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A neighborhood condition for fractional k-deleted graphs

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Abstract—Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$. Then a spanning subgraph F of G is called a k-factor if $d_F(x)=k$ for each $x \in V(G)$. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. In this paper, it is proved that G is a fractional k-deleted graph if G satisfies $\delta(G) \geq k+1$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G.

Keywords—graph, minimum degree, neighborhood union, fractional k-factor, fractional k-deleted graph.

I. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G, and $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, and $G-S = G[V(G) \setminus S]$. We say that S is independent if $N_G(S) \cap S = \emptyset$. Let S and T be disjoint subsets of $V(G)$. We use $e_G(S,T)$ to denote the number of edges joining S and T in G. The minimum vertex degree of G is denoted by $\delta(G)$.

Let k be a positive integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x)=k$ for each $x \in V(G)$. If $k=1$, then a k-factor is simply called a 1-factor. A fractional k-factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k. If $k=1$, then a fractional k-factor is a fractional 1-factor. A graph G is a fractional k-deleted graph if there exists a fractional k-factor after deleting any edge of G. If $k=1$, then a fractional k-deleted graph is a fractional 1-deleted graph. Some other terminologies and notations can be found in [1,2].

Many authors have studied graph factors [3-8]. Many authors have investigated fractional k-factors [9–12] and fractional k-deleted graphs [13,14]. The following results on k-factors, fractional k-factors and fractional k-deleted graphs are known.

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Theorem 1[15] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, kn is even, and the minimum degree is at least k. If G satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k-factor.

Theorem 2[11] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k-1-4\sqrt{2(k-1)^2+2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k-factor.

Theorem 3[16] Let $k \geq 2$ be an integer. Let G be a connected graph of order n with $n \geq 13k+1-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+2$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$ for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

The purpose of this paper is to weaken the conditions on the order, minimum degree and connectivity of G in Theorem 3. The main result is the following theorem.

Theorem 4 Let $k \geq 3$ be an integer. Let G be a graph of order n with $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $\delta(G) \geq k+1$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n+k-2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional k-deleted graph.

II. The Proof of Theorem 4

The following result is essential to the proof of our main theorem.

Lemma 2.1[17] A graph G is a fractional k-deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S,T),$$

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $\varepsilon(S,T)$ is defined as follows,

$$\varepsilon(S,T) = \begin{cases} 2, & if\ T\ is\ not\ independent, \\ 1, & if\ T\ is\ independent,\ and \\ & e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 4. Let G be a graph satisfying the hypothesis of Theorem 4, we prove the theorem by contradiction.

Suppose that G is not a fractional k-deleted graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S,T) - 1, \quad (1)$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$. Firstly, we prove the following claims.

Claim 1. $S \neq \emptyset$.

Proof. Note that $\varepsilon(S,T) \leq |T|$. If $S = \emptyset$, then by (1) we have

$$\begin{aligned}\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq (\delta(G) - k)|T| \\ &\geq |T| \geq \varepsilon(S,T).\end{aligned}$$

It is a contradiction. This completes the proof of Claim 1.

Claim 2. $|T| \geq k+1$.

Proof. Assume that $|T| \leq k$. Then from (1) and $|S| + d_{G-S}(x) - k \geq d_G(x) - k \geq \delta(G) - k \geq 1$, we get

$$\begin{aligned}\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x\in T}(|S| + d_{G-S}(x) - k) \\ &\geq |T| \geq \varepsilon(S,T).\end{aligned}$$

That is a contradiction. This completes the proof of Claim 2.

Claim 3. $|T| \geq |S| + 1$.

Proof. Let $|T| \leq |S|$. Then by (1), we obtain

$$\varepsilon(S,T) - 1 \geq k|S| + d_{G-S}(T) - k|T| \geq d_{G-S}(T). \quad (2)$$

On the other hand, according to the definition of $\varepsilon(S,T)$, we have

$$d_{G-S}(T) \geq \varepsilon(S,T),$$

which contradicts (2). The proof of Claim 3 is complete.

Claim 4. $|S| \leq \frac{n-1}{2}$.

Proof. In terms of Claim 3 and $|S| + |T| \leq n$, we have

$$n \geq |S| + |T| \geq 2|S| + 1,$$

that is,

$$|S| \leq \frac{n-1}{2}.$$

The proof of Claim 4 is complete.

In terms of Claim 2, $T \neq \emptyset$. Now we define

$$h_1 = \min\{d_{G-S}(x) : x \in T\}$$

and choose $x_1 \in T$ such that $d_{G-S}(x_1) = h_1$. Clearly, we have $0 \leq h_1 \leq k$. In the following, we consider two cases.

Case 1. $T = N_T[x_1]$.

Using Claim 2, $T = N_T[x_1]$ and $0 \leq h_1 \leq k$, we obtain

$$k \geq h_1 = d_{G-S}(x_1) \geq |T| - 1 \geq k,$$

which implies

$$h_1 = k. \quad (3)$$

In terms of (3) and Claim 1, we get

$$\begin{aligned}\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| \\ &\geq k > 2 \geq \varepsilon(S,T).\end{aligned}$$

That contradicts (1).

Case 2. $T \setminus N_T[x_1] \neq \emptyset$.

Define

$$h_2 = \min\{d_{G-S}(x) : x \in T \setminus N_T[x_1]\}.$$

We choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Obviously, $0 \leq h_1 \leq h_2 \leq k$ and $x_1x_2 \notin E(G)$. According to the hypothesis of Theorem 4, we have

$$\begin{aligned}\frac{n+k-2}{2} &\leq |N_G(x_1) \cup N_G(x_2)| \\ &\leq d_{G-S}(x_1) + d_{G-S}(x_2) + |S| \\ &= h_1 + h_2 + |S|,\end{aligned}$$

which implies

$$|S| \geq \frac{n+k-2}{2} - h_1 - h_2. \quad (4)$$

By (4) and Claim 4, we obtain

$$\frac{n-1}{2} \geq \frac{n+k-2}{2} - h_1 - h_2,$$

that is,

$$h_1 + h_2 \geq \frac{k-1}{2}. \quad (5)$$

In terms of (5), $k \geq 3$, $0 \leq h_1 \leq h_2 \leq k$ and the integrity of h_2, we get

$$h_2 \geq 1. \quad (6)$$

Claim 5. $0 \leq h_1 \leq k-1$.

Proof. If $h_1 = k$, then by (1) and Claim 1 we get

$$\begin{aligned}\varepsilon(S,T) - 1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| \geq k \\ &> 2 \geq \varepsilon(S,T),\end{aligned}$$

which is a contradiction. This completes the proof of Claim 5.

Note that

$$|N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + 1. \quad (7)$$

From (4), (7), $0 \leq h_1 \leq h_2 \leq k$ and $|S| + |T| \leq n$, we have

$$\begin{aligned}\delta_G(S,T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) \\ &\quad -k|T| \\ &= k|S| - (h_2 - h_1)|N_T[x_1]| - (k - h_2)|T| \\ &\geq k|S| - (h_2 - h_1)(h_1 + 1) \\ &\quad -(k - h_2)(n - |S|) \\ &= (2k - h_2)|S| - (h_2 - h_1)(h_1 + 1) \\ &\quad -(k - h_2)n \\ &\geq (2k - h_2)(\frac{n+k-2}{2} - h_1 - h_2) \\ &\quad -(h_2 - h_1)(h_1 + 1) - (k - h_2)n,\end{aligned}$$

that is,

$$\delta_G(S,T) \geq (2k-h_2)(\frac{n+k-2}{2} - h_1 - h_2) -(h_2-h_1)(h_1+1)-(k-h_2)n. \quad (8)$$

Let $F(h_1,h_2) = (2k-h_2)(\frac{n+k-2}{2} - h_1 - h_2) - (h_2 - h_1)(h_1+1)-(k-h_2)n$. Then by Claim 5, we have

$$\begin{aligned} F'_{h_1}(h_1,h_2) &= -(2k-h_2)+(h_1+1)-(h_2-h_1) \\ &= 2h_1-2k+1 \leq 2(k-1)-2k+1 \\ &= -1<0. \end{aligned}$$

Combining this with $h_1 \leq h_2$, we obtain

$$F(h_1,h_2) \geq F(h_2,h_2). \quad (9)$$

Using (8) and (9), we get

$$\delta_G(S,T) \geq (2k-h_2)(\frac{n+k-2}{2} - 2h_2) - (k-h_2)n. \quad (10)$$

According to (1), (10) and $\varepsilon(S,T) \leq 2$, we get

$$\begin{aligned} 1 &\geq \varepsilon(S,T)-1 \geq \delta_G(S,T) \\ &\geq (2k-h_2)(\frac{n+k-2}{2} - 2h_2)-(k-h_2)n \\ &= \frac{1}{2}(4h_2^2+(n-9k+2)h_2+2k^2-4k), \end{aligned}$$

which implies

$$4h_2^2+(n-9k+2)h_2+2k^2-4k-2 \leq 0. \quad (11)$$

Claim 6. For $k \geq 3$, we have $\sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}$.
Proof. Since $k \geq 3$, we have

$$\frac{(k-1)^2+1}{2} \geq \frac{5}{2} > \frac{9}{4},$$

that is,

$$\sqrt{\frac{(k-1)^2+1}{2}} > \frac{3}{2}.$$

Thus, we obtain

$$\sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}.$$

The proof of Claim 6 is complete.

According to (6), (11), $n \geq 9k+3-4\sqrt{2(k-1)^2+2}$, $k \geq 3$ and Claim 6, we obtain

$$\begin{aligned} 0 &\geq 4h_2^2+(n-9k+2)h_2+2k^2-4k-2 \\ &\geq 4h_2^2+(-4\sqrt{2(k-1)^2+2}+5)h_2+2k^2-4k-2 \\ &\geq 4h_2^2-8\sqrt{\frac{(k-1)^2+1}{2}}h_2+2(k-1)^2+2+5h_2-6 \\ &= 4(\sqrt{\frac{(k-1)^2+1}{2}}-h_2)^2+5h_2-6 \\ &\geq 4(\sqrt{\frac{(k-1)^2+1}{2}}-1)^2-1 \\ &> 4(\frac{1}{2})^2-1 \geq 0, \end{aligned}$$

which is a contradiction.

From all the cases above, we deduce the contradictions. Hence, G is a fractional k-deleted graph. This completes the proof of Theorem 4.

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