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A Comparison Study of a Symmetry Solution of Magneto-Elastico-Viscous Fluid along a Semi-Infinite Plate with Homotopy Perturbation Method and4th Order Runge–Kutta Method

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Abstract**—**The equations governing the flow of an electrically conducting, incompressible viscous fluid over an infinite flat plate in the presence of a magnetic field are investigated using the homotopy perturbation method (HPM) with Padé approximants (PA) and 4th order Runge–Kutta method (4RKM). Approximate analytical and numerical solutions for the velocity field and heat transfer are obtained and compared with each other, showing excellent agreement. The effects of the magnetic parameter and Prandtl number on velocity field, shear stress, temperature and heat transfer are discussed as well.

Keywords**—**Electrically conducting elastico-viscous fluid; symmetry solution; Homotopy perturbation method; Padé approximation; 4th order Runge–Kutta; Maple.

I. INTRODUCTION

THE boundary layer flow of an electrically conducting, incompressible viscous fluid over a continuously flat plate is often encountered in many engineering and industrial processes such as polymer technology, aerodynamic extrusion of plastic sheets and so on.

The problem of a fluctuating flow of a magneto-elastico-viscous fluid along an infinite flat plate under the condition of very small elastic parameter was studied in [1–3]. This type of problems may be approximated to a problem of fluctuating flow of a magneto-viscous fluid in case of consideration a very small elastic parameter. Frater [2] pointed out that the solution for the velocity should tend to the Newtonian value when the elastic parameter vanishes.

In this paper the flow of an electrically conducting, incompressible elastico-viscous fluid along a flat plate coinciding with the plane y=0 is considered, such that the flow is confined to the region y>0. The magnetic field is assumed to be normal to the plate on which the boundary layer is formed. The main purpose of this work is to investigate the effects of the magnetic field parameter and Prandtl number on the velocity and shear stress of the fluid analytically using the classical homotopy perturbation method (HPM) with the enhancement of Padé approximants (PA) and using the developed HPM as well; and numerically using the well-known 4th order Runge–Kutta method (4RKM).

The classical homotopy perturbation method, based on series approximation, is one among the newly developed analytical methods for strongly nonlinear problems and has been proven successful in solving a wide class of nonlinear differential equations [5–9]. The developed HPM can be achieved by introducing addition linear operator(s) with unknown parameter(s) that can be chosen suitably to fulfill certain desirable criteria and identified optimally [10–12].

In this paper, we are interested in applying the classical HPM with PA technique, developed HPM and 4RKM for obtaining analytical and numerical solutions of the boundary layer flow of an electrically conducting elastico-viscous fluid along an infinite flat plat with heat transfer in presence of a magnetic field normal to the plate. The comparison of the analytical solutions with the numerical solution has been made and excellent agreement noted.

II. GOVERNING EQUATIONS

In terms of the stream function ψ the governing equations of a steady two-dimensional incompressible flow of an electrically conducting elastico-viscous fluid over a semi-infinite flat plate coinciding with the plane y=0, such that the flow is confined to the region y>0 under the influence of a constant transverse applied magnetic field normal to the plate on which the boundary layer is formed are given in [2, 3]. The magnetic Reynolds number is assumed to be small and negligible in comparison to the applied magnetic filed.

The governing equations describe fluid motion and temperature are given by

$$\frac{\partial^3\psi}{\partial y^3}-\frac{M}{x}\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}$$

$$=k\left[\frac{\partial\psi}{\partial y}\frac{\partial^4\psi}{\partial x\partial y^3}-\frac{\partial\psi}{\partial x}\frac{\partial^4\psi}{\partial y^4}+\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}\right], \quad (1)$$

$$\frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2}=\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}, \quad (2)$$

where M is the magnetic parameter, k is a small elastic parameter representing the non-Newtonian character of the fluid and Pr is the Prandtl number.

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The boundary conditions of the problem are:

$$y=0: \quad \frac{\partial\psi}{\partial y}=0, \quad \frac{\partial\psi}{\partial x}=0, \quad T=T_0, \tag{3}$$

$$y\to\infty: \quad \frac{\partial\psi}{\partial y}\to\psi_0, \quad T\to 0, \tag{4}$$

where T_0 and ψ_0 are constants.

Because of being the elastic parameter k is small and may be neglected, the solution of the problem described by Eqs. (1) and (2) may be approximated to the solution of the Newtonian fluid described by the following equations:

$$\frac{\partial^3\psi}{\partial y^3}-\frac{M}{x}\frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}=0, \tag{5}$$

$$\frac{\partial^2 T}{\partial y^2}+\Pr\left(\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}\right)=0, \tag{6}$$

under the same boundary conditions.

This approximation gives excellent results in case of small values of k as we will see in next sections.

III. Invariant transformation

Using one-parametric group transformation included in PDEtools package of Maple software, the two-independent variables PDEs (5) and (6) will be transformed into ODEs in only one-independent similarity variable.

A. The complete set of invariants

The invariants set obtained by Maple 11 are:

$$\eta(x,y)=x, \quad F(\eta)=\frac{g(x)\psi(x,y)-y}{g(x)}, \quad \theta(\eta)=T(x,y), \tag{7}$$

$$\eta(x,y)=x, \quad F(\eta)=\psi(x,y), \quad \theta(\eta)=\frac{g(x)T(x,y)-y}{g(x)}, \tag{8}$$

$$\eta(x,y)=x, \quad F(\eta)=\psi(x,y), \quad \theta(\eta)=T(x,y)\exp\left(-\frac{y}{g(x)}\right), \tag{9}$$

$$\eta(x,y)=\frac{y}{\sqrt{x}}-\int\frac{g(x)}{\sqrt{x^3}}dx, \quad F(\eta)=\frac{\psi(x,y)}{\sqrt{x}}, \quad \theta(\eta)=T(x,y), \tag{10}$$

where η is the similarity variable, F and θ are invariants of the dependent variables ψ and T respectively and g is an arbitrary function.

From the invariants set (7)–(10), it is clear that the invariants in Eq. (10) are the only which makes both of ψ and T a function in x and y. Therefore, we used Eq. (10) for doing the similarity transformation of PDEs (5) and (6).

B. The ordinary differential equations invariant transformation

Substituting Eq. (10) into PDEs (5) and (6) yields the following system of ODEs:

$$\frac{d^3}{d\eta^3}F(\eta)+\frac{1}{2}F(\eta)\frac{d^2}{d\eta^2}F(\eta)-M\frac{d}{d\eta}F(\eta)=0, \tag{11}$$

$$\frac{d^2}{d\eta^2}\theta(\eta)+\frac{1}{2}\Pr F(\eta)\frac{d}{d\eta}\theta(\eta)=0. \tag{12}$$

By examining invariants in Eq. (10) and boundary conditions (3) and (4), function $g(x)$ should be equal to zero in order to make the left boundary point constant at y=0. Therefore, the suitable similarity invariants are:

$$\eta(x,y)=\frac{y}{\sqrt{x}}+c, \quad F(\eta)=\frac{\psi(x,y)}{\sqrt{x}}, \quad \theta(\eta)=T(x,y), \tag{13}$$

where c is an arbitrary constant (left boundary point of the similarity boundary problem). Hence, the appropriate corresponding conditions are:

$$\eta=c: \quad \frac{\partial F(\eta)}{\partial\eta}=0, \quad F(\eta)=0, \quad \theta(\eta)=T_0, \tag{14}$$

$$\eta\to\eta_\infty: \quad \frac{\partial F(\eta)}{\partial\eta}\to\psi_0, \quad \theta(\eta)\to 0. \tag{15}$$

It is obvious that Eq.(11) is the Blasius equation in case of $M = 0$ [4].

For convenience and comparison with results in [3], let $\psi_0 = T_0 = 1$, c=0 and η_∞=6.

IV. Analytical solution using the classical HPM with PA technique

Following the standard procedures of the HPM described in [5–9], the system (11) and (12) should be written in the classical homotopy form,

$$(1-p)\left(\frac{d^3}{d\eta^3}U(\eta)-\frac{d^3F_0}{d\eta^3}\right)$$
$$+p\left(\frac{d^3}{d\eta^3}U(\eta)+\frac{1}{2}U(\eta)\frac{d^2}{d\eta^2}U(\eta)-M\frac{d}{d\eta}U(\eta)\right)=0, \tag{16}$$

$$(1-p)\left(\frac{d^2}{d\eta^2}V(\eta)-\frac{d^2\theta_0}{d\eta^2}\right)$$
$$+p\left(\frac{d^2}{d\eta^2}V(\eta)+\frac{1}{2}\Pr U(\eta)\frac{d}{d\eta}V(\eta)\right)=0, \tag{17}$$

where

$U\equiv F,\ V\equiv\theta,\ F_0=U_0(c)=F(c)$ and $\theta_0=V_0(c)=\theta(c)$.

One can now try to obtain a solution of system (21) and (22) in form of,

$$U(\eta)=U_0(\eta)+pU_1(\eta)+p^2U_2(\eta)+..., \tag{18}$$

$$V(\eta)=V_0(\mu)+p\,V_1(\eta)+p^2\,V_2(\eta)+..., \tag{19}$$

where U_n and V_n, n=0, 1, 2, … are functions yet to be determined.

Substituting Eqs. (18) and (19) into system (16) and (17), and arranging the coefficients of "p" powers yields:

$$p^0: \ U_0'''=0, \qquad V_0''=0,$$
$$p^1: \ U_1'''+\frac{1}{2}U_0U_0''-MU_0'=0, \qquad V_1''+\frac{1}{2}\Pr U_0V_0'=0,$$
$$p^2: \ U_2'''-MU_1'+\frac{1}{2}U_0U_1''+\frac{1}{2}U_1U_0''=0,$$
$$V_2''+\frac{1}{2}\Pr U_0V_1'+\frac{1}{2}\Pr U_1V_0'=0,$$
$$\vdots, \tag{20}$$

with corresponding initial conditions,

$$U_0(0)=0,\ \ U_0'(0)=0,\ \ U_0''(0)=\alpha,\ \ V_0(0)=1,\ \ V_0'(0)=\beta,$$

$$U_n=U_n'=U_n''=V_n=V_n'=0,\ \text{at}\ \ \eta=0,\ \ \text{for}\ n=1,2,3,...,\quad (21)$$

where unknown initial values α and β can be calculated using the boundary conditions in Eq. (15) after obtaining a closed form expression to the solution.

We continued solving system (20) corresponding to initial conditions (21) for U_n and V_n, n=0, 1, 2, … until n=6 and hence obtained a six-term approximations $F_6(\eta)=\sum_{n=0}^{6}U_n$ and $\theta_6(\eta)=\sum_{n=0}^{6}V_n$.

It is known that Padé approximations (PA) [13] have the advantage of manipulating the polynomial approx-imation into a rational function of polynomials. This manipulation provides us with more information about the mathematical behavior of the solution. Besides that, a power series solution is not useful for large value of η. Therefore, the combination of the series solution through HBM or any other series solution method with the Padé approximation provides an effective tool for handling boundary value problems on semi-infinite domains. It is a known fact that Padé approximation converges on the entire real axis if the solution is free of singularities on the real axis.

So, the more accurate analytical solutions will be obtained after applied PA [M/N] to both of F_6 and θ_6 such that $M+N\le$ (highest power of η in the series solution). We have been applied PA [10/10] to obtain the analytical solution for the problem, say $F_{6[10/10]}$ and $\theta_{6[10/10]}$.

V. Analytical solution using developed HPM

According to the developed HPM [10–12], a homotopy of the system (11) and (12) may be written as

$$\left(\frac{d^3}{d\eta^3}F(\eta)+a\right)+p\left(\frac{1}{2}F(\eta)\frac{d^2}{d\eta^2}F(\eta)-M\frac{d}{d\eta}F(\eta)-a\right)=0,\quad (22)$$

$$\left(\frac{d^2}{d\eta^2}\theta(\eta)+b\right)+p\left(\frac{1}{2}\Pr F(\eta)\frac{d}{d\eta}\theta(\eta)-b\right)=0,\quad (23)$$

where a and b are unknown constants to be further identified.

Using p as an expanding parameter as that in the classic perturbation method, we have

$$p^0: F_0'''+a=0,\qquad F_0(0)=F_0'(0)=0,\ F_0'(6)=1,$$
$$\theta_0''+b=0,\qquad \theta_0(0)=1,\ \theta_0(6)=0,$$
$$p^1: F_1'''+\frac{1}{2}F_0F_0''-MF_0'-a=0,\qquad F_1(0)=F_1'(0)=F_1'(6)=0,$$
$$\theta_1''+\frac{1}{2}\Pr F_0\theta_0'-b=0,\qquad \theta_1(0)=\theta_1(6)=0.\quad (24)$$

Solving the system (24) and setting p = 1, we obtain a first-order approximate solution which reads

$$F(\eta)=F_0(\eta)+F_1(\eta)=\left(\frac{7}{48}+\frac{9}{20}a-\frac{27}{20}a^2-\frac{9}{2}Ma-\frac{1}{2}M\right)\eta^2$$
$$+\left(\frac{1}{144}M+\frac{1}{8}Ma\right)\eta^4-\left(\frac{1}{120}Ma+\frac{1}{240}a+\frac{1}{8640}+\frac{3}{80}a^2\right)\eta^5$$
$$+\left(\frac{1}{2160}a+\frac{1}{120}a^2\right)\eta^6-\frac{1}{2520}a^2\eta^7,\quad (25)$$

$$\theta(\eta)=\theta_0(\eta)+\theta_1(\eta)=1-\left(\frac{1}{6}+\frac{27}{10}ba\Pr+\frac{27}{20}\Pr a+\frac{9}{20}\Pr b+\frac{1}{8}\Pr\right)\eta$$
$$-\frac{1}{360}\Pr ba\eta^6+\left(\frac{1}{1728}\Pr+\frac{1}{96}\Pr a-\frac{1}{96}\Pr b-\frac{3}{16}ba\Pr\right)\eta^4$$
$$+\left(\frac{1}{20}ba\Pr-\frac{1}{1440}\Pr a+\frac{1}{480}\Pr b\right)\eta^5.\quad (26)$$

There are many approaches for identification of the unknown parameters in the obtained solution. One of those methods is weighted residuals, especially the least squares method [10–12]. For the present problem, we set

$$\int_0^6 R_F\frac{\partial R_F}{\partial a}d\eta=0,\quad \text{and}\quad \int_0^6 R_\theta\frac{\partial R_\theta}{\partial b}d\eta=0,\quad (27)$$

to identify the unknown constants a and b,where R_F and R_θ are the residuals

$$R_F=F'''+\frac{1}{2}FF''-MF',\ \text{and}\ \ R_\theta=\theta''+\frac{1}{2}\Pr F\theta'.\quad (28)$$

VI. Results and discussion

With the analytical solution given by $F_{6[10/10]}$ and $\theta_{6[10/10]}$ using the classical HPM with PA technique, approximate values of $\alpha=F''(0)$ and $\beta=\theta'(0)$ can be calculated using the conditions in Eq. (15). Some numerical results of α and β that obtained from $F'_{6[10/10]}(\eta_\infty)=\psi_0$ and $\theta_{6[10/10]}(\eta_\infty)=0$ are presented in Table I for different values of M and Pr when $\eta_\infty=6$ and $\psi_0=1$.

With the first-order approximate solution arises in Eqs. (25) and (26), approximate values of unknown constants a and b are optimally identified using Eqs. (27) and (28) and presented in Table II for M=0 and M=0.5 at Pr=0.7.

In order to obtain a numerical solution, we have solved the initial value problem of Eqs. (11) and (12) corresponding to conditions in Eq. (14) and the numerical values arise in Table I using the well-known 4RKM.

Figs. 1(a), (b) and (c) show the variations of the fluid stream function, velocity and shear stress with η. As shown in Figs. 1(a) and (b), the stream function F and fluid velocity F' decrease and come near to each other as the magnetic parameter M increases. In addition, Fig. 1(b) shows that the smaller the value of M, the faster it reaches the maximum value of F'. From Fig. 1(c), it is clear that the behavior of the shear stress F'' depends on the magnetic parameter and the distance. In case of M=0, the shear stress starts with the high value, and then decreases with increasing distance. Oppositely for M>0, the shear stress starts with a lower value, and then increases with the distance.

Figs. 2(a) and (b) show the variations of the temperature and heat transfer with η. As shown in Fig. 2(a), the temperature θ increases with the increasing of M. For M=1, the temperature almost linearly depends on η. From Fig. 2(b), it is clear that the heat transfer θ' starts with a higher value for the lower values of M and then decreases. In addition, for the higher values of M, the behavior of the heat transfer with η tends to be uniform and takes a horizontal shape.

Figs. 3(a) and (d) illustrate the effect of Prandtl number Pr on the temperature and heat transfer at M=0. The results are obtained for Pr = 0.5, 1, 2 and 3. Form Fig. 3, it clear that the temperature and heat transfer rapid decrease as the Prandtl number increases. Moreover, the rapid decrease of θ and $-\theta'$ becomes more obvious for larger values of Pr.

To demonstrate the acceptability and accuracy of developed HPM results, even though we used only the first-order approximate solution, the behaviors of the fluid stream function, velocity, temperature and heat transfer using the closed form solutions in Eqs. (30) and (31), with the values in Table II, are illustrated in Figs. 4(a), (b), (c) and (d) in a comparison with 4RKM results.

It is obvious that the results of α and β obtained by the classical HPM with PA technique are used for obtaining the numerical solution using 4RKM by converting the boundary value problem to an initial value one. Moreover, the analytical solutions using the classical HPM with PA technique and developed HPM in great agree with the numerical solution using the 4[th] order Runge–Kutta method.

The results obtained in this investigation, in case of the elastic parameter k=0, agree with that obtained in [3] in case of k=0.2. Hence, the problem of fluctuating flow of a magneto-elastico-viscous fluid over a semi-infinite flat plate under the condition of a very small elastic parameter k can be approximated to the problem of fluctuating flow of a magneto-viscous fluid, i.e. k=0. The present results of F for M=0 agree with that obtained in [4] as well.

TABLE II: Numerical values of a and b for M=0 and M=0.5 at Pr=0.7

M	a	b
0.0	0.07060198460	–0.0608015174
0.5	–0.07567076914	–0.02106461217

TABLE I: Numerical values of α= F″(0) and β=Θ′(0) for different values of M and Pr

M	α	β				
		Pr=0.5	Pr=0.7	Pr=1.0	Pr=2.0	Pr=3.0
0.0	0.33465139	–0.26444536	–0.29502831	–0.33329657	–0.42496497	–0.49297215
0.5	0.02846081	–0.18468413	–0.19143043	–0.20104155	–0.22873935	–0.25055399
1.0	0.00571327	–0.17366347	–0.17639173	–0.18040127	–0.19294501	–0.20402326

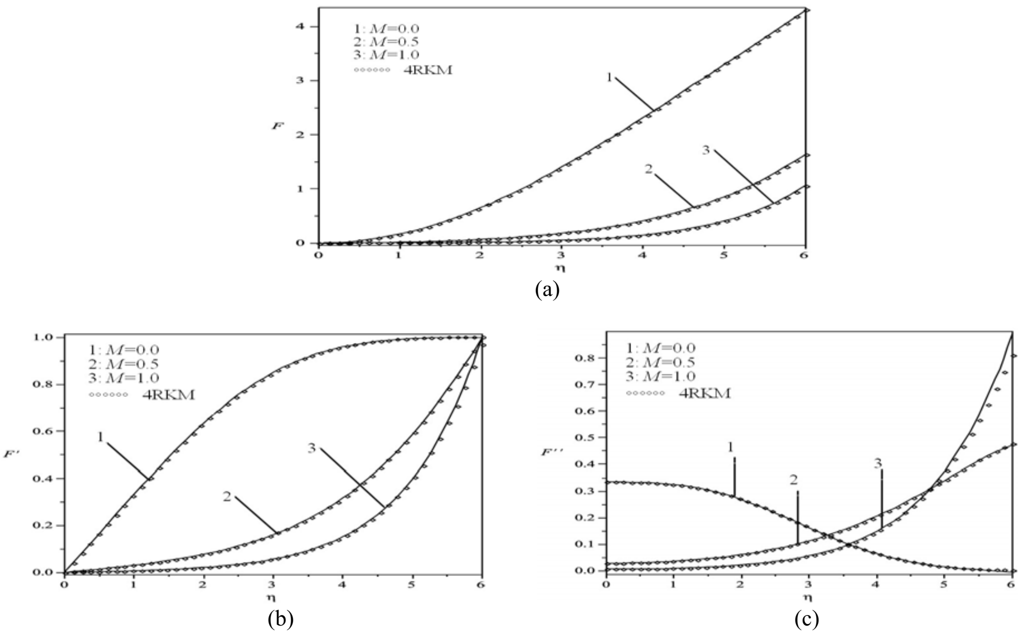

Fig. 1. Profiles of (a) stream function; (b) velocity and (c) shear stress using analytical results of $F_{6[10/10]}$; $F'_{6[10/10]}$ and $F''_{6[10/10]}$ respectively and numerical results of 4RKM for various values of M at Pr=0.7

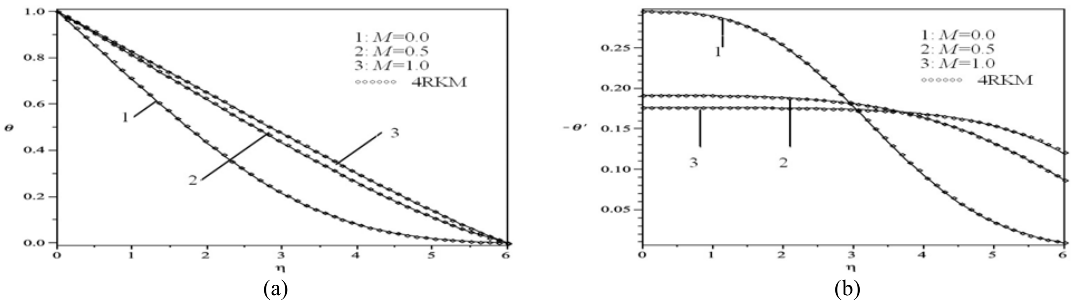

Fig. 2. Profiles of (a) temperature and (b) heat transfer using analytical results of $\theta_{6\,[10/10]}$ and $-\theta'_{6[10/10]}$ respectively and numerical results of 4RKM for various values of *M* at Pr=0.7

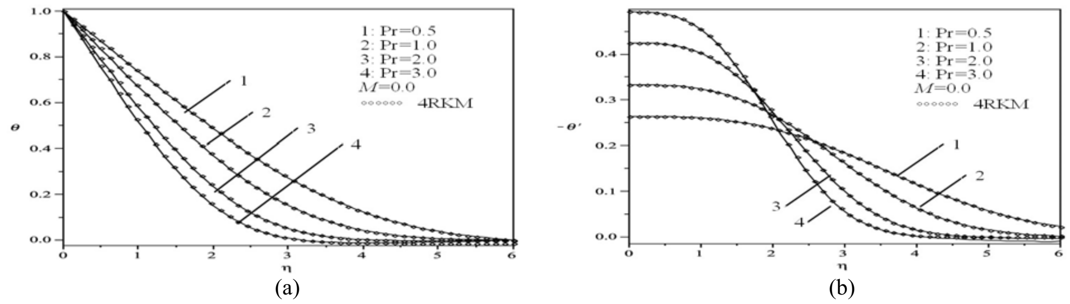

Fig. 3. Profiles of (a) temperature and (b) heat transfer using analytical results of $\theta_{6\,[10/10]}$ and $-\theta'_{6[10/10]}$ respectively and numerical results of 4RKM for various values of Pr at *M*=0

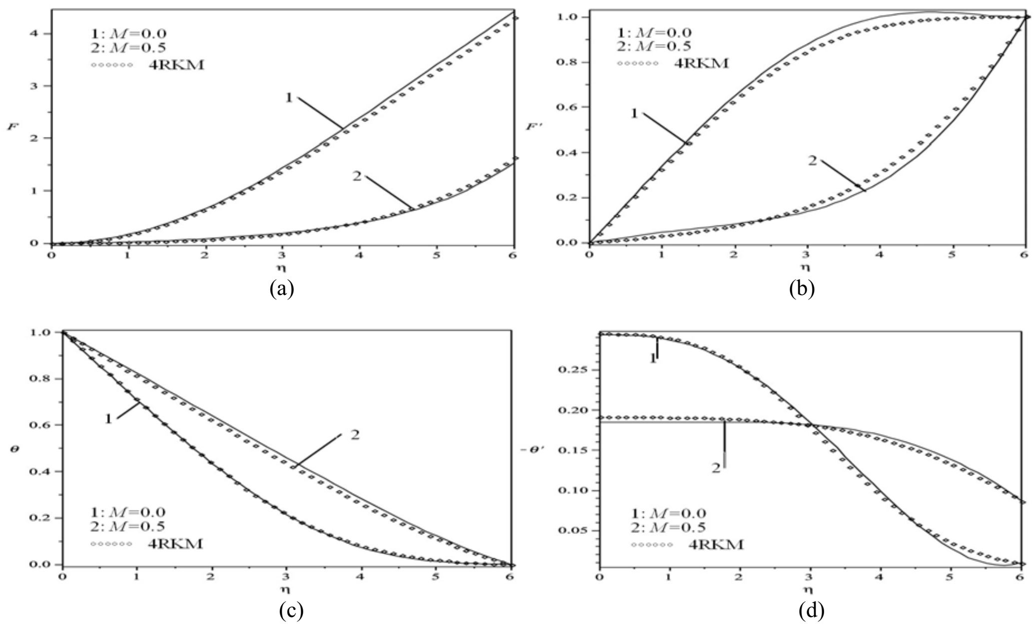

Fig. 4. Profiles of (a) stream function; (b) velocity; (c) temperature and (d) heat transfer using developed HPM analytical results and 4RKM numerical results for *M*=0 and *M*=0.5 at Pr=0.7

VII. Conclusion

The homotopy perturbation method is applied to the system of nonlinear differential equations that describe a magneto-viscous fluid along a semi-infinite flat plate in presence of a magnetic field. The excellent agreement of the analytical solution with the 4RKM numerical one shows the reliability and efficiency of the HPM. The behaviors of fluid stream function, velocity, shear stress, temperature and heat transfer illustrated by the graphs are consistent with the graphs obtained in [3, 4] and therefore further establish the reliability and effective-ness of the HPM. It has been demonstrated that the HPM can be applied advantageously even when the flow is governed by a BVP consisting of more than one differential equation.

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