

A Contractor Iteration Method Using Eigenpairs for Positive Solutions of Nonlinear Elliptic Equation

Hailong Zhu Zhaoxiang Li Kejun Zhuang

Abstract—By means of Contractor Iteration Method, we solve and visualize the Lane-Emden(-Fowler) equation

$$\Delta u + u^p = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

It is shown that the present method converges quadratically as Newton's method and the computation of Contractor Iteration Method is cheaper than the Newton's method.

Keywords—Positive Solutions; Newton's Method; Contractor Iteration Method; Eigenpairs.

I. INTRODUCTION

IN this paper, we study semilinear boundary value problems(BVPs) of the form

$$\Delta u + u^p = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega. \quad (1)$$

Eq.(1), first introduced by Lane in the mid-19th century, is called Lane-Emden(-Fowler) equation as a model of the distribution of clusters of stars in Astrophysics^[1,2]. From then on, (1) has been extensively investigated. See for example [3,4] for a survey.

Concerning the algorithmic development for solutions of semilinear elliptic BVPs, there are mainly six numerical methods for computing such kinds of problems: the Monotone Iterative Scheme (MIS)^[5,6], the Mountain Pass Algorithm (MPA)^[7], the High Linking Algorithm (HLA)^[8], the Min-Max Algorithm (MMA)^[9,10], the Search Extension Method (SEM)^[11] and the Bifurcation method(BM)^[12].

MIS is based on the monotone iterative methods in the ordered Banach space; MPA, MMA and HLA are based on the numerical implementation of the mountain pass lemma and the min-max theorem in the critical point theory. MPA was proposed by Choi and McKenna to compute the solutions with the Morse Index (MI) 0 or 1. Ding, Costa and Chen established HLA for sign-changing solution (MI=2) of semilinear elliptic problems. Li and Zhou designed a new min-max algorithm (MMA) to find multiple saddle points with any Morse index which is more constructive than the traditional min-max theorem. Chen and Xie proposed SEM, which searches the initial guess based on the linear combination of the eigenfunctions of the linearized problem

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and then gets the better initial guess by the continuation method for the discretized problem by the finite element method. Yang used BM to solve the BVPs with any Morse index and different regions.

It is well known that nonlinear problems are usually solved by the Newton's method. Nevertheless, when the grid becomes fine, the direct implementation of the Newton method leads to a costly computation. In this paper, a contractor iteration method will be used for such problems^[13,14,15], which converges quadratically as Newton's method. This method mainly consists of the matrix-matrix and the matrix-vector multiplications.

This method makes the computation of the numerical solution of nonlinear equations become very cheap. In order to illustrate that the present method is better than the Newton's Method, we will compute and visualize solutions of (1) in various domains and compare it with Newton's Method.

II. CONTRACTOR ITERATION ALGORITHM

Assume $\Omega = \Omega_0 = (-1, 1) \times (-1, 1)$, then (1) turns into

$$\begin{cases} \Delta u + u^p = 0, & (x, y) \in \Omega_0, \\ u = 0, & (x, y) \in \partial\Omega_0. \end{cases} \quad (2)$$

In two dimensions, the mesh is the set of points

$$(x_i, x_j) = (ih, jh)$$

that lie within the region Ω_0 . Approximating the partial derivatives with centered second differences gives the 5-point discrete Laplacian

$$\Delta_h u(x, y) = \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2} + \frac{u(x, y+h) - 2u(x, y) + u(x, y-h)}{h^2}$$

Substitute Δu of (2) with $\Delta_h u$. Let $F(u) = \Delta_h u + u^p$, the standard Newton's method is of form

$$u_h^{n+1} = u_h^n - DF^{-1}(u_h^n)F(u_h^n) \quad (3)$$

where n is the number of iterations and $DF(u_h^n)$ is the Jacobian matrix of F at u_h^n which is of the form

$$DF(u_h^n) = \Delta_h + \text{spdiags}(3h^2 u_h^2, [0], n, n) \quad (4)$$

with *spdiags* is a MATLAB code.

We will give the algorithm by applying the concept of contractors, then we shall use it to compute (2) in (III) with this algorithm.

ALGORITHM 1

1. Select a starting value of solution u_0 and a matrix L_0 close to $DF(u_0)$.
2. $K_0 = L_0^{-1}$, $\Gamma_0 = I_N$, where I_N is the identity matrix.
3. $u_{n+1} = u_n - K_n F(u_n)$.
4. $\Lambda_n = DF(u_{n+1})\Gamma_n - L_n$.
5. $L_{n+1} = L_n M_n + L_n + \Lambda_n$, where M_n is a matrix such that $\|M_n\| \leq c\|\Lambda_n\|$ with $c \geq 1$ independent of n .
6. $\Gamma_{n+1} = \Gamma_n(I_N + M_n)$.
7. $K_{n+1} = \Gamma_{n+1}^{-1} L_{n+1}^{-1}$.
8. Check the convergence of the iteration. Unless convergent, go to step 3.

The sequence $\{u_n\}$ converges quadratically if K_n are sufficiently good approximations for $DF(u_n)^{-1}$ for all n ^[15]. This scheme seems to be complicated and the determination of M_n is not unique, As remarked by Scheurle^[15] who took $M_n = 0$ for all n , we have $\Gamma_{n+1} = I_N$, $L_{n+1} = DF(u_{n+1})$ and hence $K_{n+1} = DF(u_{n+1})^{-1}$. This is Newton's method. By taking $M_n = -L_n^{-1}\Lambda_n$, we have $L_{n+1} = L_0$. We consider the latter further. It is rather simple and L_0 is the only matrix that has to be inverted during the iterations.

We are paying particular attention to $DF(u_h^n)$ of (4). Let h be small enough(in fact, we have to finely partition the interval), Δ_h is main part of $DF(u_h^n)$. The complete matrix Δ_h has -4 's on its diagonal, four 1 's off the diagonal in most of its rows, two or three 1 's off the diagonal in some of its rows, and zeros elsewhere. For the example of region above, Δ_h would be 25 by 25(see Fig.1).

So we define step 1' instead of step 1, and Steps 2-8 can

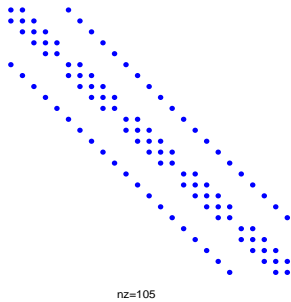


Fig.1. Sparsity structure of Δ_h

be reduced to the following steps 2'-5'.

ALGORITHM 2

- 1'. Select a starting value of solution u_0 and a matrix $L_0 = \Delta_h$.
- 2'. $K_0 = L_0^{-1}$.
- 3'. $u_{n+1} = u_n - K_n F(u_n)$.
- 4'. $K_{n+1} = K_n(2I_N - DF(u_{n+1})K_n)$.
- 5'. Check the convergence of the iteration. Unless convergent, go to step 3'.

III. METHOD & VISUALIZATION

To overcome the difficulty of the local convergence of Newton's Method or the Contractor Iteration Method and to compute positive solutions of (2) as many as possible, bifurcation and eigenpairs have been used in our article^[12,16]. A close description is briefly given below.

Let $p = 3$, We embed (2) to the nonlinear bifurcation problems with parameter λ of the following form:

$$\begin{cases} \Delta u + \lambda u + u^3 = 0, & (x, y) \in \Omega_0, \\ u = 0, & (x, y) \in \partial\Omega_0. \end{cases} \quad (5)$$

Considering the linearized equation of (5) at $u = 0$, we get

$$\begin{cases} \Delta \varphi + \lambda \varphi = 0, & (x, y) \in \Omega_0, \\ \varphi = 0, & (x, y) \in \partial\Omega_0. \end{cases} \quad (6)$$

It is well known that (6) always has a trivial solution. Further more, Eq.(6) has eigenpairs, namely eigenvalues $\lambda_{n,m} = (n^2 + m^2)\pi^2$ and corresponding eigenfunctions $\varphi_{n,m} = \sin(n\pi x)\sin(m\pi y)$. Therefore, $\varphi_{n,m} = \sin(n\pi x)\sin(m\pi y)$ are roots of (6) when $\lambda = \lambda_{n,m} = (n^2 + m^2)\pi^2$.

From the analysis above, we know that the solution branch which bifurcates from the first eigenvalue $\lambda_{1,1} = 2\pi^2$ is a positive solution branch. Bifurcation method will be applied to compute the positive solution of (2).

For $\lambda_0 = \lambda_{1,1} = 2\pi^2$, $\varphi_{1,1} = \sin(\pi x)\sin(\pi y)$, then let $L = \Delta + 2\pi^2$, $X = \{u | u \in C^2(\Omega_0), u|_{\partial\Omega_0} = 0\}$, $Y = \{u | u \in C^0(\Omega_0)\}$. we define inner product by

$$\langle u, v \rangle = 4 \int_0^1 \int_0^1 uv dx dy,$$

L is a Fredholm self-adjoint operator with index zero, and

$$N(L^*) = N(L) = \text{span}\{\varphi_{1,1}\}, \quad (7)$$

where $N(L)$ and $N(L^*)$ are the null space of L and L^* respectively. Space X and Y have the decomposition

$$X = N(L) \oplus M, \quad Y = N(L) \oplus R(L),$$

where $M = N(L)^\perp \cap X$, $R(L)$ is the range of L .

Let P be the orthogonal projector from Y to $R(L)$

$$Pz = z - (z, \varphi_0)\varphi_0. \quad z \in Y$$

Eq.(5) is equivalent to

$$PF(\tau\varphi_0 + \omega, \mu + \lambda_0) = 0, \quad \tau \in \mathbb{R}, \omega \in M \quad (8)$$

$$\langle \varphi_0, F(\tau\varphi_0 + \omega, \mu + \lambda_0) \rangle = 0. \quad (9)$$

where $\mu = \lambda - \lambda_0$, $u = \tau\varphi_0 + \omega$. Since $PF_\omega(0, \lambda_0) = PF_u(0, \lambda_0) = PL = L$, and L restricted in M is regular, (8) has a unique solution $\omega = \omega(\tau, \mu)$ which satisfies $\omega(0, 0) = 0$ by the implicit function theorem^[17].

Substituting $\omega(\tau, \mu)$ into (9) yields

$$g(\tau, \mu) = \langle \varphi_0, F(\tau\varphi_0 + \omega(\tau, \mu), \mu + \lambda_0) \rangle = 0. \quad (10)$$

Then we get

$$F(u, \lambda) = F(\tau\varphi_0 + \omega, \mu + \lambda_0) = \Delta\omega + \lambda_0\omega + h(\tau, \mu), \quad (11)$$

where $h(\tau, \mu) = \mu(\tau\varphi_0 + \omega) + (\tau\varphi_0 + \omega)^p, \omega = \omega(\tau, \mu)$. (5) is transformed into (12).

$$\begin{cases} \Delta\omega + (\mu + \lambda_0)\omega + \eta\tau\omega + (\tau\varphi_0 + \omega)^3 = 0, & (x, y) \in \Omega_0, \\ \omega = 0, & (x, y) \in \partial\Omega_0, \\ (\varphi_0, \omega) = 0. \end{cases} \quad (12)$$

Newton's Method and Contractor Iteration Method are used to solve the φ_0 and τ , then we get $u = \tau\varphi_0 + \omega$.

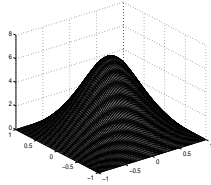


Fig.2. positive solution on square.

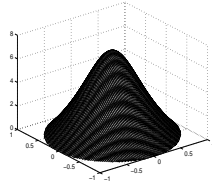


Fig.3. positive solution on disk.

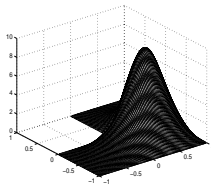


Fig.4. positive solution on L-shaped domain.

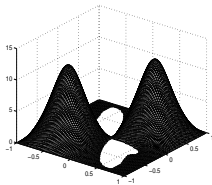


Fig.5. positive solution on the exterior of a "Butterfly".

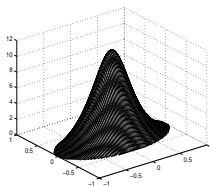


Fig.6. positive solution on ellipse.

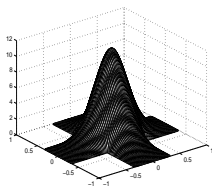


Fig.7. positive solution on crisscross.

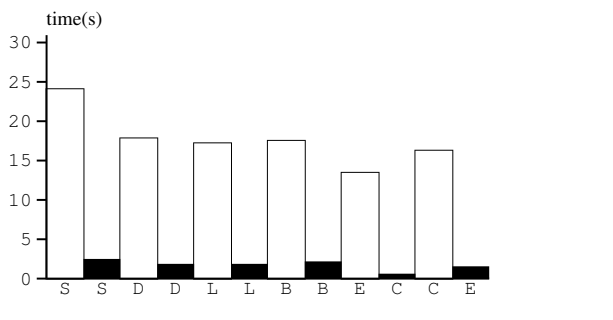


Fig.8 Newton's Method Contractor Iteration Method
 'S' denotes square; 'D' denotes disk; 'L' denotes L-shaped domain; 'B' denotes the exterior of a "Butterfly"; 'C' denotes crisscross; 'E' denotes ellipse. Y-axis denotes the time(s) which Newton's Method and Contractor Iteration Method spend in various domains.

We visualize the time(s) which Newton's Method and Contractor Iteration Method spend in various domains with the interval $h = 1/100$ in Fig.8. We can see that the direct implementation of the Newton method leads to a costly

computation and the computation of Contractor Iteration Method is very cheap.

Some different symmetric positive solutions of (2) are visualized from Fig.2 to Fig.7 on some different domains. TABLE I illustrates the symmetric properties.

TABLE I THE SOLUTIONS OF (2) WITH DIFFERENT SYMMETRIES.

Shape of the domain	Symmetry
Square(Fig.2)	D_4
Disk(Fig.3)	$O(2)$
L-shaped domain(Fig.4)	Σ_2'
The exterior of a "Butterfly"(Fig.5)	Σ_d
Crisscross(Fig.6)	D_4
Ellipse(Fig.7)	Σ_M

To solve the nonsymmetric solution of (2), the eigenvalues of Jacobian $DF(u)$ are monitored. Eq(12) must be computed with the μ being carefully continued. We can find symmetric-breaking solution of (2) by continuation(see Fig.9). Because of μ being carefully continued, the Contractor Iteration Method is better than Newton's Method.

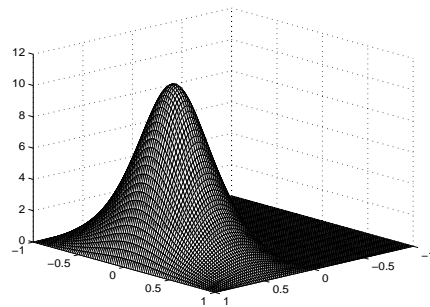


Fig.9. symmetric-breaking solution of (2).

IV. CONCLUSION

A contractor iteration method is presented and is implemented numerically for solving positive solutions of Lane-Emden(-Fowler) equation. Computation times of the present method and Newton's method are both discussed. By comparing the time in Fig.8, it is shown that Contractor Iteration Method works more efficiently than Newton's Method. Some similar equation, such as Henon's equation^[18], Chandrasekhar equation^[19], and generalized Lane-Emden system^[20], can also use this method to solve positive solutions or multiple solutions.

ACKNOWLEDGMENT

This work was supported by the Excellent Youth Scholars Foundation and the Natural Science Foundation of Anhui Province of P.R.China(No.2009SQRZ083, KJ2009B076Z, 090416222), Shanghai Normal University Academic Project(NO.SK200936).

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