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A Homogenisation Procedure for the Free Vibration Analysis of Functionally Graded Beams at Large Vibration Amplitudes

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***Abstract*—**The purpose of the present paper is to show that the problem of geometrically nonlinear free vibrations of functionally graded beams (FGB) with immovable ends can be reduced to that of isotropic homogeneous beams with effective bending stiffness and axial stiffness parameters by using an homogenization procedure. The material properties of the functionally graded composites examined are assumed to be graded in the thickness direction and estimated through the rule of mixture. The theoretical model is based on the Euler-Bernouilli beam theory and the Von Kármán geometrical nonlinearity assumptions. Hamilton's principle is applied and a multimode approach is derived to calculate the fundamental nonlinear frequency parameters, which are found to be in a good agreement with the published results. The non-dimensional curvatures associated to the nonlinear fundamental mode are also given for various vibration amplitudes in the case of clamped-clamped FGB.

***Keywords*—**Nonlinear vibrations, functionally graded materials, homogenization procedure.

I. Introduction

RECENTLY, the developments in materials engineering have led to consideration of special composites called "functionally graded materials" (FGMs). They are characterized by smooth and continuous variations in their thermomechanical properties along the thickness of a structure obtained by gradually varying the volume fraction of the constituent materials. FGMs possess various advantages over the conventional composite laminates, such as smaller thermal stresses and stress concentrations and they can be designed to achieve specific properties for different applications. Indeed, FGMs are generally made of a mixture of ceramic and metal to satisfy the demand of ultra-high-temperature environment and to eliminate the interface problems. Hence, this new kind of materials has been employed in the design of many engineering structures such as aircrafts, space vehicles, defense industries, electronics and biomedical equipments.

Because of the wide applications of FGMs, it is important to study the dynamic behavior of FGM structures such as beams which are used extensively as structural members and often subjected to relatively large vibration amplitudes, compared to their thickness. Consequently, many structural problems are encountered in these severe work conditions due to material fatigue which is accelerated leading to structural damages and catastrophic failures. These phenomena are more significant around the natural frequencies of the structure, for which the nonlinear vibration analysis becomes essential for a reliable structural design.

Recently, the dynamic response of FG beams has been the purpose of many research works. In [1], fundamental frequency analysis of FG beams having different boundary conditions has been made within the framework of the classical, the first-order and different higher-order shear deformation beam theories. Exact solutions for the bending vibration problem of FG beams with variation of material properties in a one-layer beam or in the layers of a multilayered sandwich beam have been proposed in [2]. The linear beam theory has been used for establishing the equilibrium and kinematical equations, taking into account the effect of the shear deformation and the effect of consistent mass distribution and mass inertia moment. In [3], the dynamic characteristics of a functionally graded beam with axially or transversally material graduation through the thickness have been presented. The finite element method has been employed under the assumptions of the Euler–Bernoulli beam theory to develop the discretized model and obtain a numerical approximation of the motion equation. A mixed method for forced vibration of functionally graded beams subjected to moving loads has been developed in [4]. The theoretical formulation is based on the Euler–Bernoulli beam theory, and the governing equations of motion of the system have been derived using Lagrange's equations. Then, the Rayleigh–Ritz method has been employed to discretize the spatial partial derivatives and the differential quadrature method has been used for the discretization of the temporal derivatives. Static and free vibration analysis of functionally graded layered beams have been conducted in [5] using a third order zigzag theory based model. Two systems, Al/SiC and Ni/Al2O3, fabricated using powder metallurgy and thermal spraying techniques respectively, have been considered for the experimental validation. The detection of cracks in structural members made of a functionally graded material has been a significant subject due to their increasing applications in various important engineering industries. A model-based approach has been developed in [6] to determine the location and size of an open edge crack in an FGM beam. The p-

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version of finite element method has been employed to estimate the transverse vibration characteristics of a cracked FG beam. In [7], approximate analytical expressions have been presented for geometrically nonlinear vibration and post-buckling analysis of FG beams on a nonlinear elastic foundation subjected to an axial force. The Euler-Bernoulli assumptions together with The Von Kaman's strain displacement relation have been employed to derive the governing partial differential equation of motion, the analytical solution of which has been obtained using one parameter Galerkin solution in conjunction with a variational approach.

In the present paper, the problem of geometrically nonlinear free and steady state periodic forced vibrations of clamped-clamped FG beams with immovable ends is investigated using Hamilton's principle and spectral analysis. Based on the governing axial equation of the beam in which the axial inertia and damping are ignored, a homogenisation procedure is proposed which reduces the problem studied to that of isotropic homogeneous beams with effective bending stiffness and axial stiffness parameters.

II. Modeling Functionally Graded Materials

In this section, we consider a clamped-clamped FG beam having the geometrical characteristics shown in Fig. 1. It is assumed that the FG beam is made of ceramic and metal, and the effective material properties of the FG beam, i.e., Young's modulus E and mass densityρ, are functionally graded in the thickness direction according to a function of the volume fractions V of the constituents.

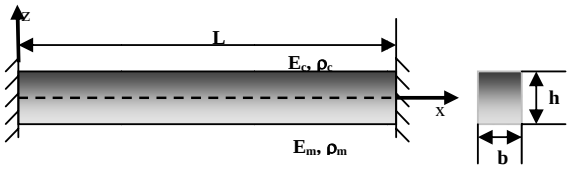


Fig. 1 FG beam notation

According to the rule of mixture, the effective material properties P, can be expressed as:

$$P = P_m V_m + P_c V_c \tag{1}$$

where subscripts "m" and "c" refer to the metal and ceramic constituents, respectively.

Various types of functions are used in the literature to describe the variation of the volume fraction of the constituents. Here, a simple power law is considered to describe the variation of the volume fraction of the metal and the ceramic constituents as follows:

$$V_m = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{2}$$

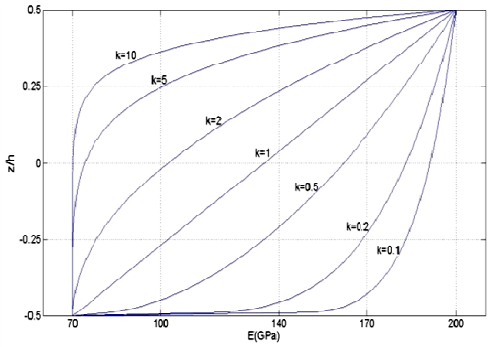
with $V_m + V_c = 1$

where k is a non-negative parameter (power-law exponent) which dictates the material variation profile through the thickness of the beam.

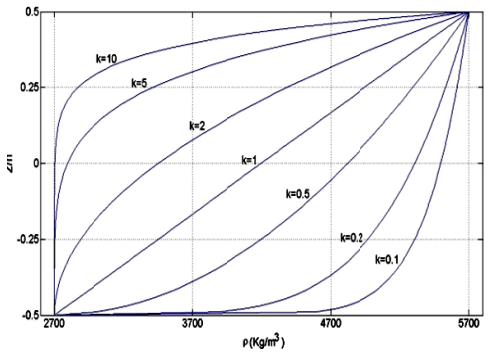
Effective material properties of the FG beam such as Young's modulus (E) and mass density (ρ) can be determined by substituting (2) into (1), which gives:

$$E(z) = E_m + (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{3}$$

$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{4}$$



(a)



(b)

Fig. 2 Variation of the material properties of the FG beam through the thickness: (a) Young Modulus, (b) mass density

III. Nonlinear Vibration Analysis

The Von Kármán type nonlinear strain–displacement relationships are given by:

$$\varepsilon_x^a = \frac{\partial u^a}{\partial x} + \frac{1}{2}\left(\frac{\partial W^a}{\partial x}\right)^2 \tag{5}$$

$$K_x = \frac{\partial^2 W^a}{\partial x^2} \quad (6)$$

where ε_x^a is the nonlinear axial strain, K_x is the curvature of the beam. The total elastic strain energy V_T of Euler-Bernoulli beams is given by:

$$V_T = \frac{1}{2}\int_0^i N_x \varepsilon_x^a + M_y K_x dx \quad (7)$$

in which N_x and M_y are the axial internal force and the bending moment respectively, which are related to the strains as follows:

$$N_x = bA_{11}\varepsilon_x^a + bB_{11}K_x \quad (8)$$

$$M_y = bB_{11}\varepsilon_x^a + bD_{11}K_x \quad (9)$$

where A_{11}, B_{11} and D_{11} are the extension-extension, bending-extension, bending-bending coupling coefficients respectively, and can be evaluated using the classical laminate plate theory.

By considering the following change of variable $z' = z - d$, with $d = B_{11}/A_{11}$, then B_{11} vanishes and D_{11} becomes $D_{11} - (B_{11}^2/A_{11})$, whereas A_{11} doesn't change in this new coordinate system. Consequently, the total strain energy takes the following new expression, involving the effective bending stiffness: $D_{eff} = D_{11} - (B_{11}^2/A_{11})$ and the effective axial stiffness $A_{eff} = A_{11}/4$

$$V_T = \frac{(ES)_{eff}}{8l}\left(\int_0^l \left(\frac{\partial W^a}{\partial x}\right)^2 dx\right)^2 + \frac{(Ei)_{eff}}{2}\int_0^l \left(\frac{\partial^2 W^a}{\partial x^2}\right)^2 dx \quad (10)$$

where $(ES)_{eff} = bA_{11}$ and $(EI)_{eff} = b(D_{11} - (B_{11}^2/A_{11}))$

Expression (10) for the total strain energy is the basis of the proposed homogenisation procedure for replacing the FGM beam problem with an equivalent classical isotropic beam problem [8].

The kinetic energy, in which the axial and rotary inertia are ignored, is given by:

$$T = \frac{1}{2}\int_{-h/2}^{h/2} \rho(z) dz \int_0^l \left(\frac{\partial W^a}{\partial t}\right)^2 dx \quad (11)$$

For a general parametric study, we use the following non dimensional formulation by putting:

$$x^* = \frac{x}{l},\ W^{a^*} = \frac{W^a}{r},\ \text{where}\ r = \sqrt{\frac{I}{S}}$$

Applying Hamilton's principle and expanding the displacement W^a in the form of a finite series, the following set of nonlinear amplitude equations is obtained:

$$a_i k_{ir}^* + a_i a_j a_k b_{jkir}^* - \omega^{*2} a_i m_{ir}^* = 0, \quad i = 1,...,n \quad (12)$$

where k_{ij}^*, b_{ijkl}^* and m_{ij}^* stand for the non dimensional classical rigidity tensor, the nonlinear rigidity tensor and the mass tensor, respectively, which are defined as:

$$k_{ij}^* = \int_0^i \frac{\partial^2 w_i^*}{\partial x^{*2}} \frac{\partial^2 w_j^*}{\partial x^{*2}} dx^* \quad (13)$$

$$b_{ijkl}^* = \alpha \int_0^i \frac{\partial w_i^*}{\partial x^*} \frac{\partial w_j^*}{\partial x^*} dx^* \int_0^i \frac{\partial w_k^*}{\partial x^*} \frac{\partial w_l^*}{\partial x^*} dx^* \quad (14)$$

where α is given by: $\alpha = \frac{h^2}{4}\frac{(ES)_{eff}}{(EI)_{eff}}$

$$m_{ij}^* = \int_0^i w_i^* w_j^* dx^* \quad (15)$$

To obtain the nonlinear vibration response of a clamped-clamped FG beam in the neighborhood of its first resonant frequency, the values of the linear rigidity matrix k_{ij}^* and the nonlinear geometrical rigidity tensor b_{ijkl}^* have been calculated using the first six normalized symmetric linear beam function w_1^*, w_3^*, …, w_{11}^*. The functions w_i^* have been normalized in such a manner that the obtained mass matrix equals the identity matrix.

IV. Numerical Results and Discussions

In the present work, the functionally graded material of the beam is that considered in reference [7]. The top surface of the FG beam is ceramic rich (E_c=322.03 GPa, ρ_c =2370 Kg/m^3), whereas the bottom surface of the FG beam is metal rich (E_m=207.08 GPa, ρ_m =8166 Kg/m^3).

In Table I, It is noted that the first nonlinear frequency ratios ω_{nl}/ω_l, calculated in the present work at various vibration amplitudes in the case of an isotropic clamped-clamped beam, agree very well with the results obtained in [8], since the percentage error does not exceed 0.3%. However, the solutions given in [7] overestimate the frequencies of the clamped beams, especially for high values of dimensionless amplitude.

TABLE I
NONLINEAR TO LINEAR FREQUENCY RATIOS ω^*_{NL}/ω^*_L OF AN ISOTROPIC CLAMPED-CLAMPED BEAM AT VARIOUS VIBRATION AMPLITUDES

$W^{a^*}(x^*=0.5)$	Present	[8]	[7]
1	1.0225	1.0221	1.0552
2	1.0870	1.0856	1.2056
3	1.1869	1.1831	1.4214
4	1.3105	1.3064	1.6776

In Table II, the fundamental nonlinear to linear Frequency ratios ω^*_{nl}/ω^*_l of the FG clamped-clamped beam, considered in the present numerical simulation, are given and compared with the published results in [7] for various vibration amplitudes.

TABLE II
NONLINEAR TO LINEAR FREQUENCY RATIOS ω^*_{NL}/ω^*_L OF A FG CLAMPED-CLAMPED BEAM AT VARIOUS AMPLITUDES

$W^{a^*}(x^*=0.5)$	n=0.5		n=1.0		n=2.0	
	present	[7]	present	[7]	present	[7]
0	1.000	1.000	1.000	1.000	1.0000	1.000
1	1.023	1.056	1.023	1.056	1.022	1.055
2	1.088	1.210	1.087	1.208	1.085	1.203
3	1.187	1.429	1.186	1.426	1.181	1.417
4	1.313	1.689	1.311	1.685	1.303	1.671
5	1.457	1.974	1.454	1.968	1.443	1.949

The same discrepancy is noted in this case, which is mainly due to the inadequate one trial function assumed in [7], which leads to an overestimate of the natural frequencies in comparison with those obtained in the present work in which a multimodal approach is adopted .

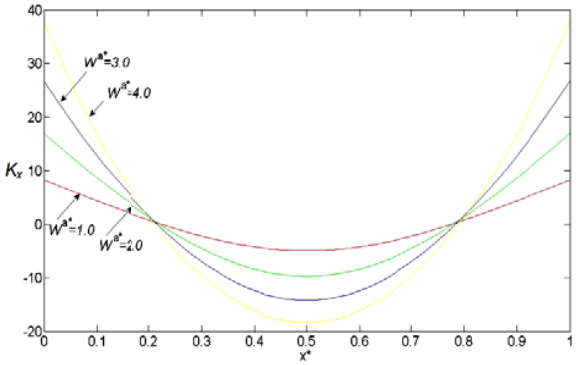


Fig 3 Nonlinear free vibration curvatures of a FG clamped- clamped beam at amplitudes W^{a^*}(x*=0.5)=1, 2, 3 and 4

In Fig. 3, the curvature distributions associated with the first nonlinear mode of the clamped-clamped FG beam considered here for various values of the maximum vibration amplitudes. It is noted that the curvature is strongly affected by the vibration amplitude, especially at the centre and in the regions of the clamps.

V. CONCLUSION

The present study deals with the problem of geometrically nonlinear free vibrations of FG beams with immovable ends. Using Hamilton's principle and spectral analysis, and neglecting the axial inertia and damping, an homogenization procedure has been proposed which reduces the problem studied to that of isotropic homogeneous beams with effective bending stiffness and axial stiffness parameters. Using a multimode approach, nonlinear frequency ratios ω_{nl}/ω_l have been compared to the results published in [7] in the case of isotropic and FG beams. An overestimate of the natural frequencies has been noted in the results given [7], due to the assumed approximate solution adopted. The non-dimensional curvature of the FG clamped beam, associated with the first nonlinear mode, has also been given for different non dimensional amplitudes. It has been clearly shown that the nonlinear effect can be significant, especially at the centre and in the regions of the clamps.

The main feature of the present contribution is the fact that the existing analytical solutions, numerical techniques and software developed over the years for the nonlinear analysis of isotropic beams can be easily used for FG beams case, leading to more accurate estimates of the nonlinear dynamic stress, especially near to the clamps, which is a sensitive region in the clamped beam.

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