International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:1, 2015

# A Further Study on the 4-Ordered Property of Some Chordal Ring Networks

Shin-Shin Kao, Hsiu-Chunj Pan

*Abstract*—Given a graph $G$. A cycle of $G$ is a sequence of vertices of $G$ such that the first and the last vertices are the same. A hamiltonian cycle of $G$ is a cycle containing all vertices of $G$. The graph $G$ is $k$-ordered (resp. $k$-ordered hamiltonian) if for any sequence of $k$ distinct vertices of $G$, there exists a cycle (resp. hamiltonian cycle) in $G$ containing these $k$ vertices in the specified order. Obviously, any cycle in a graph is 1-ordered, 2-ordered and 3-ordered. Thus the study of any graph being $k$-ordered (resp. $k$-ordered hamiltonian) always starts with $k = 4$. Most studies about this topic work on graphs with no real applications. To our knowledge, the chordal ring families were the first one utilized as the underlying topology in interconnection networks and shown to be 4-ordered. Furthermore, based on our computer experimental results, it was conjectured that some of them are 4-ordered hamiltonian. In this paper, we intend to give some possible directions in proving the conjecture.

*Keywords*—Hamiltonian cycle, 4-ordered, Chordal rings, 3-regular.

## I. Introduction

We consider finite, undirected and simple graphs only. Let $G = (V, E)$ be a graph, where $V$ is the set of vertices of $G$ and $E = \{(u, v) \mid u, v \in V\}$ is the set of edges of $G$, respectively. Let $u, v$ be two vertices of $G$. If $e = (u, v) \in E$, then we say that the vertices $u$ and $v$ are *adjacent* in $G$. The *degree* of any vertex $u$ is the number of distinct vertices adjacent to $u$. $N(u)$ denotes the set of vertices which are adjacent to $u$. If $|N(u)| = 3$ for any vertex $u$ of $G$, then we call $G$ a 3-*regular* graph. A *path* $P$ between two vertices $v_0$ and $v_k$ is represented by $P = \langle v_0, v_1, \ldots, v_k\rangle$, where each pair of consecutive vertices are connected by an edge. We also write the path $P = \langle v_0, v_1, \ldots, v_k\rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k\rangle$, where $Q$ denotes the path between $v_i$ and $v_j$. A *hamiltonian path* between $u$ and $v$, where $u$ and $v$ are two distinct vertices of $G$, is a path joining $u$ to $v$ that visits every vertex of $G$ exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of $G$ is a cycle that traverses every vertex of $G$ exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G$ is $k$-*ordered* (or $k$-*ordered hamiltonian*, resp.) if for any sequence of $k$ distinct vertices of $G$, there exists a cycle (or a hamiltonian cycle, resp.) in $G$ containing these $k$ vertices in the specified order. Obviously, any cycle in a graph is 1-ordered, 2-ordered and 3-ordered. Thus the study of $k$-orderedness (or $k$-ordered hamiltonicity) of any graph always starts with $k = 4$.

Corresponding author: S.-S. Kao is a professor in the Department of Applied Mathematics, Chung-Yuan Christian University, Chong-Li City, Tao-Yuan County, 32023 Taiwan (e-mail: shin2kao@gmail.com).

H.-C. Pan is a Ph.D. student under the guidance of Professor Kao.

A graph $G = (V, E)$ is a $k$-*ordered hamiltonian-connected* graph if for any sequence of $k$ vertices of $G$, there exists a hamiltonian path $P$ between $u$ and $v$ such that $P$ passes these vertices in the specified order. It can be seen that $k$-ordered hamiltonicity and $k$-ordered hamiltonian-connectedness do not imply each other.

The concept of $k$-orderedness and $k$-ordered hamiltonicity was first introduced by Ng and Schultz [2] in 1997. See [2], [3], [4], [5], [6] for example. In [2], the authors posed the question of the existence of 4-ordered 3-regular graphs other than the complete graph $K_4$ and the complete bipartite graph $K_{3,3}$. In [5], Meszaros answered the question by proving that the Petersen graph and the Heawood graph are non-bipartite, 4-ordered 3-regular graphs. Hsu et al. in [6] provided examples of bipartite non-vertex-transitive 4-ordered 3-regular graphs of order n for any sufficiently large even integer $n$. In 2013, Hung et al. further gave a complete classification of generalized Petersen graphs, $GP(n, 4)$, and showed the following theorems[7]:

**Theorem 1.** [7] Let $n \geq 9$. $GP(n, 4)$ is 4-ordered hamiltonian if and only if $n \in \{18, 19\}$ or $n \geq 21$.

**Theorem 2.** [7] Let $n \geq 9$. $GP(n, 4)$ is 4-ordered hamiltonian-connected if and only if $n \geq 18$.

Since Petersen graphs have been well-known and often provide examples or counterexamples for interesting graphic properties, the results of [7] and Theorems 1-2 might leave readers an impression that most 4-ordered graphs are 4-ordered hamiltonian, and most 4-ordered hamiltonian graphs are 4-ordered hamiltonian-connected. It could be misleading. Therefore, the authors intend to study this topic on graphs with real applications, and the chordal ring networks turn out to be a good subject. The chordal ring family has been adopted as the underlying topology of certain interconnection networks [8] and is studied for the real architecture for parallel and distributed systems due to the advantage of a built-in hamiltonian cycle, symmetry, easy routing and robustness. See [9] and its references.

This paper is organized as follows. In Section II, the formal definition of the chordal ring networks is given. We introduce some known results and techniques, which are related to the study of the 4-ordered properties. Then we propose two possible methods of proving the following conjecture in [1], and the difficulties it may encounter. Finally, a brief conclusion is given in Section III.

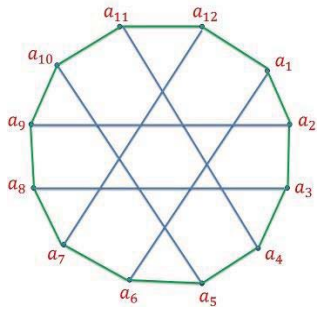


Fig. 1. $CR(12;1,5)$

**Conjecture 1.** [1] $CR(n;1,5)$ is a 4-ordered hamiltonian graph if $n = 14$, $n = 12k+2$ or $n = 12k+10$ with $k \geq 2$.

## II. About $CR(n;1,q)$ and its 4-ordered property

The chordal rings $CR(n;1,q)$, where $n$ is an even integer with $n \geq 6$ and $q$ an odd integer with $3 \leq q \leq \frac{n}{2}$, is defined as follows. Let $G(V,E) = CR(n;1,q)$, where $V = \{a_1, a_2, \ldots, a_n\}$ and $E = \{(a_i, a_{(i+1)\bmod n}) : 1 \leq i \leq n\} \cup \{(a_i, a_{(i+q)\bmod n}) : i$ is odd and $1 \leq i \leq n\}$. An illustration of $CR(12;1,5)$ is given in Fig. 1. Obviously, it is a 3-regular graph, has a built-in hamiltonian cycle $\langle a_1, a_2, \ldots, a_n, a_1\rangle$, and is vertex symmetric. The following theorems were proved in [1].

**Theorem 3.** [1] $CR(n;1,5)$ is 4-ordered for any even integer $n$ with $n \geq 14$.

**Theorem 4.** [1] $CR(n;1,7)$ is 4-ordered for any even integer $n$ with $n \geq 18$.

The first method we propose to prove Conjecture 1 is applying the similar technique for the above two theorems. Thus, before we present our possible methods of proving Conjecture 1, a short review of [1] about how the above two theorems are derived is necessary. In particular, the techniques required for Theorem 3 is recalled below. It can be observed that Theorem 3 is a combination of the following three theorems.

**Theorem 5.** [1] $CR(20+6k;1,5)$ is 4-ordered for any integer $k$ with $k \geq 0$.

**Theorem 6.** [1] $CR(22+6k;1,5)$ is 4-ordered for any integer $k$ with $k \geq 0$.

**Theorem 7.** [1] $CR(24+6k;1,)$ is 4-ordered for any integer $k$ with $k \geq 0$.

Since Theorems 5-7 are proved with the similar method, here we only recall that of Theorem 5. Based on the computer experimental results, it was shown that $CR(20;1,5)$ is 4-ordered. Now, we define a function $f$, which maps $R$ from $CR(20+6k;1,5)$ into $CR(26+6k;1,5)$ in the following way:

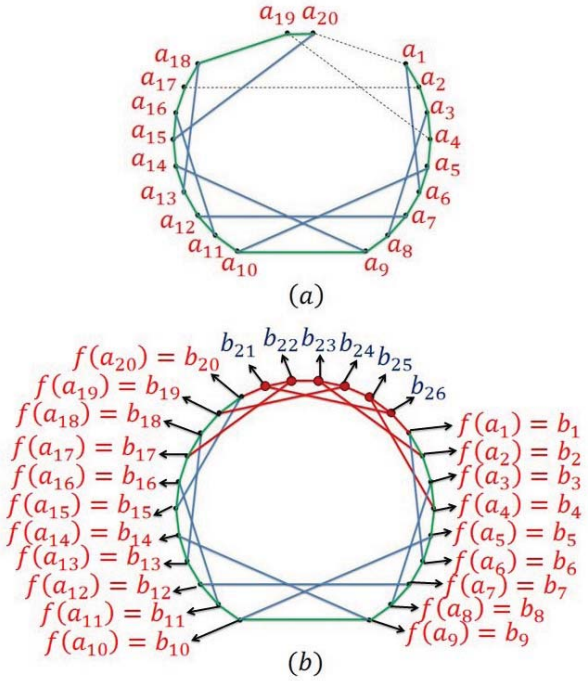


Fig. 2. From $CR(20;1,5)$ into $CR(26;1,5)$

(1) If $a_i \in R \cap V(CR(20+6k;1,5))$, where $1 \leq i \leq 20+6k$, then $f(a_i) = b_i$.

(2) If $(a_i, a_j) \in R \cap E(CR(20+6k;1,5))$, where $1 \leq i, j \leq 20+6k$, then

$$f((a_i,a_j)) = \begin{cases} (b_i, b_{i+1}) & 1 \leq i \leq 19+6k, j = i+1; \\ (b_i, b_{i+5}) & i = \text{ odd}, 1 \leq i \leq 15+6k, j = i+5; \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to see that $f((a_{20+6k}, a_1)) = \emptyset$, $f((a_{17+6k}, a_2)) = \emptyset$ and $f((a_{19+6k}, a_4)) = \emptyset$. Therefore, $CR(26+6k;1,5) - f(CR(20+6k;1,5))$ consists of the vertex set $\{b_{21+6k}, b_{22+6k}, b_{23+6k}, b_{24+6k}, b_{25+6k}, b_{26+6k}\}$ and the edge set $\{(b_{20+6k}, b_{21+6k}), (b_{21+6k}, b_{22+6k}), (b_{22+6k}, b_{23+6k}), (b_{23+6k}, b_{24+6k}), (b_{24+6k}, b_{25+6k}), (b_{25+6k}, b_{26+6k}), (b_{26+6k}, b_1), (b_{17+6k}, b_{22+6k}), (b_{19+6k}, b_{24+6k}), (b_{21+6k}, b_{26+6k}), (b_{23+6k}, b_2), (b_{25+6k}, b_4)\}$. Fig. 2 gives an illustration, in which $f$ maps $R$ from $CR(20;1,5)$ into $CR(26;1,5)$. We can see the following things happen.

(1) $f(a_i) = b_i$ for $1 \leq i \leq 20$, denoted by black vertices on both graphs.

(2) $f((a_i, a_{i+1})) = (b_i, b_{i+1})$ for $1 \leq i \leq 19$, denoted by green edges on both graphs.

(3) $f((a_i, a_{i+5})) = (b_i, b_{i+5})$ for $i$ is odd with $1 \leq i \leq 15$, denoted by blue edges on both graphs.

(4) $f((a_{20}, a_1)) = \emptyset$,$f((a_{17}, a_2)) = \emptyset$ and $f((a_{19}, a_4)) = \emptyset$,

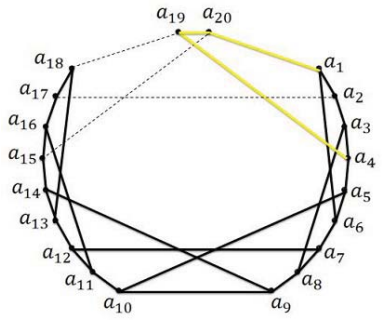


Fig. 3. $CR(20;1,5)$

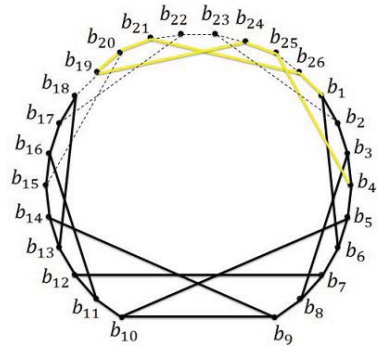


Fig. 4. $CR(26;1,5)$

denoted by dashed edges on $CR(20;1,5)$.

(5) $CR(26;1,5) - f(CR(20;1,5))$ consists of the vertex set $\{b_{21}, b_{22}, b_{23}, b_{24}, b_{25}, b_{26}\}$ and edge set $\{(b_{20}, b_{21}), (b_{21}, b_{22}), (b_{22}, b_{23}), (b_{23}, b_{24}), (b_{24}, b_{25}), (b_{25}, b_{26}), (b_{26}, b_1), (b_{17}, b_{22}), (b_{19}, b_{24}), (b_{21}, b_{26}), (b_{23}, b_2), (b_{25}, b_4)\}$.

As an example, we show the construction of the required cycle in $CR(26;1,5)$ using the known cycle of $CR(20;1,5)$. There are 20 vertices $a_1, a_2, ..., a_{20}$ in $CR(20;1,5)$, and 26 vertices $b_1, b_2, ..., b_{26}$ in $CR(26;1,5)$. To prove the theorem, we do case studies by considering different situations. Take $G = CR(26;1,5)$. Let $x_1, x_2, x_3$ and $x_4$ be four arbitrary vertices of $G$. We want to construct a cycle $C$ in $G$ that visits $x_i{}'s$ in the given order. Note that we can always find at least one set of six consecutive vertices, denoted by $S = \{b_i, b_{i+1}, b_{i+2}, ..., b_{i+5}\}$, such that $S \cap \{x_1, x_2, x_3, x_4\} = \phi$. Without loss of generality, let $x_1 = b_1$ and $S = \{b_{21}, b_{22}, ..., b_{26}\}$. Removing the vertices of $S$ and all edges adjacent to $S$ in $G$, we obtain a subgraph of $CR(20;1,5)$. Obviously, $S \cap f(CR(20;1,5)) = \phi$. Note that $CR(20;1,5)$ is 4-ordered and hence contains a cycle that visits $x_i{}'s$ in the given order, denoted by $C'$. We will obtain $C$ by embedding $CR(20;1,5)$ into $CR(26;1,5)$ and rerouting the cycle $C'$. One of the possible rerouting cases are illustrated in Fig. 3 and 4, where we reroute a path in $CR(20;1,5)$ to obtain a path in $CR(26;1,5)$ without changing the corresponding location of any other vertex in $CR(26;1,5)$.

Once such an embedding and rerouting method is realized,

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 0 |
| 0 | 5 | 6 | 11 | 12 | 17 | 18 | 23 | 24 | 3 | 4 | 9 | 10 | 15 | 16 | 21 | 22 | 1 | 2 | 7 | 8 | 13 | 14 | 19 | 20 | 25 | 0 |
| 0 | 5 | 6 | 7 | 2 | 3 | 4 | 9 | 8 | 13 | 12 | 11 | 10 | 15 | 14 | 19 | 18 | 17 | 16 | 21 | 20 | 25 | 24 | 23 | 22 | 1 | 0 |

Fig. 5. Simulation Results

we can talk about the general construction method as follows. Given any four vertices in $G = CR(n;1,5)$, denoted by $a_1, a_i, a_j, a_k$ with $1 < i < j < k$, we want to construct a hamiltonian cycle which passes through these four vertices in any specified order. The computer experimental results show that the conjecture holds for $k = 2, 3, 4$ by giving the required hamiltonian cycles concretely. Note that when $k = 4$, the total number of vertices of $G$ is 50 or 58, and we want to verify the conjecture for $k \geq 5$. It is easy to see that there exists a set $S$ of at least 12 consecutive vertices on $G$ such that $S \cap \{a_1, a_i, a_j, a_k\} = \emptyset$. Therefore, we can embed the graph $G = CR(n;1,5)$ into $\bar{G} = CR(n+12;1,5)$ by a proper function $f$, which will map the known hamiltonian cycle $C$ in $G$ into $\bar{G}$ as well. Since there are additional 12 vertices in $\bar{G}$ not included in the cycle $C$, we must develop a routing algorithm that relies on $C$ to keep the original order of $\{a_1, a_i, a_j, a_k\}$ and extends $C$ to go through the 12 vertices of $\bar{G}$. Compared with the work in [1], it becomes rather complicated for the 4-ordered hamiltonicity. Since some of the paths on $C$ has to be broken in order for rerouting in $\bar{G}$, the requirement that all of these extra 12 vertices must be covered makes the routing scheme very difficult. In many cases, one must give up the known hamiltonian cycle $C$ provided by the computer and reconstruct another hamiltonian cycle $C'$ in $G$ for the purpose of "extension". The working load of case-study analysis as in [1] could be doubled or even tripled.

The second method we propose requires some combinatorial knowledge. Since $CR(n;1,q)$ is vertex symmetric, without loss of generality, we can always let $a_1$ be one of the four given vertices which need to be visited in order and let the other three vertices be $x = a_i, y = a_j$ and $z = a_k$, where $1 < i < j < k$. It is natural to ask the question regarding the number of hamiltonian cycles required to show the 4-ordered hamiltonicity. Given any four vertices, since they lie on a circle, if we can find three hamiltonian cycles on which these four vertices appear as $a_1 \to x \to y \to z \to a_1$, $a_1 \to y \to x \to z \to a_1$, and $a_1 \to x \to z \to y \to a_1$, then it is done. In fact, theoretically, three hamiltonian cycles which are "independent" of each other should be enough. However, the following example tells us that the construction for such three hamiltonian cycles could be very difficult.

In Fig. 5, any vertex of $G = CR(26;1,5)$ is labeled by an integer $i$ with $0 \leq i \leq 25$ for simplicity. Any $(i,j)$ is an edge of $G$ if and only if $j = (i \pm 1)$ **mod** 26 or $j = (i+5)$ **mod** 26. Suppose that the given four vertices are $0, 1, 2, 3$, it is observed that the three hamiltonian cycles presented in Fig. 2 provide the required cycles with $0, 1, 2, 3$ in any specified order. However, if the given four vertices in $G$ are $0, 12, 17, 23$, the three hamiltonian cycles in Fig. 5 gives no help other than the natural built-in hamiltonian cycle

of the chordal ring graphs. Thus the following two questions arise.

**Q.1.** Is it possible to construct three and only three hamiltonian cycles in $CR(n;1,5)$, denoted by $C_1, C_2$ and $C_3$, which are "totally independent" so that for any given four vertices with a specific order, one could find the required one among them? Does the structure of $CR(n;1,5)$ and its algebraic pattern, after excluding the repetitions due to symmetry, hinder the optimum to occur?

**Q.2.** If the answer to the first question is yes, then how can we find these three hamiltonian cycles? If the answer is no, then what's the least number of hamiltonian cycles we need to include all possible 4-ordered cases? Once again, if we only need $k$ hamiltonian cycles for $CR(n;1,5)$ for 4-ordered hamiltonicity, where are these $k$ ones? A concrete construction scheme must be developed.

## III. Conclusion

In this paper, we continue the study of 4-ordered properties of the chordal ring networks. A rigorous proof of the 4-ordered hamiltonicity of any graph family has been shown a difficult work (see [6], [7] for example). For $CR(n;1,5)$, Conjecture 1 has been proposed based on computer experimental results. We not only propose two possible techniques for verifying the conjecture, but also present the corresponding complexities. Finally, two interesting topics following this discussion are raised for future studies.

## Acknowledgment

This work was supported in part by Ministry of Science and Technology of R.O.C. under Contract MOST103-2115-M-033 -003 -.

## References

[1] S. S. Kao, S. C. Wey and H. C. Pan, *The 4-ordered property of some chordal ring networks*, Mathematics Methods in Engineering and Economics, Proceedings of the 2014 International Conference on Applied Mathematics and Computational Methods in Engineering, ISBN: 978-1-61804-230-9, 2014, pp. 44–48.
[2] Lenhard Ng and Michelle Schultz, *k-ordered hamiltonian graphs*, Journal of Graph Theory, 24, 1997, No. 1, pp. 45–57.
[3] Ralph J. Faudree, *Survey of results on k-ordered graphs*, Discrete Mathematics, 229, 2001, pp. 73–87.
[4] Ruijuan Li, Shengjia Li, and Yubao Guo, *Degree conditions on distance* 2 *vertices that imply k-ordered hamiltonian*, Discrete Applied Mathematics, 158, 2010, pp. 331–339.
[5] Karola Meszaros, *On 3-regular 4-ordered graphs*, Discrete Mathematics, 308, 2008, pp. 2149–2155.
[6] Lih-Hsing Hsu, Jimmy J.M. Tan, Eddie Cheng, Laszlo Liptak, Cheng-Kuan Lin, and Ming Tsai, *Solution to an open problem on 4-ordered Hamiltonian graphs*, Discrete Mathematics, 312, 2012, pp. 2356–2370.
[7] Chun-Nan Hung, David Lu, Randy Jia, Cheng-Kuan Lin, Laszlo Liptak, Eddie Cheng, Jimmy J.M. Tan, and Lih-Hsing Hsu, *4-ordered-Hamiltonian problems of the generalized Petersen graph GP(n,4)*, Mathematical and Computer Modelling 57, 2013, pp. 595–601.
[8] R.N. Farah, M.Othman, and M.H. Selamat, *Combinatorial properties of modified chordal rings degree four networks*, Journal of Computer Science 6, 3, 2010, pp. 279–284.
[9] Xianbing Wang and Yong Meng Teo, *Global data computation in chordal rings*, J. Parallel and Distributed Computing, 69, 2009, pp. 725–736.

**Shin-Shin Kao** received her Ph.D. in Dept. of Mathematics in University of California, Los Angeles, USA, in 1995. She has been a faculty in Dept. of Applied Mathematics, Chung-Yuan Christian University, Taiwan, since September, 1995. She is one of the board members of Taiwan Mathematics Society (the Mathematics Society of the Republic of China) since 2010, and is currently the editor of the electronic newspaper of TMS. Her main researches are about graph theory and its applications.

**Hsiu-Chunj Pan** is currently a first-year graduate student in the Ph.D. program in Dept. of Applied Mathematics, Chung-Yuan Christian University, Taiwan. She is most interested in combinatorics, graph theory and their applications.