. R. China 264209.

Many algorithms have been proposed for numerically solving inverse problems, such as GPST method[7][8], regularized nonlinear least squares and iterative methods[9][10] and convexification algorithm[11][12], etc.. However, such methods are extremely time-consuming and of some assumptions and restrictions about known conditions. In this paper, we present a new algorithm for determining the time-dependent leading coefficient in a parabolic equation. In order to solve the coefficient inverse problem, we define several reproducing kernel spaces, in which the general form of the solution $u(x,t)$ is given. The identification of the time-dependent leading coefficient $a(t)$ is solved by a lower triangular linear system. Some numerical examples are studied to demonstrate the accuracy of the present method.

II. Reproducing kernel spaces

In this section we define several reproducing kernel spaces based on smoothness requirements on the solution function u(x,t) and the given boundary value condition.

The inner product space $W_1[\,0,T\,]$ is defined as

$$W_1[\,0,T\,] = \{u(x)|u \text{ is absolutely continuous function}, \; u' \in L^2[\,0,T\,])\}.$$

endowed with the inner product

$$< u(x), v(x) >_{W_1} = u(0)v(0) + \int_0^1 u'(x)v'(x)\,dx,$$

and with the norm $\|u\|_{W_1} = \sqrt{(u,u)_{W_1}}$.

It is proved that $W_1[\,0,T\,]$ is a reproducing kernel space [14], that is, for every $u(\xi) \in W_1[\,0,T\,]$, and every fixed $t \in [\,0,T\,]$, there exists $P_t(s) \in W_1[\,0,T\,]$ such that

$$< u(t), P_t(s) >_{W_1} = u(t).$$

where

$$P_t(s) = \begin{cases} 1+s, & s \le t, \\ 1+t, & s > t. \end{cases} \tag{5}$$

$P_t(s)$ is called the reproducing kernel of $W_1[0,T]$. The following are some reproducing kernel spaces are described similar to $W_1[\,0,T\,]$.

$$W_2[\,0,T\,] = \{u(t)|u' \text{ are an absolutely continuous function}, \; u'' \in L^2[\,0,T\,]\}.$$

It's inner product and norm are defined as

$$< u(t), v(t) >_{W_2}= \sum_{i=0}^{1} u^{(i)}(a)v^{(i)}(a) + \int_0^1 u''(t)v''(t)dt,$$

$$\|u\|_{W_2} = \sqrt{(u,u)_{W_2}}.$$

$$W_3[\,0,1\,] = \{u(x)|u'' \text{ are absolutely continuous function}, \\ u''' \in L^2[\,0,1\,], u(0) = u'(1) = 0\}.$$

endowed with the inner product

$$< u(x), v(x) >_{W_3}= \sum_{i=0}^{2} u(0)v(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)\,dx,$$

and with the norm $\|u\|_{W_3} = \sqrt{< u, u >_{W_3}}$.

$$W_3^0[\,0,1\,] = \{u(x)|u \in W_3[\,0,1\,], \int_0^1 u(x) = 0\}.$$

In [15], the author gives the general method of solving reproducing kernels. We can use the method described in the book to prove that $W_2[\,0,T\,]$ and $W_3[\,0,1\,]$ are reproducing spaces and solve their reproducing kernels $R_t^{\{2\}}(s)$ and $R_t^{\{3\}}(\eta)$, respectively.

$$R_t^{\{2\}}(s) = \begin{cases} 1 - \frac{t^3}{6} + \frac{1}{2}st(2+t) & t \le s, \\ 1 - \frac{s^3}{6} + \frac{1}{2}st(2+s) & t > s, . \end{cases} \tag{6}$$

$$R_x^{\{3\}}(y) = \begin{cases} \frac{xy}{7}(y+xy) - \frac{xy}{14}(3x+3y+x^2+y^2) + \frac{x^4y}{56} \\ -\frac{x^3y^2}{28} + \frac{x^2y^2}{112} + \frac{x^2y^3}{21} - \frac{x^3y^3}{84} + \frac{x^3y^3}{336}(x+y) \\ \quad -\frac{xy^4}{42} - \frac{x^4y^4}{1344} + \frac{y^5}{120}, \qquad y \le x, \\ \frac{xy}{7}(x+xy) - \frac{xy}{14}(3x+3y+x^2+y^2) + \frac{xy^4}{56} \\ -\frac{x^2y^3}{28} + \frac{y^2y^2}{112} + \frac{x^3y^2}{21} - \frac{x^3y^3}{84} + \frac{x^3y^3}{336}(x+y) \\ \quad -\frac{x^4y}{42} - \frac{x^4y^4}{1344} + \frac{x^5}{120}, \qquad y > x. \end{cases} \tag{7}$$

$W_3^0[\,0,1\,]$ is a subspace of $W_3^0[\,0,1\,]$, and we also can solve prove that it is a reproducing kernel space and solve for its reproducing kernel. We denoted it by $R0_x^{\{3\}}(y)$.

Now we consider a reproducing kernel space $W(D)$ based on the region $D = [\,0,1\,] \times [\,0,T\,]$

$$W(D) = W_3[\,0,1\,] \otimes W_2[\,0,T\,].$$

In terms of $W(D)$ and its inner product, we have the following fact, see [13].

$$W(D) = \{u(x,t)|u(x,t) = \textstyle\sum_{i,j=1}^{\infty} c_{ij}p_i(x)q_j(t),\, c_{ij} \in l^2, \\ i,j = 1,2,\ldots n,\},$$

where $p_i(x) \in W_3[\,0,1\,], q_j(t) \in W_2[\,0,T\,]$. If

$$u(x,t) = \sum_{i,j=1}^{\infty} c_{ij}p_i(x)q_j(t),$$

$$v(x,t) = \sum_{i',j'=1}^{\infty} \bar{c}_{i'j'}p_{i'}(x)q_{j'}(t),$$

where $\{p_i(t)\}_{i=1}^{\infty}$ is the complete normal orthogonal system of $W_3[0,1]$ and $\{q_i(x)\}_{i=1}^{\infty}$ is the complete normal orthogonal system of $W_2[\,0,T\,]$. The inner product is defined as

$$< u(x,t), v(x,t) >_W= \sum_{i,j=1}^{\infty} \sum_{i',j'=1}^{\infty} c_{ij}\bar{c}_{i'j'}.$$

For the inner product of two separable functions $u(x,t) = u_1(x)u_2(t), v(x,t) = v_1(x)v_2(t) \in W(D)$, it follows that [13]

$$<u(x,t), v(x,t)>_W=<u_1(x), v_1(x)>_{W_3}<u_2(t), v_2(t)>_{W_2} .$$

$W(D)$ is a reproducing kernel space with the reproducing kernel[13]

$$K_{x,t}(\xi,\eta) = R_x(\xi)Q_t(\eta). \tag{8}$$

For every $u(x,t) \in$ W(D), the following is obvious

$$< u(\xi,\eta), R_{x,t}(\xi,\eta) >_W= u(x,t).$$

For the reproducing kernels of $W_2[\,0,T\,]$, $W_3[\,0,1\,]$ and $W_2(D)$, obviously we have the following properties:

$$R_\eta^{\{2\}}(t) = R_t^{\{2\}}(\eta), R_\xi^{\{3\}}(x) = R_x^{\{3\}}(\xi), R_{\xi,\eta}(x,t) = R_{x,t}(\xi,\eta).$$

It should be observed that any function $u(x,t) \in W(D)$ automatically satisfies the boundary conditions of (3).

III. The coefficient inverse problem in reproducing kernel spaces

In this section, we discuss the inverse problem of parabolic equation (1-4) in the reproducing kernel space $W(D)$. The inverse problem (1-4) can be reduced to solving the operator equation

$$(Lu)(t) = F(t) \tag{9}$$

with the initial condition

$$u(x,0) = h(x) \qquad h(x) \in W_3[\,0,1\,] \tag{10}$$

and additional condition

$$u'_x(0,t) = g(t) \qquad g(t) \in W_2[\,0,T\,], \tag{11}$$

where $u(x,t) \in W(D), F(t) \in W_1[\,0,T\,]$, and $L : W(D) \to W_1[\,0,T\,]$ is defined as follows:

$$(Lu)(t) = \int_0^1 u'_t(x,t)\,dx \tag{12}$$

and

$$F(t) = -a(t)g(t) + \int_0^1 f(x,t)\,dx. \tag{13}$$

It is readily to prove that L is a bounded operator from $W(D)$ to $W_1[\,0,T\,]$. It is worth noting that the boundary condition have been put into the reproducing space $W(D)$

In order to express all solutions of the operation equation (9), we discompose the space $W(D)$. For a fixed dense set $\{s_i\}_{i=1}^{\infty}$ of time interval $[\,0,T\,]$, let

$$\varphi_i(t) = R_t^{\{1\}}(s_i).$$

So from the property of $R_t^{\{1\}}(\eta)$, for every $u(t) \in W_1[\,0,T\,]$, it follows that

$$<u(t), \varphi_i(t)>_{W_1}= u(s_i) \qquad i = 1,2,\ldots \tag{14}$$

Let L^* denote the conjugate operator of L, and we introduce the following notation

$$\psi_i(x,t) = (L^*\varphi_i)(x,t) \qquad i = 1,2,\ldots$$

Lemma III.1. *$\psi_i(x,t)$ can be expressed in the form*

$$\psi_i(x,t) = \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_i}\int_0^1 R_x^{\{3\}}(\xi)\,d\xi \qquad i = 1,2,\ldots \tag{15}$$

and

$$<\psi_i(x,t),\psi_j(x,t)>_W = C_1\frac{\partial^2 R_t^{\{2\}}(\eta)}{\partial t\partial \eta}\Big|_{\substack{\eta=s_j,\\ t=s_i}} \tag{16}$$

where

$$C_1 = \int_0^1 dx\int_0^1 R_x^{\{3\}}(\xi)\,d\xi. \tag{17}$$

Proof: Since L is bounded, It is nature to expect L^* is bounded. By the properties of the reproducing kernels $R_t^{\{1\}}(\eta)$, $R_x^{\{3\}}(\xi)$, $R_t^{\{2\}}(\eta)$, and $R_{x,t}(\xi,\eta)$, we have

$$\begin{aligned}
\psi_i(x,t) &= <(\psi_i(\xi,\eta), R_{x,t}(\xi,\eta)>_W\\
&= <(L^*\varphi_i)(\xi,\eta), R_x^{\{3\}}(\xi)R_t^{\{2\}}(\eta)>_W\\
&= <\varphi_i(\cdot), L\left(R_x^{\{3\}}(\xi)R_t^{\{2\}}(\eta)\right)(\cdot)>_{W_1}\\
&= L\left(R_x^{\{3\}}(\xi)R_t^{\{2\}}(\eta)\right)(s_i)\\
&= \frac{\partial R_t^{\{2\}}(\eta)}{\partial\eta}\Big|_{\eta=s_i}\int_0^1 R_x^{\{3\}}(\xi)d\xi
\end{aligned}$$

and

$$\begin{aligned}
&<\psi_i(x,t),\psi_j(x,t)>_W\\
&=<(L^*\varphi_i)(x,t), \frac{\partial R_t^{\{2\}}(\eta)}{\partial\eta}\Big|_{\eta=s_j}\int_0^1 R_x^{\{3\}}(\xi)\,d\xi>_W\\
&=<\varphi_i(\cdot), \left(L\left[\frac{\partial R_t^{\{2\}}(\eta)}{\partial\eta}\Big|_{\eta=s_j}\int_0^1 R_x^{\{3\}}(\xi)\,d\xi\right]\right)(\cdot)>_{W_1}\\
&= L\left(\frac{\partial R_t^{\{2\}}(\eta)}{\partial\eta}\Big|_{\eta=s_j}\int_0^1 R_x^{\{3\}}(\xi)\,d\xi\right)(s_i)\\
&= \frac{\partial^2 R_t^{\{2\}}(\eta)}{\partial t\partial\eta}\Big|_{\substack{\eta=s_j\\ t=s_i}}\int_0^1 dx\int_0^1 R_x^{\{3\}}(\xi)\,d\xi\\
&= C_1\frac{\partial^2 R_t^{\{2\}}(\eta)}{\partial t\partial\eta}\Big|_{\substack{\eta=s_j\\ t=s_i}}.
\end{aligned}$$

■

Let $\{\overline{\psi}_i(x,t)\}_{i=1}^\infty$ denote an orthonormal system that derives from Gram-Schmidt orthonormalization process of $\{\psi_i(x,t)\}_{i=1}^\infty$. Therefore we can express $\overline{\psi}_i(x,t)$ in the following form:

$$\overline{\psi}_i(x,t) = \sum_{k=1}^{i}\beta_{ik}\psi_k(x,t) \qquad i = 1,2,\ldots, \tag{18}$$

where β_{ik} are coefficients of orthonomalization. Let

$$\begin{aligned}
S = span\left(\{\overline{\psi}_i(x,t)\}_{i=1}^\infty\right) &= \{u(x,t)|u(x,t)\\
&= \textstyle\sum_{i=1}^\infty c_i\overline{\psi}_i(x,t), c_i \in l^2\},
\end{aligned}$$

and $S^\perp$ denote the orthcomplement space of S in $W(D)$, so $W(D) = S\oplus S^\perp$.

Lemma III.2.

$$S^\perp = Null(L)$$

where $Null(L)$ denotes the null space of L.

Proof: For every $u(x,t)\in S^\perp$, we find

$$(Lu)(s_i) = <(Lu)(t),\varphi_i(t)>_{W_1} = <u(x,t),\psi_i(x,t)>_W = 0 \quad i = 1,2,\ldots.$$

Since $\{s_i\}_{i=1}^\infty$ is dense in the time interval $[0,T]$, then it means that

$$(Lu)(t) = 0$$

for arbitrary $t\in[0,T]$. That proved $u(x,t)\in Null(L)$. On the other hand, if $u(x,t)\in Null(L)$, that is, it satisfy $(Lu)(t) = 0$, we can conclude that

$$<u(x,t),\psi_i(x,t)>_W = ((Lu)(s_i)) = 0.$$

Thus $u(x,t)\in S^\perp$. ■

Let

$$\rho_i(x,t) = R0_x^{\{3\}}(x_i)R_t^{\{2\}}(t_i).$$

Since $R0_x^{\{3\}}(x_i)$ and $R_t^{\{2\}}(t_i)$ are the reproducing kernel of $W_3^0[0,1]$, $W_2[0,T]$, respectively, $\rho_i(x,t)$ is the kernel of $S^\perp$

Lemma III.3. *$\{\psi_i(x,t)\}_{i=1}^\infty$ is a complete system of S, and $\{\rho_i(x,t)\}_{i=1}^\infty$ is a complete system of $S^\perp$.*

Proof: According to the definition of S, The first part of Lemma can be proved. As to the second part, if $u(x,t)\in S^\perp$ and $<u(x,t),\rho_i(x,t)>= 0$ holds, then

$$u(x,t) = \sum_{k,l=1}^\infty c_{kl}p_k(x)q_l(t),$$

where $\{p_k(x)\}_{k=1}^\infty$ is the complete normal orthogonal system of $W_3^0[0,1]$ and $\{q_l(t)\}_{l=1}^\infty$ is the complete normal orthogonal system of $W_2[0,T]$. By **(??)**,**(??)** and the definitions of $\rho_i(x,t)$, we have

$$\begin{aligned}
&<\textstyle\sum_{k,l=1}^\infty c_{kl}p_k(x)q_l(t),\rho_i(x,t)>_W\\
&=\textstyle\sum_{k,l=1}^\infty c_{kl}<p_k(x),R_x^{\{3\}}(x_i)>_{W_3}<q_l(t),R_t^{\{2\}}(t_i)>_{W_2} = 0.
\end{aligned}$$

So

$$u(x_i,t_j) = \sum_{k,l=1}^\infty c_{kl}p_k(x_i)q_l(t_j) = 0$$

Since $\{(x_i,t_i)\}_i^\infty$ is dense in the domain of D, we can conclude that $u(x,t) = 0$. ■

The orthonormal system $\{\overline{\rho}_i(x,t)\}_{i=1}^\infty$ can be derived from the Gram-Schmidt orthonormalization process of $\{\rho_i(x,t)\}_{i=1}^\infty$, We can infer that $\{\overline{\rho}_i(x,t)\}_{i=1}^\infty$ also constitutes a complete system of $S^\perp$.

Lemma III.4.

$$<\frac{\partial\psi_i(0,t)}{\partial x},\frac{\partial\psi_j(0,t)}{\partial x}>_{W_2} = \frac{C_2^2}{C_1}<\psi_i(x,t),\psi_j(x,t)>_W \tag{19}$$

and

$$<\frac{\partial\overline{\psi}_i(0,t)}{\partial x},\frac{\partial\overline{\psi}_j(0,t)}{\partial x}>_{W_2} = \begin{cases} 0 & i\neq j\\ \frac{C_2^2}{C_1} & i = j,\end{cases} \tag{20}$$

International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:3, No:7, 2009

where C_1 is defined in (17) *and*

$$C_2 = \int_0^1 \frac{\partial R_x^{\{3\}}(\xi)}{\partial x}|_{x=0}\, d\xi. \tag{21}$$

Proof: Note that

$$\begin{aligned}
&< \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_i}, \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_j} >_{W_2} \\
&= \; < \frac{\partial R_t^{\{2\}}(\eta_1)}{\partial \eta_1}\Big|_{\eta_1=s_i}, \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_j} >_{W_2} \\
&= \; \frac{\partial}{\partial \eta_1} < R_t^{\{2\}}(\eta_1), \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_j} >_{W_2} \Big|_{\eta_1=s_i} \\
&= \; \frac{\partial^2 R_{\eta_1}^{\{2\}}(\eta)}{\partial \eta_1 \partial \eta}\Big|_{\substack{\eta=s_j \\ \eta_1=s_i}} = \frac{\partial^2 R_t^{\{2\}}(\eta)}{\partial t \partial \eta}\Big|_{\substack{\eta=s_j \\ t=s_i}}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&< \frac{\partial \psi_i(0,t)}{\partial x}, \frac{\partial \psi_j(0,t)}{\partial x} >_{W_2} \\
&= C_2^2 < \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_i}, \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_j} >_{W_2} \\
&= C_2^2 \frac{\partial^2 R_t^{\{2\}}(\eta)}{\partial t \partial \eta}\Big|_{\substack{\eta=s_j \\ t=s_i}}.
\end{aligned} \tag{23}$$

From (16), (22) and (23) we can conclude (19). Further (20) is a nature result of (19) and the orthonormality of $\{\overline{\psi}_i(x,t)\}_{i=1}^{\infty}$. ■

IV. Implementation of the numerical procedure

In this section the solution $u(x,t)$ of (9) is expressed in the form of series and a numerical procedure for solving the time-depedent leading coefficient $a(t)$ is discussed.

Theorem IV.1. *The solution $u(x,t)$ of* (9) *can be expressed in the following form*

$$\begin{aligned}
&u(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left(-a(s_k)g(s_k) + \int_0^1 f(\xi, s_k) d\xi\right)\overline{\psi}_i(x,t) \\
&+ \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t),
\end{aligned} \tag{24}$$

where $g(t)$ and $f(x,t)$ are given by (1-4), $\overline{\rho}_i(x,t)$ and β_{ik} are defined in Section 3, and $a(s_k)$ and α_i are to be solved for according to the initial and additional conditions.

Proof: From the definition of $\overline{\psi}_i(x,t)$ and $\overline{\rho}_i(x,t)$, we have

$$\begin{aligned}
&u(x,t) \\
&= \sum_{i=1}^{\infty} < u, \overline{\psi}_i >_W \overline{\psi}_i(x,t) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < u, \psi_k >_W \overline{\psi}_i(x,t) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} < Lu, \varphi_k >_W \overline{\psi}_i(x,t) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t),
\end{aligned}$$

where α_i are unknown coefficients. According to (9) and (14), we then get

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(s_k)\overline{\psi}_i(x,t) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t). \tag{25}$$

From (13) and (25) we can obtain (24). ■

In terms of unknowns α_i, we can get them by applying the initial condition (2). From (13), (15), and (18), (25) can be written in the following form as

$$\begin{aligned}
&u(x,t) \\
&= \int_0^1 R_x^{\{3\}}(\xi)\, d\xi \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} \beta_{ik} F(s_k)\right)\left(\sum_{k=1}^{i} \beta_{ik} \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}\Big|_{\eta=s_k}\right) \\
&+ \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t).
\end{aligned}$$

For convenience, we denote $\int_0^1 R_x^{\{3\}}(\xi)\, d\xi$ by $M(x)$ and $\sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} \beta_{ik} F(s_k)\right)\left(\sum_{k=1}^{i} \beta_{ik} \frac{\partial R_t^{\{2\}}(\eta)}{\partial \eta}|_{\eta=s_k}\right)$ by $N(t)$, thus

$$u(x,t) = M(x)N(t) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,t). \tag{26}$$

Setting $t = 0$ in (26), and applying the initial condition (1.2) gives us

$$h(x) = M(x)N(0) + \sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,0). \tag{27}$$

Integrating both sides of (27) and Noticing the fact that $\rho_i(x,0) \in W_3^0[0,1]$, that is, $\int_0^1 \rho_i(x) = 0$. Further we have $\int_0^1 \overline{\rho}_i(x) = 0$. So

$$N(0) = \int_0^1 M(x)\, dx \Big/ \int_0^1 h(x)\, dx$$

We denote $N(0)$ by a constant C_3, so

$$\sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x,0) = h(x) - C_3 M(x). \tag{28}$$

Taking $x_k \in [0,T], k = 1,2,...$, we get the infinite linear system about α_i:

$$\sum_{i=1}^{\infty} \alpha_i \overline{\rho}_i(x_k,0) = h(x_k) - C_3 M(x_k).$$

Theorem IV.2. *The coefficients $a(s_k)$ can be solved for by the following lower triangular system of equations*

$$\begin{aligned}
&\tfrac{C_2}{C_1}\textstyle\sum_{k=1}^{j} \beta_{jk}\left(-a(s_k)g(s_k) - \int_0^1 f(\xi, s_k)\, d\xi\right) \\
&= < g(t), \tfrac{\partial \overline{\psi}_j(0,t)}{\partial x} >_{W_2} - < \tfrac{\partial \overline{\rho}_j(0,t)}{\partial x}, \tfrac{\partial \overline{\psi}_j(0,t)}{\partial x} >_{W_2} \\
&\qquad j = 1, 2, \ldots,
\end{aligned} \tag{29}$$

where $g(t), h(x)$ and $f(x,t)$ are given by (1-4), C_1, C_2 are given by (17) *and* (21)*, and*

$$C_3 = \int_0^1 \frac{\partial R_x^{\{3\}}(\xi)}{\partial x}\, d\xi\Big|_{x=0}. \tag{30}$$

Proof: Differentiating with respect to x in (24) and applying the additional specification (4), we have

$$g(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left(-a(s_k)g(s_k) + \int_0^1 f(\xi, s_k)\,d\xi\right)\frac{\partial\overline{\psi}_i(0,t)}{\partial x} + \sum_{i=1}^{\infty}\alpha_i\frac{\partial\overline{\rho}_i(0,t)}{\partial x}. \quad (31)$$

Making inner product with $\frac{\partial\overline{\psi}_j(0,t)}{\partial x}$ on both sides of (31) and applying Lemma III.4, the lower triangular system of equation has been built. ■

V. Numerical examples

In this section, we present some results of numerical experiments using the numerical procedure described above. The following is a parabolic equation with initial, boundary, and additional conditions.

$$\begin{cases} u'_t = a(t)u''_x + f(x,t) \\ u(x,0) = h(x) \\ u(0,t) = 0 \qquad u'_x(1,t) = 0 \\ u'_x(0,t) = g(t) \qquad (x,t) \in [0,1] \times [0,T], \end{cases}$$

where $h(x) = (x^2 - 2x)$, $g(t) = -2e^{-t}$, and $f(x,t) = e^{-t}(20t + x^2 - 2x)$. The true coefficient $a(t) = 10t$, and the true solution of the parabolic equation $u(x,t) = (x^2 - 2x)e^{-t}$. Results of determination of the leading coefficient $a(t)$ illustrated in Tables 1 and 2 are obtained by truncating the two series in (24). The second example have been done to control the sensitivity of method to errors. Artificial errors 10^{-4} were introduced into the right end and conditional condition. As seen from table 2 that the error almost never affects the results of the method. The method of solving the problem was tried on different tests and the results we observed indicate that the method is stable and gives excellent approximation to the solution.

TABLE I: THE ERROR OF COEFFICIENT $a(t)$

t	true solution	approximate solution	absolute error	relative error
0.1	1	1.0137048	0.0137048	0.0135195
0.2	2	2.0074319	0.0074319	0.0037022
0.3	3	2.9998994	0.0001006	0.00003354
0.4	4	3.9912472	0.0087528	0.00219299
0.5	5	4.9826268	0.0173732	0.00348675
0.6	6	5.9743981	0.0256019	0.00428527
0.7	7	6.9668120	0.0331881	0.00476374
0.8	8	7.9588223	0.0411777	0.00517385
0.9	9	8.9499818	0.0500182	0.00558863
1	10	9.9402011	0.0597989	0.00601586

TABLE II: THE ERROR OF COEFFICIENT $a(t)$
(with artificial $10^{(-4)}$ to the right end and $g(t)$)

t	true solution	approximate solution	absolute error	relative error
0.1	1	1.0136403	0.0136403	0.0134568
0.2	2	2.0072072	0.0072072	0.0035907
0.3	3	2.9995021	0.0004979	0.0001660
0.4	4	3.9906627	0.0093373	0.0023398
0.5	5	4.9818380	0.0181620	0.0036456
0.6	6	5.9733852	0.0266148	0.0044556
0.7	7	6.9655522	0.0344478	0.0049455
0.8	8	7.9572898	0.0427102	0.0053674
0.9	9	8.9481470	0.0518530	0.0057948
1	10	9.93802995	0.0619701	0.0062356

VI. Conclusions

In this paper, we consider solving one-dimensional inverse parabolic problem. We presented a stable numerical algorithm for identifying the time-dependent leading coefficient in a parabolic equation. Numerical results show that the proposed method is effective. It will be very interesting to expand our work to higher dimensional cases.

References

[1] M.I.Ivanchov, *Inverse problem of heat condition with nonlocal conditions*, Dop.Nats.Akad.Nauk Ukrainy,NO.5,15-21, 1997
[2] A.I.Prilepko and A.B. Kostin, *On inverse problems of determination of a coefficients in a parabolic equation*, I, Sib. Mat. Zh.33,No.3, 146-155(1992)
[3] Akhundov A. Ya., *An inverse problem for linear parabolic equations*, Dokl. Akad. Nauk AzSSR, 39, No. 5, 3-6 (1983).
[4] Ivanchov N. I., *On the inverse problem of simultaneous determination of thermal conductivity and specific heat capacity*, Sibirsk. Mat. Zh., 35, No. 3, 612-621 (1994).
[5] N.I.Ivanchov and N. V. Pabyrivska, *On determination of two time-dependent coefficients in a parabolic equation*, Siberian Mathematical Journal, Vol. 43, No. 2, pp. 323-329, 2002
[6] I. B. Bereznyts'ka, *Determination of the free term and leading coefficient in a parabolic equation*, Ukrainian Mathematical Journal, Vol. 55, No. 1, 2003
[7] Y.M.Chen, Generalized Pulse-Spectrum Technique, Geophysics, 50(1985)1664-1675
[8] B.Han, M.L.Zhang, and J,Q.Liu, *A widely convergent Genevalized Pulse-Spectrum Technique for the coefficient converse problem of Differential equations*, Applied Mathematics and Computation, 81(1997)97-112 12
[9] A.B.Bakushinsky, A.V.Goncharsky, *Ill-posed problem:Theory and Applications*, Kluwer Academic, Dordrecht,1994
[10] A.N.Tikhonov,A.S.Leonov,A.G.Yagola, *Nonlinear Ill-posed Problems*, Chapman and Hall, London, 1998
[11] M.V.Klibanov, A.Timonov, *A new slant on the inverse problems of electromagnetic frequency sounding:'convexification' of a multi-extremal objective function via the CarlemanWeight functions*, Inverse Problem,17(2001)1865-1887
[12] M.V.Klibanov, A.Timonov, *A globally convergent convexification algorithm for the inverse problem of electromagnetic frequency sounding in one dimension*, Numer. Methods Programming, 4(2003)52-81
[13] N. Aronszajn, *Theory of reproducing kernels*, Trans. A.M.S.,68,1950:337-404
[14] Zhong Chen, YingZhen Lin, *The exact solution of a linear integral equation with weakly singular kernel*, J. Math. Anal. Appl. 344:726-734(2008).
[15] MingGen Cui, Yingzhen Lin, *Nonlinear numercial Analysis in the Reproducing kernel space*, Nova Science Publisher, New York, 2008.