International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:3, No:11, 2009

$[a,b]$-Factors Excluding Some Specified Edges In Graphs

Sizhong Zhou, Bingyuan Pu

Abstract—Let G be a graph of order n, and let a, b and m be positive integers with $1 \leq a < b$. An $[a,b]$-factor of G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. In this paper, it is proved that if $n \geq \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$ and $\delta(G) > n+a+b-2\sqrt{bn+1}$, then for any subgraph H of G with m edges, G has an $[a,b]$-factor F such that $E(H) \cap E(F) = \emptyset$. This result is an extension of that of Egawa [2].

Keywords—graph, minimum degree, $[a,b]$-factor.

I. Introduction

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on [1]. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term graph instead of network.

The graphs considered here will be finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, the neighborhood set $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. Furthermore, the minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we denote by $G-S$ the subgraph obtained from G by deleting all the vertices of S together with the edges incident with the vertices of S. Let S and T be two disjoint vertex subsets of G, we use $E_G(S,T)$ to denote the set of edges with one end in S and the other end in T and set $S - S' = S \setminus S'$.

Let a and b be nonnegative integers with $a \leq b$. An $[a,b]$-factor of G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. Note that if $a = b = k$, then an $[a,b]$-factor is a k-factor.

Let us first introduce a known result which proves a minimum degree condition for the existence of k-factors in graphs.

Sizhong Zhou is with the School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, People's Republic of China, e-mail: zsz_cumt@163.com.

Bingyuan Pu is with the Department of Fundamental Course, Chengdu Textile College, Chengdu, Sichuan 611731, People's Republic of China, e-mail: skypuby@163.com

Manuscript received April 25, 2009.

Theorem 1. (Egawa [2]). Let $k \geq 2$ be an integer, and let G be a graph of order n with kn even. If $\delta(G) > n + 2k - 2\sqrt{kn+1}$, then G has a k-factor.

The following results on the existence of an $[a,b]$-factor which excludes some specified edges are known.

Theorem 2. (Zhou [5]). Let G be a graph, and let a and b be two non-negative integers with $a < b$, M is a maximum matching of G. If $a \leq d_G(x)$ and $b(d_G(y)-1) \geq ad_G(x)$ for each $x, y \in V(G)$, then G has an $[a,b]$-factor excluding M.

Theorem 3. (Zhou [6]). Let G be a graph of order n, and let a, b and m be nonnegative integers with $1 \leq a < b$. If $n \geq \frac{(b-1)(a+b-1)(a+b-2)+2bm}{b(b-1)}$ and $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-2m+2}$, then for any subgraph H of G with m edges, G has an $[a,b]$-factor F excluding H.

Theorem 4. (Kano [3]). Let a, b, m and k be positive integers such that $1 \leq a < b$ and $2 \leq k < \frac{a+b+1-m}{a}$. Let G be a graph of order $n > \frac{(a+b)((k+m)(a+b-1)-1)}{b}$. If

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{an}{a+b}$$

for every independent set $\{x_1, x_2, \cdots x_k\} \subseteq V(G)$, then for any subgraph H of G with m edges and $\delta(G-H) \geq a$, G has an $[a,b]$-factor F excluding H.

Theorem 5. (Kano [3]). Let a, b and m be integers such that $1 \leq a < b$ and $m \geq 1$. Suppose that G is a graph of order $n > \frac{(a+b)((m+1)(a+b+1)-5)}{b}$ and $\delta(G) \geq \frac{an}{a+b}$. Then for any subgraph H of G with m edges, G has an $[a,b]$-factor F excluding H.

In this paper, we give a new minimum degree condition for the existence of an $[a,b]$-factor which excludes some specified edges. Our result is an extension of Theorem 1.

Theorem 6. Let a, b and m be positive integers with $1 \leq a < b$, and let G be a graph of order $n \geq \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$. If $\delta(G) > n+a+b-2\sqrt{bn+1}$, then for any subgraph H of G with m edges, G has an $[a,b]$-factor F excluding H $(i.e., E(H) \cap E(F) = \emptyset)$.

We do not know whether the condition $\delta(G) > n+a+b-2\sqrt{bn+1}$ in Theorem 6 can be replaced by $\delta(G) \geq n+a+b-2\sqrt{bn+1}$.

In Theorem 6, if $m = 1$, then we get the following corollary.

Corollary 1. Let a and b be integers with $1 \leq a < b$, and let G be a graph of order $n \geq \frac{(a+b-1+\sqrt{a+b-1})^2-1}{b}$. If $\delta(G) > n+a+b-2\sqrt{bn+1}$, then for any $e \in E(G)$, G has an $[a,b]$-factor F excluding e.

II. Proof of Main Theorems

In order to prove Theorem 6, we depend on the following lemmas.

Lemma 2.1. (Lam etc. [4]). Let $1 \le a < b$ be integers, and let G be a graph and H a subgraph of G. Then G has an $[a,b]$-factor F such that $E(H)\cap E(F)=\emptyset$ if and only if

$$b|S|+\sum_{x\in T} d_{G-S}(x)-a|T| \ge \sum_{x\in T} d_H(x)-e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

Lemma 2.2. Let a, b, m and n be positive integers. If $n \ge \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$, then $n+a+b-2\sqrt{bn+1}-m \ge a-1+\frac{a}{b}$.

Proof. Since $n \ge \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$, then we obtain

$$\begin{aligned}
& n+a+b-2\sqrt{bn+1}-m \\
\ge\ & \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}+a \\
& +b-2(a+b-1+\sqrt{(a+b+1)m-2})-m \\
=\ & \frac{(a+b-1)^2+(a+b+1)m-3}{b} \\
& +\frac{2(a+b-1)\sqrt{(a+b+1)m-2}}{b} \\
& -(a+b-2)-2\sqrt{(a+b+1)m-2}-m \\
\ge\ & \frac{(a+b-1)^2+(a+b+1)m-3}{b}-(a+b-2)-m \\
\ge\ & \frac{(a+b-1)^2+(a+b+1)-3}{b}-(a+b-2)-1 \\
=\ & \frac{(a+b)(a+b-1)-1}{b}-(a+b-1) \\
=\ & \frac{a(a+b-1)-1}{b} \\
=\ & \frac{(a-1)(a+b-1)+a+b-2}{b} \\
\ge\ & a-1+\frac{a}{b}.
\end{aligned}$$

Completing the proof of Lemma 2.2.

Proof of Theorem 6. Suppose a graph G satisfies the condition of the theorem, but has no desired $[a,b]$-factor for some subgraph H with m edges. Note that $\delta(G-H) > n+a+b-2\sqrt{bn+1}-m \ge a-1+\frac{a}{b}$ holds by Lemma 2.2. According to the integrity of $\delta(G-H)$, we obtain $\delta(G-H)\ge a$. Then by Lemma 2.1, there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S|+\sum_{x\in T}(d_{G-S}(x)-d_H(x)+e_H(x,S)-a)\le -1. \quad (1)$$

We choose such subsets S and T so that $|T|$ is minimum.

At first, we prove the following claims.

Claim 1. $|S|\ge 1$.

Proof. If $S=\emptyset$, then by (1), we obtain

$$\begin{aligned}
-1 &\ge \sum_{x\in T}(d_G(x)-d_H(x)-a) \\
&\ge \sum_{x\in T}(\delta(G-H)-a)\ge 0,
\end{aligned}$$

which is a contradiction.

Claim 2. $|T|\ge b+1$.

Proof. If $|T|\le b$, then by (1), Claim 1 and $\delta(G-H)\ge a$, we have

$$\begin{aligned}
-1 &\ge b|S|+\sum_{x\in T}(d_{G-S}(x)-d_H(x)+e_H(x,S)-a) \\
&\ge |T||S|+\sum_{x\in T}(d_{G-S}(x)-d_H(x)+e_H(x,S)-a) \\
&= \sum_{x\in T}(|S|+d_{G-S}(x)-d_H(x)+e_H(x,S)-a) \\
&\ge \sum_{x\in T}(|S|+d_{G-S}(x)-d_H(x)-a) \\
&\ge \sum_{x\in T}(d_G(x)-d_H(x)-a) \\
&\ge \sum_{x\in T}(\delta(G-H)-a) \\
&\ge 0.
\end{aligned}$$

This is a contradiction.

Claim 3. $d_{G-S}(x)-d_H(x)+e_H(x,S)\le a-1$ for each $x\in T$.

Proof. If $d_{G-S}(x)-d_H(x)+e_H(x,S)\ge a$ for some $x\in T$, then the subsets S and $T\setminus\{x\}$ satisfy (1). This contradicts the choice of S and T.

In view of Claim 2, we may define

$$h=\min\{d_{G-S}(x)-d_H(x)+e_H(x,S)|x\in T\},$$

and choose $x_1\in T$ such that $d_{G-S}(x_1)-d_H(x_1)+e_H(x_1,S)=h$. Then by Claim 3, we obtain

$$0\le h\le a-1.$$

According to the condition of Theorem 6, the following inequalities hold:

$$\begin{aligned}
n+a+b-2\sqrt{bn+1} &< \delta(G)\le d_G(x_1) \\
&\le d_{G-S}(x_1)+|S| \\
&\le h+d_H(x_1)-e_H(x_1,S)+|S|,
\end{aligned}$$

that is,

$$|S| > n+a+b-2\sqrt{bn+1}-(h+d_H(x_1)-e_H(x_1,S)). \quad (2)$$

In the following, we shall consider various cases according to the value of h and derive contradictions.

Case 1. $h=0$.

According to (1) and $|S|+|T|\le n$, we obtain

$$\begin{aligned}
-1 &\ge b|S|+\sum_{x\in T}(d_{G-S}(x)-d_H(x)+e_H(x,S)-a) \\
&\ge b|S|+h|T|-a|T| = b|S|-a|T| \\
&\ge b|S|-a(n-|S|) \\
&= (a+b)|S|-an,
\end{aligned}$$

which implies

$$|S|\le \frac{an-1}{a+b}. \quad (3)$$

On the other hand, by (2), (3) and $h = 0$, we have

$$\frac{an-1}{a+b} \geq |S| > n+a+b-2\sqrt{bn+1}-(d_H(x_1)-e_H(x_1,S)),$$

that is,

$$\begin{aligned} d_H(x_1) - e_H(x_1, S) &> n + a + b - 2\sqrt{bn+1} - \frac{an-1}{a+b} \\ &= \frac{bn+1}{a+b} + a + b - 2\sqrt{bn+1} \\ &= \frac{(\sqrt{bn+1} - (a+b))^2}{a+b} \geq 0. \end{aligned}$$

In view of the integrity of $d_H(x_1) - e_H(x_1, S)$, we obtain

$$d_H(x_1) - e_H(x_1, S) \geq 1.$$

For $x \in T \setminus \{x_1\}$, if $d_{G-S}(x) - d_H(x) + e_H(x, S) = 0$, then we have $d_H(x) - e_H(x, S) \geq 1$ by the analogous method of arguing $h = 0$. Hence, one of (a) and (b) holds for each $x \in T \setminus \{x_1\}$:

(a) $d_{G-S}(x) - d_H(x) + e_H(x, S) \geq 1$, or

(b) $d_{G-S}(x) - d_H(x) + e_H(x, S) = 0$ and $d_H(x) - e_H(x, S) \geq 1$.

Thus, we obtain

$$\sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S)) \geq |T| - 2m. \qquad (4)$$

According to (1), (2), (4), $h = 0$, $|S| + |T| \leq n$ and $n \geq \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$, we obtain

$$\begin{aligned} 0 &\geq b|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - a) + 1 \\ &\geq b|S| + |T| - 2m - a|T| + 1 \\ &= b|S| - (a-1)|T| - 2m + 1 \\ &\geq b|S| - (a-1)(n - |S|) - 2m + 1 \\ &= (a+b-1)|S| - (a-1)n - 2m + 1 \\ &> (a+b-1)(n + a + b - 2\sqrt{bn+1} \\ &\quad -(h + d_H(x_1) - e_H(x_1, S))) - (a-1)n - 2m + 1 \\ &\geq (a+b-1)(n + a + b - 2\sqrt{bn+1} - m) \\ &\quad -(a-1)n - 2m + 1 \\ &= bn + 1 + (a+b-1)(a+b) - 2(a+b-1)\sqrt{bn+1} \\ &\quad -(a+b+1)m \\ &= (\sqrt{bn+1} - (a+b-1))^2 + (a+b-1) \\ &\quad -(a+b+1)m \\ &\geq (a+b+1)m - 2 + (a+b-1) - (a+b+1)m \\ &= a + b - 3 \geq 0, \end{aligned}$$

a contradiction.

Case 2. $1 \leq h \leq a - 1$.

In view of (1), (2) and $|S| + |T| \leq n$, we have

$$\begin{aligned} 0 &\geq b|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - a) + 1 \\ &\geq b|S| + h|T| - a|T| + 1 \\ &= b|S| - (a-h)|T| + 1 \\ &\geq b|S| - (a-h)(n - |S|) + 1 \\ &= (a+b-h)|S| - (a-h)n + 1 \\ &> (a+b-h)(n + a + b - 2\sqrt{bn+1} \\ &\quad -(h + d_H(x_1) - e_H(x_1, S))) - (a-h)n + 1 \\ &= bn + (a+b-h)(a+b) - 2(a+b-h)\sqrt{bn+1} \\ &\quad -(a+b-h)(h + d_H(x_1) - e_H(x_1, S))) + 1 \\ &\geq bn + 1 + (a+b-h)(a+b) \\ &\quad -2(a+b-h)\sqrt{bn+1} - (a+b-h)(h+m) \\ &= (\sqrt{bn+1} - (a+b-h))^2 - (a+b-h)m, \end{aligned}$$

that is,

$$0 > (\sqrt{bn+1} - (a+b-h))^2 - (a+b-h)m. \qquad (5)$$

Let $f(h) = (\sqrt{bn+1} - (a+b-h))^2 - (a+b-h)m$. Obviously, the function $f(h)$ attains its minimum value at $h = 1$ since $1 \leq h \leq a-1$. Then we have

$$f(h) \geq f(1).$$

Combining this with (5) and $n \geq \frac{(a+b-1+\sqrt{(a+b+1)m-2})^2-1}{b}$, we obtain

$$\begin{aligned} 0 &> f(h) \geq f(1) \\ &= (\sqrt{bn+1} - (a+b-1))^2 - (a+b-1)m \\ &\geq (a+b+1)m - 2 - (a+b-1)m \\ &= 2m - 2 \geq 0, \end{aligned}$$

that is a contradiction.

Completing the proof of Theorem 6.

Acknowledgment

This research was sponsored by Qing Lan Project of Jiangsu Province and was supported by Jiangsu Provincial Educational Department (07KJD110048) and Sichuan Provincial Educational Department (08zb068).

References

[1] B. Alspach, K. Heinrich, G. Liu, Contemporary design theory-A collection of surveys, Wiley, New York, 1992, 13–37.

[2] Y. Egawa, H. Enomoto, Sufficient conditions for the existence of k-factors, Recent Studies in Graph Theory, Vishwa International Publication, India, 1989, 96–105.

[3] M. Kano, H. Matsuda, A neighborhood condition for graphs to have $[a, b]$-factors, LNCS, 4381(2007), 70–78.

[4] P. B. C. Lam, G. Liu, G. Li, W. C. Shiu, Orthogonal (g, f)-factorizations in networks, Networks, 35(2000), 274–278.

[5] Sizhong Zhou, Some sufficient conditions for graphs to have (g, f)-factors, Bulletin of the Australian Mathematical Society, 75(2007), 447–452.

[6] Sizhong Zhou, Binding numbers of graphs and the existence of $[a, b]$-factors, Journal of Systems Science and Mathematical Sciences (in Chinese), 29(4)(2009), 484–489.