

# A Generalized Approach for State Analysis and Parameter Estimation of Bilinear Systems using Haar Connection Coefficients

Monika Garg, *Life Member, ISTE*, and Lillie Dewan, *Life Member, ISTE*

**Abstract**—Three novel and significant contributions are made in this paper. Firstly, non-recursive formulation of Haar connection coefficients, pioneered by the present authors is presented, which can be computed very efficiently and avoid stack and memory overflows. Secondly, the generalized approach for state analysis of singular bilinear time-invariant (TI) and time-varying (TV) systems is presented; vis-à-vis diversified and complex works reported by different authors. Thirdly, a generalized approach for parameter estimation of bilinear TI and TV systems is also proposed. The unified framework of the proposed method is very significant in that the digital hardware once-designed can be used to perform the complex tasks of state analysis and parameter estimation of different types of bilinear systems single-handedly. The simplicity, effectiveness and generalized nature of the proposed method is established by applying it to different types of bilinear systems for the two tasks.

**Keywords**—Bilinear Systems, Haar Wavelet, Haar Connection Coefficients, Parameter Estimation, Singular Bilinear Systems, State Analysis.

## I. INTRODUCTION

**A**NALYSIS of bilinear systems is useful to study approximately the behaviour of complicated non-linear systems. A wide range of physical, chemical, biological, and social systems, which cannot be effectively modelled under the assumption of linearity, are modelled by bilinear systems.

In the domain of control systems, these are the systems whose dynamics are jointly linear in the state and control variables. Bilinear systems represent a mathematically tractable structure over Volterra models for a nonlinear system and can obviously represent the dynamics of a nonlinear system more accurately than a linear model. Hence, modelling and control of nonlinear systems in a bilinear framework are fundamental problems in engineering.

Analytical methods to analyze the bilinear systems often, are too complicated or fail completely, especially when the model is singular or time-varying. The task of parameter estimation is equally complex. Numerical methods can provide answer to these problems. Several works, regarding piecewise constant functions, or their approximations, based computational methods such as Block Pulse Functions (BPF) [1,2], Walsh Functions [3-7], Orthogonal Function Series [8,9], Haar Wavelet [10,11], have been reported in the literature for the

analysis and parameter estimation of singular or non-singular bilinear TI and TV systems.

Wavelet based numerical methods have attracted much attention in the recent times for various application in the disciplines of engineering, physics and mathematical research due to their nice properties of multiresolution and compact support [12]. Haar wavelet, the first member of Daubechies family of wavelets, is most convenient for computer implementations due to the availability of explicit expression for the Haar scaling and wavelet functions. Specifically, Haar wavelet based numerical methods have gained more prominence after Hsiao et. al pioneered and reported several applications of operational approach in the analysis, identification and optimization of different types of control systems [10,11]. In this approach, the integro-differential equations are converted into linear matrix-algebraic equations by replacing the mathematical operations of integration and differentiation etc. by corresponding operational matrices and hence the analysis and parameter estimation of bilinear systems have been either reduced or much simplified.

Hitherto, recursive formulations of various operational matrices derived by Hsiao et. al have been used invariably, for different applications, in the literature [13-14]. However, these recursive formulations are computationally costlier as higher resolution matrices are to be computed using all matrices at lower resolutions.

Moreover, different algorithms have been reported for the analysis and parameter estimation or identification of singular bilinear TI and TV systems [1-11,13-16]. Hsiao et. al studied the analysis and parameter estimation of bilinear systems [10] and analysis of singular bilinear systems via Haar wavelet [11]. They have dealt with time-invariant and time-variant cases separately. These studies have made the whole domain complex and diversified.

In this paper, an attempt has been made to overcome the above problems. Firstly, non-recursive formulation of Haar connection coefficients, pioneered by the present authors, are used resulting in the computationally efficient algorithms. Secondly, a generalized method is proposed which is capable of analyzing singular or non-singular bilinear systems both of TI and TV types, making the whole domain simple and unified. Thirdly, parameter estimation of both bilinear TI and TV systems is presented through a single unified approach vis-à-vis separate approaches reported in the literature so far. The significance of the proposed unified approach is that the digital hardware once-designed can be used to perform the

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complex tasks of state analysis and parameter estimation of different types of bilinear systems single-handedly.

The work presented in this paper is organised as follows; a brief review of some nice properties of Haar wavelet is presented in Section II alongwith the non-recursive formulation of Haar connection coefficients. In Section III, the generalized approach for the state analysis of singular bilinear systems of both TI and TV types is presented. Generalized approach for estimating the parameters of both TI and TV bilinear systems is presented in Section IV. The operating unified nature of the Proposed method for state analysis and parameter estimation is demonstrated by taking several Illustrative examples relating to different types of bilinear systems in Section V, followed by conclusions in the end.

II. SOME PROPERTIES OF HAAR WAVELET

Haar wavelet series  $h_n(t)$  is a group of square waves with magnitude of  $\pm 1$  in certain intervals and zeros elsewhere [12] with first function known as Haar scaling function  $h_0(t)$  followed by Haar wavelet function  $h_1(t)$  as the second function. All the other functions are dilations and translations of Haar wavelet function. In general, Haar wavelet series is defined as

$$h_0(t) = 1, \quad 0 \leq t < 1, h_1(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t < 1 \end{cases} \quad (1)$$

$$h_n(t) = h_1(2^j t - l), \quad n = 2^j + l, \quad j \geq 0, \quad 0 \leq l < 2^j$$

where  $j$  &  $l$  indicate dilations and translations respectively. The resolution  $m$  is given by  $m = 2^j$  and  $n = 0, 1, \dots, m-1$

The symbolic form of the Haar wavelet matrix  $H_m(t)$  is defined as

$$H_m(t) = [h_0(t) \ h_1(t) \ \dots \ h_{m-1}(t)]^T \quad (2)$$

The numeric form of Haar wavelet matrix  $H_m$  is the sampled values of Haar wavelet series, arranged as rows. For example,

$$H_4 \text{ is written as } H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

**Lemma 1.** The non-recursive expression for Haar product matrix, defined as  $H_m(t)H_m^T(t)$ , is expressed as

$$H_m(t)H_m^T(t) = H_m \text{diag}(B_m(t))H_m^T \quad (3)$$

where  $B_m(t)$  are block pulse functions defined to be unity in an unit interval of time and zero elsewhere and are expressed collectively as  $B_m(t) = [b_0(t) \ b_1(t) \ \dots \ b_{m-1}(t)]^T$  where  $b_i(t)$ 's are individual BPF. The product matrix in (3) is the non-recursive formulation pioneered by us [17].

**Lemma 2.** When the product of two Haar wavelet matrices operate on Haar expansion coefficients of any square integrable function  $f(t)$ , then it can be expressed as simple Haar expansion via connection coefficients as

$$H_m(t)H_m^T(t)c_h = C_h H_m(t) \quad (4)$$

where  $C_h$  are Haar connection coefficients and  $c_h = [c_{h0} \ c_{h1} \ \dots \ c_{h(m-1)}]^T$  are Haar expansion

coefficients of  $f(t)$ . Using the BPF expansion coefficients  $c_b$  of  $f(t)$ , value of Haar connection coefficients  $C_h$  can be evaluated non-recursively as

$$C_h = H_m \text{diag}(c_b)H_m^{-1} \quad (5)$$

The Haar connection coefficients in (5) is the non-recursive formulation pioneered by us [17].

These non-recursive formulations have the advantage of computing the Haar connection coefficients directly at the required resolution  $m$ , thereby obviating the need of computing all the matrices at lower resolutions. The reported advantage of recursive formulations of avoiding inverse of large matrices [10,11] is of not much relevance today in the era of abundant cheap computing capability at-hand and the need for avoiding recursive computer implementations in general.

A generalized approach for the state analysis of bilinear systems is proposed in the next Section.

III. PROPOSED GENERALIZED APPROACH FOR STATE ANALYSIS OF BILINEAR SYSTEMS

Consider a generalized bilinear system of the following form:

$$E(t)\dot{x}(t) = A(t)x(t) + \sum_{i=1}^q N_i(t)x(t)u_i(t) + B(t)u(t), \quad x(0) = x_0 \quad (6)$$

where the singular matrix  $E(t) \in R^{n \times n}$ , the state  $x(t) \in R^n$ , the control  $u(t) \in R^q$ ,  $A(t) \in R^{n \times n}$  and  $B(t) \in R^{n \times q}$ ,  $N_i(t) \in R^{n \times n}$  and  $u_i(t)$ ,  $i = 1, 2, \dots, q$ , are components of  $u(t)$ .

The system in (6) represents the generalized bilinear system because if, any or all, system matrices  $A(t), B(t), N_i(t)$  and singular matrix  $E(t)$  contain time-varying terms, it is singular time-varying bilinear system and it is singular time-invariant if all the above matrices are constants. Also, if  $E(t) = I_{n \times n}$  (Identity matrix), then (6) represents bilinear system without singularity.

The generalized system in (6) can be expressed in alternative form as

$$E(t)\dot{x}(t) = A(t)x(t) + K(t)x(t) + B(t)u(t), \quad x(0) = x_0 \quad (7)$$

$$\text{where } K(t) = \sum_{i=1}^q K_i(t) = \sum_{i=1}^q u_i^T(t)N_i(t)$$

In Haar transform domain, the final simulation time  $t_f$  is normalized to be unity by substituting  $t = t_f \sigma$  in (7) where  $0 \leq \sigma < 1$ .

After normalization of time scale, (7) becomes

$$E(\sigma)\dot{x}(\sigma) = t_f (A(\sigma)x(\sigma) + K(\sigma)x(\sigma) + B(\sigma)u(\sigma)) \quad (8)$$

Next each term in (8) is expanded into the Haar transform as

$$\dot{x}(\sigma) = d^T H_{mn} \quad (9)$$

where  $d^T = [d_1^T \ d_2^T \ \dots \ d_n^T]$ ,  $H_{mn}(\sigma) = I_{n \times n} \otimes H_m(\sigma)$  and  $I_{n \times n}$  is the identity matrix,  $\otimes$  is the kronecker

product. Each  $d_i^T, i = 1, 2, \dots, n$  are the expansion coefficients of each component of  $\dot{x}(\sigma)$  arranged in a row. Similarly,

$$x(\sigma) = c^T H_{mn} \quad (10)$$

where  $c^T = [c_1^T \ c_2^T \ \dots \ c_n^T]$  and each  $c_i^T, i = 1, 2, \dots, n$  are the expansion coefficients of each component of  $x(\sigma)$  arranged in a row.

Relation between  $d^T$  and  $c^T$  are obtained by integrating (9) on both sides

$$x(\sigma) = \int_0^\sigma c^T H_{mn}(\sigma) d\sigma + x(0) \quad (11)$$

Using forward operational matrix of integration  $Q_{hm}$  derived by Lin Wu et. al [18] and substituting the expansions from (9) and (10), (11) becomes

$$c^T H_{mn}(\sigma) = d^T Q_{hmn} H_{mn}(\sigma) + c_0^T H_{mn}(\sigma) \quad (12)$$

where  $Q_{hmn} = I_{n \times n} \otimes Q_{hm}$ . And Initial states  $x(0)$  are expanded in Haar domain as  $x(0) = c_0^T H_{mn}(\sigma)$ ,  $c_0^T = [c_{01}^T \ c_{02}^T \ \dots \ c_{0n}^T]$ . The effect of the term  $t_f$ , arising out of normalization of time scale, is incorporated in  $Q_{hm}$ .

Solving (12) for  $d^T$  yields

$$d^T = (c^T - c_0^T) Q_{hmn}^{-1} \quad (13)$$

On the similar lines, the expansions of  $B(\sigma)$  and  $u(\sigma)$  are expressed as

$$B(\sigma) = b^T H_{mn} \quad (14)$$

where  $b^T = [b_1^T \ b_2^T \ \dots \ b_n^T]$  and each  $b_i^T, i = 1, 2, \dots, q$  are the expansion coefficients of each component of  $B(\sigma)$  arranged in a row. And

$$u(\sigma) = u^T H_{mn} \quad (15)$$

where  $u^T = [u_1^T \ u_2^T \ \dots \ u_n^T]$  and each  $u_i^T, i = 1, 2, \dots, q$  are the expansion coefficients of each component of  $u(\sigma)$  arranged in a row.

Next, the functions  $A(\sigma), K(\sigma)$  and  $E(\sigma)$  are expanded in Haar domain as

$$A(\sigma) = a H_{mn}(\sigma) \quad (16)$$

where  $a = \begin{pmatrix} a_{11}^T & \dots & a_{1n}^T \\ \vdots & \ddots & \vdots \\ a_{n1}^T & \dots & a_{nn}^T \end{pmatrix}$  are Haar expansion coefficients of elements of matrix  $A(\sigma)$  and each  $a_{ij}^T = [a_{ij0} \ \dots \ a_{ij(m-1)}]$  &  $i, j = 1, 2, \dots, n$  at the resolution  $m$ , resulting in  $nm \times nm$  coefficients matrix. Similarly

$$\left. \begin{aligned} K(\sigma) &= k H_{mn}(\sigma) \\ E(\sigma) &= e H_{mn}(\sigma) \end{aligned} \right\} \quad (17)$$

where  $k$  and  $e$  are the expansion coefficients matrix of appropriate dimensions as defined in (16).

Expansions from (9)-(17) are substituted in (8) to get

$$e H_{mn}(\sigma) d^T H_{mn}(\sigma) = a H_{mn}(\sigma) c^T H_{mn}(\sigma) + k H_{mn}(\sigma) c^T H_{mn}(\sigma) + b^T H_{mn}(\sigma) u^T H_{mn}(\sigma) \quad (18)$$

Collecting the terms in (18) results in

$$d^T H_{mn}(\sigma) H_{mn}^T(\sigma) e^T = c^T H_{mn}(\sigma) H_{mn}^T(\sigma) a^T + c^T H_{mn}(\sigma) H_{mn}^T(\sigma) k^T + u^T H_{mn}(\sigma) H_{mn}^T(\sigma) b \quad (19)$$

Each of the term  $H_{mn}(\sigma) H_{mn}^T(\sigma)$  in (19) is simplified using Haar connection coefficients defined in (4) and evaluated non-recursively using (5), as

$$d^T \hat{e}^T H_{mn}(\sigma) = c^T \hat{a}^T H_{mn}(\sigma) + c^T \hat{k}^T H_{mn}(\sigma) + u^T \hat{b} H_{mn}(\sigma) \quad (20)$$

where  $\hat{(\cdot)}$  denotes the connection coefficients for the corresponding matrices.

Right multiplying each term in (20) by  $[H_{mn}(\sigma)]^{-1}$ , we get

$$d^T \hat{e}^T = c^T \hat{a}^T + c^T \hat{k}^T + u^T \hat{b} \quad (21)$$

Substituting the value of  $d^T$  from (13) and collecting the terms results in

$$c^T Q_{hmn}^{-1} \hat{e}^T - c^T \hat{a}^T - c^T \hat{k}^T = c_0^T Q_{hmn}^{-1} \hat{e}^T + u^T \hat{b} \quad (22)$$

where orthogonality of Haar transform is used.

$c^T$  is obtained from (22) as

$$c^T = [c_0^T Q_{hmn}^{-1} \hat{e}^T + u^T \hat{b}] [Q_{hmn}^{-1} \hat{e}^T - \hat{a}^T - \hat{k}^T]^{-1} \quad (23)$$

It is trivial to calculate the inverse of the term  $[Q_{hmn}^{-1} \hat{e}^T - \hat{a}^T - \hat{k}^T]$  due to sparse nature of the matrix – a key characteristics of Haar wavelet [12]. The desired values of states  $x(\sigma)$  are evaluated using (10) from the value of  $c^T$  obtained from (23).

It is clear that the proposed method is simple, elegant and generalized. And, non-recursive formulation of the Haar connection coefficients enables the use of simple rules of matrix linear algebra to solve the complex analysis problems of bilinear systems of all types.

A generalized approach for parameter estimation of bilinear systems is proposed in the next Section.

#### IV. PROPOSED GENERALIZED APPROACH FOR PARAMETER ESTIMATION OF BILINEAR SYSTEMS

The parameters  $A(\sigma), N(\sigma)$  and  $B(\sigma)$  of a bilinear system are to be estimated from the given states and input. To accomplish this, Haar expansions of various functions, given in Section III, are substituted in (6) to obtain

$$e H_{mn}(\sigma) d^T H_{mn}(\sigma) = a H_{mn}(\sigma) c^T H_{mn}(\sigma) + \sum_{i=1}^q n_i H_{mn}(\sigma) c^T H_{mn}(\sigma) u_i^T H_{mn}(\sigma) + b^T H_{mn}(\sigma) u^T H_{mn}(\sigma) \quad (24)$$

where  $N_i(\sigma) = n_i H_{mn}(\sigma)$ ,  $n_i$  are the expansion coefficients matrix of appropriate dimensions as defined in (16).

Rearranging (24) yields

$$dH_{mn}(\sigma)H_{mn}^T(\sigma)e^T = aH_{mn}(\sigma)H_{mn}^T(\sigma)c + \sum_{i=1}^q n_i H_{mn}(\sigma)c^T H_{mn}(\sigma)H_{mn}^T(\sigma)u_i + b^T H_{mn}(\sigma)H_{mn}^T(\sigma)u \quad (25)$$

Each of the term  $H_{mn}(\sigma)H_{mn}^T(\sigma)$  in (25) is simplified using Haar connection coefficients defined in (4) and evaluated non-recursively in (5), as

$$d^T \hat{e}^T H_{mn}(\sigma) = a\hat{c}H_{mn}(\sigma) + \sum_{i=1}^q n_i H_{mn}(\sigma)c^T \hat{u}_i H_{mn}(\sigma) + b^T \hat{u} H_{mn}(\sigma) \quad (26)$$

where  $\widehat{(\cdot)}$  denotes the connection coefficients for the corresponding matrices.

Similarly the term under summation is simplified using the connection coefficients as above to obtain

$$d^T \hat{e}^T H_{mn}(\sigma) = a\hat{c}H_{mn}(\sigma) + \sum_{i=1}^q n_i (c^T \hat{u}_i)^T H_{mn}(\sigma) + b^T \hat{u} H_{mn}(\sigma) \quad (27)$$

Right multiplying each term in (27) by  $[H_{mn}(\sigma)]^{-1}$ , we get

$$d^T \hat{e}^T = a\hat{c} + \sum_{i=1}^q n_i (c^T \hat{u}_i)^T + b^T \hat{u} \quad (28)$$

For proper estimation (28) is simulated for a number of combination of inputs and initial conditions so as to generate sufficient information.

Let for  $k^{th}$  combination of input and initial conditions (28) is represented as

$$(d^T \hat{e}^T)_k = a\hat{c}_k + \sum_{i=1}^q n_i \left[ (c^T \hat{u}_i)^T \right]_k + b^T \hat{u}_k \quad (29)$$

Rearranging (29), for the  $k^{th}$  combination of input and initial conditions, yields

$$(d^T \hat{e}^T)_k = \begin{bmatrix} a & n_1 & \dots & n_q & b^T \end{bmatrix} \begin{bmatrix} \hat{c} \\ \left[ (c^T \hat{u}_1)^T \right] \\ \vdots \\ \left[ (c^T \hat{u}_q)^T \right] \\ \hat{u} \end{bmatrix}_k \quad (30)$$

For proper estimation,  $k$  should be equal to number of elements in  $k^{th}$  column i.e.  $n + (n + 1)q$  in addition to the conditions imposed on the resolution  $m$  of Haar basis by Hsiao et al. [10]. Hence, the required estimated values of the unknown parameters are finally obtained, from (30), as

$$\begin{bmatrix} a & n_1 & \dots & n_q & b^T \end{bmatrix} = (d^T \hat{e}^T)_j \begin{bmatrix} \hat{c} \\ \left[ (c^T \hat{u}_1)^T \right] \\ \vdots \\ \left[ (c^T \hat{u}_q)^T \right] \\ \hat{u} \end{bmatrix}_j^{-1} \quad (31)$$

It is trivial to evaluate inverse in (31) due to sparse nature of the Haar wavelet.

## V. ILLUSTRATIVE EXAMPLES

The generalized nature of the proposed method is established by taking several examples one each for Singular and non-singular bilinear TI and TV systems for state analysis and bilinear TI and TV system for parameter estimation.

### A. State Analysis

#### Example 1 (Bilinear TI system):

Consider the bilinear time-invariant (TI) system [10] of the form (6), where

$$E(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, B(t) = 0, \\ N(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, t_f = 8, u(t) = e^{-t} \text{ and } x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (32)$$

The values of various time-varying functions are obtained by normalizing the final time to unity by substituting  $t = t_f \sigma = 8\sigma$ .

Analytical solution for states  $x(t)$  is reported to be [10]

$$x(t) = \frac{1}{2} \begin{bmatrix} e^{-t-e^{-t}+1} + e^{-3t-e^{-t}+1} \\ e^{-t-e^{-t}+1} - e^{-3t-e^{-t}+1} \end{bmatrix} \quad (33)$$

Comparison between the Haar solutions obtained using the proposed method from (10) - (23) and Analytical solution is shown in Fig. 1

**Example 2 (Bilinear TV system):** Consider the bilinear time-varying (TV) system [1] of the form (6), where

$$E(t) = 1, A(t) = -t, B(t) = 1, N(t) = e^{(1-t^2)/2} \\ t_f = 3, u(t) = e^{-(t-1)^2/2} \text{ and } x(0) = 0 \quad (34)$$

The values of various time-varying functions are obtained by normalizing the final time to unity by substituting  $t = t_f \sigma = 3\sigma$ .

Analytical solution for state  $x(t)$  is reported to be [1]

$$x(t) = te^{-(t-1)^2/2} \quad (35)$$

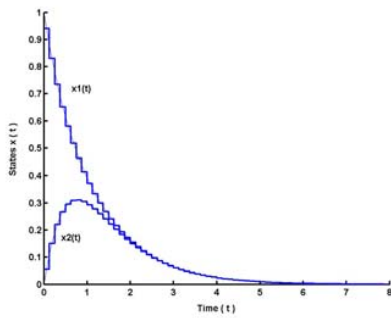


Fig. 1. Haar and analytic solution of the TI bilinear system in Example 1

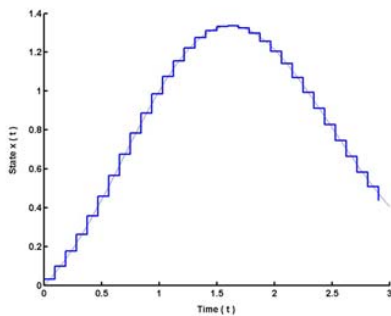


Fig. 2. Haar and analytic solution of the TV bilinear system in Example 2

Comparison between the Haar solutions obtained using the proposed method from (10) - (23) and Analytical solution is shown in Fig. 2

It is evident from Fig 1. and Fig. 2 that the Haar solution obtained using the proposed method agrees well with the analytical solution. More accurate results can be obtained for higher values of resolution  $m$ .

**Example 3 (Singular bilinear TV system):**

Consider the singular bilinear time-varying (TV) system [11] of the form (6), where

$$E(t) = \begin{pmatrix} 0 & -t & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{pmatrix}, A(t) = \begin{pmatrix} -2 & t & 1 \\ 0 & -4 & 2 \\ -2t & 0 & 1 \end{pmatrix},$$

$$N(t) = \begin{pmatrix} 1 & -t & 1 \\ 0 & 3 & -2 \\ 2t & 0 & -2 \end{pmatrix}, B(t) = [ 2 \ 1 \ 3 ]^T,$$

$$t_f = 1, u(t) = 1 \text{ and } x(0) = [ 12 \ 2 \ 5 ]^T \tag{36}$$

The values of various time-varying functions are obtained by normalizing the final time to unity by substituting  $t = t_f \sigma = \sigma$ .

Analytical solution for state  $x(t)$  is reported to be [11]

$$x(t) = \begin{bmatrix} (2-t)(e^{-t/2} + e^t) + 8 \\ 2e^{-t/2} - e^t + 1 \\ e^{-t/2} + e^t + 3 \end{bmatrix} \tag{37}$$

Comparison between the Haar solutions obtained using the proposed method from (10) - (23) and Analytical solution is shown in Fig. 3

It is evident from Fig. 3 that the Haar solutions obtained using the proposed method agrees well with the analytical solution. More accurate results can be obtained for higher values of resolution  $m$ .

**B. Parameter Estimation**

**Example 4 (Bilinear TI system):**

Consider the bilinear time-invariant (TI) system [1] of the form (6) as

$$\dot{x}(t) = A(t)x(t) + N(t)x(t)u(t) + B(t)u(t), x(0) = x_0 \tag{38}$$

The simulated response data due to three different combinations of inputs and initial conditions  $u(t) = e^{-0.5t}, \cos t$  and  $\sin t$  are tabulated in Table I.

TABLE I  
SIMULATED RESPONSE DATA FOR BILINEAR SYSTEM IN EXAMPLE 4

$t$	0	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	$\frac{9}{16}$	$\frac{11}{16}$	$\frac{13}{16}$	$\frac{15}{16}$	
$u(t)$		States $x(t)$								
$e^{-0.5t}$	2	2.1734	2.4896	2.7433	2.9327	3.0591	3.1262	3.1393	3.1053	
$\cos(t)$	1	1.1868	1.5545	1.9022	2.2129	2.4678	2.6486	2.7402	2.7337	
$\sin(t)$	3	2.6958	2.2025	1.9061	1.7520	1.7057	1.7454	1.8572	2.0316	

The unknown parameters  $A(t), N(t)$  and  $B(t)$  are estimated using (31) as shown in (39), alongwith the true values in (40). In this example, the average of the samples of the estimated values is taken due to time-invariant nature of the system.

$$[ \tilde{A} \ \tilde{N} \ \tilde{B} ] = [ -1.9989 \ : \ 1.9984 \ : \ 3.0003 ] \tag{39}$$

$$[ A \ N \ B ] = [ -2.0000 \ : \ 2.0000 \ : \ 3.0000 ] \tag{40}$$

Comparing the averaged estimated values with true values, it is evident that the estimated values agrees well with the true values.

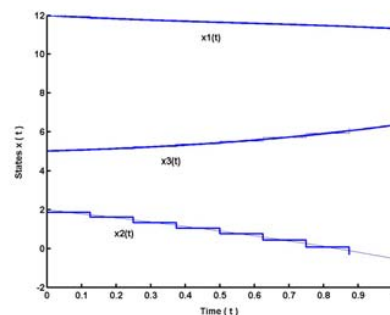


Fig. 3. Haar and analytic solution of the singular bilinear system in Example 3

**Example 5 (Bilinear TV system):**

The 1st order bilinear time-varying (TV) system, taken in Example 2, is simulated to obtain response data due to three different combinations of input and initial conditions given in Table II.

TABLE II  
SIMULATED RESPONSE DATA FOR BILINEAR SYSTEM IN EXAMPLE 2

$t$	0	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$	$\frac{9}{16}$	$\frac{11}{16}$	$\frac{13}{16}$	$\frac{15}{16}$	
$u(t)$		States $x(t)$								
$e^{-(t-1)^2/2}$	0	0.1590	0.5321	0.9443	1.2479	1.3300	1.1784	0.8799	0.5566	
$\cos(t)$	1	1.4109	1.9523	1.9562	1.5759	1.0662	0.6203	0.3166	0.1457	
$\sin(t)$	2	2.0627	2.2356	2.4376	2.5676	2.5839	2.4883	2.2724	1.8045	

The unknown parameters  $A(t)$ ,  $N(t)$  and  $B(t)$  are to be estimated. True values of these unknowns are given in (34).

The estimated parameters  $\hat{A}(t)$ ,  $\hat{N}(t)$  and  $\hat{B}(t)$  along with the true values are shown in Fig. 4.

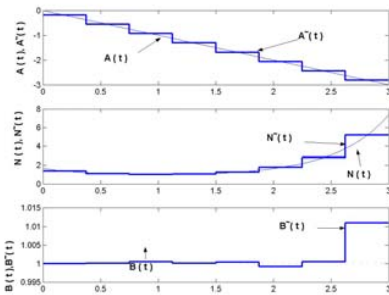


Fig. 4. Estimated and True values of unknown Parameters of Example 2

It is evident from Fig 4. that the estimated values of parameters agree well with the true values. Better estimation is expected for larger values of resolution  $m$ .

## VI. CONCLUSION

The importance of different types of bilinear systems arising in various disciplines of Engineering and Sciences cannot be understated. Rather complex problem of State analysis of singular bilinear time-invariant and time-varying systems is achieved by applying the same proposed generalized approach to different types of bilinear systems. The proposed method, which is based on the non-recursive formulation of Haar Connection Coefficients, enables the repeated use of proposed method very efficient computationally. Also, the parameter estimation of bilinear TI and TV systems is accomplished successfully by applying the proposed generalized estimation algorithm. The future scope lies in formulating the unified approaches, on the similar lines, for the state analysis and parameter estimation or identification of different types of systems especially in the presence of measurement noise.

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