

θ -Euclidean k-Fuzzy Ideals of Semirings

D.R Prince Williams

Abstract— In this paper, we introduce the notion θ -Euclidean k-fuzzy ideal in semirings and to study the properties of the image and pre image of a θ -Euclidean k-fuzzy ideal in a semirings under epimorphism.

Keywords—semiring, fuzzy ideal, k-fuzzy ideal, θ -Euclidean L-fuzzy ideal, θ -Euclidean fuzzy k-ideal, θ -Euclidean k-fuzzy ideal.

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I. INTRODUCTION

L. A. Zadeh [1] introduced the notion of a fuzzy subset μ of a set X as a function from X into the closed unit interval $[0, 1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld [2]. W.J. Liu [3] introduced and studied fuzzy ideals of rings. T.K. Dutta and B.K. Biswas [4] studied fuzzy ideals, fuzzy prime ideals of semirings and they defined fuzzy k-ideal and fuzzy prime k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k-ideals. S.I. Baik and H.S Kim [6] studied more about the fuzzy k-ideals in semirings and investigated their properties. Y.B. Jun *et al* [5] extended the concept of L-fuzzy ideal of rings to semirings. Ayten Koç, Erol Balkanay [7, 8] introduced a concept of θ -Euclidean L-fuzzy ideals, θ -Euclidean level subset in rings and studied the properties of ideals θ -Euclidean L-fuzzy ideals, θ -Euclidean level subset in rings. C.B Kim *et al* [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a k-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k-ideals in k-semirings.

The purpose of this paper is to introduce θ -Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a θ -Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a θ -Euclidean k-fuzzy ideal.

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II. PRELIMINARIES

An algebra $(S; +, \cdot)$ is said to be a semiring if $(S; +)$ and $(S; \cdot)$ are semigroup satisfying $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$, for all $a, b, c \in S$. A semiring S may have an identity 1, defined by $1 \cdot a = a = a \cdot 1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in S$. A non-empty subset I of S is said to be left (*resp.*, right) ideal if $x, y \in I$ and $r \in S$ imply that $x + y \in I$ and $rx \in I$ (*resp.*, $xr \in I$). If I is both left and right ideal of S , we say I is a two-sided ideal, or simply ideal, of S . A left ideal I of a semiring S is said to be a left k-ideal if $a \in I$ and $x \in S$ and if $a + x \in I$ or $x + a \in I$ then $x \in I$. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.

Definition 2.1 [10]: Let K and S be any sets and let $f : K \rightarrow S$ be a function. A fuzzy subset μ of K is called f -invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, where $x, y \in K$.

Definition 2.2 [2]: A fuzzy subset μ of a semiring S is said to be fuzzy left (*resp.*, right) ideal of S if

$$(i) \mu(x+y) \geq \min \{ \mu(x), \mu(y) \} \text{ and}$$

$$(ii) \mu(xy) \geq \mu(y) \text{ (resp., } \mu(xy) \geq \mu(x) \text{)}$$

for all $x, y \in S$. If μ is a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S .

Definition 2.3 [10]: A fuzzy ideal μ of a semiring S is said to be a k-fuzzy ideal of S $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$ imply $\mu(x) = \mu(0)$, for all $x, y \in S$.

Definition 2.4 [8]: Let $\theta : S \rightarrow [0, 1]$ and $\mu : S \rightarrow [0, 1]$ be a fuzzy subsets of S . For any, $0 \neq y \in S$ the set

$$\mu_{\theta_y} = \left\{ \begin{array}{l} x \in S \mid \text{there exists } q, r \in S \text{ such that } x = yq + r \\ \text{where either } r = 0 \text{ or else } \mu(r) \geq \max \{ \mu(y), \theta(y) \} \end{array} \right\}$$

is called a θ -Euclidean level subset of μ .

II. θ -EUCLIDEAN K-FUZZY IDEALS

Definition 3.1: Let S be a semiring and let $\theta : S \rightarrow [0,1]$ be a non-constant fuzzy subset of S . A fuzzy ideal $\mu : S \rightarrow [0,1]$ is called a θ -Euclidean k -fuzzy ideal if μ satisfies the following axioms

(i) $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$ imply $\mu(x) = \mu(0)$, for all x, y in R .

(ii) For any $x, y \in R$ with $y \neq 0$, there exists elements $q, r \in R$ such that $x = yq + r$, where either $r = 0$ or else $\max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \}$.

Example 3.2: Let S be the set of Natural Numbers including zero and $\mu : S \rightarrow [0,1]$ be a fuzzy subset defined by

$$\mu(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases}$$

Let $\theta : S \rightarrow [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a = 3, 5, 7, \dots \\ \frac{1}{|a|} & \text{otherwise.} \end{cases}$$

Clearly μ is a k -fuzzy ideal of S , also μ is a θ -Euclidean k -fuzzy ideal of S .

Example 3.3: Let S be the set of Natural Numbers including zero and $\mu : S \rightarrow [0,1]$ be a fuzzy set defined by

$$\mu(a) = \begin{cases} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases}$$

Let $\theta_1 : S \rightarrow [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & \text{if } a = 0 \\ \frac{1}{|a|} & \text{otherwise.} \end{cases}$$

So μ is a k -fuzzy ideal but μ is not a θ_1 -Euclidean k -fuzzy ideal of S .

Theorem 3.4: Let A be a non empty subset of S . Let μ be a fuzzy subset of a semiring S such that μ is into $\{0,1\}$, so that μ is the characteristic function of A . Then μ is a θ -Euclidean k -fuzzy ideal of a semiring S then A is a left ideal of S .

Proof: The proof is easy and straight forward. \square

Theorem 3.5: Let μ be a θ -Euclidean k -fuzzy ideal of a semiring S . Then for $0 \neq y \in S$, (i) μ_{θ_y} is an ideal of S

(ii) θ_{μ_y} is an ideal of S . and (iii) μ_t is a θ -Euclidean k -fuzzy ideal of S , for $t \in [0,1]$.

Proof: The proof is similar to [8, Theorem 3.3]. \square

Theorem 3.6: Let μ be a fuzzy ideal of a semiring S . If μ_{θ_y} and θ_{μ_y} is the Euclidean level set of μ and θ respectively. Then μ is a θ -Euclidean k -fuzzy ideal of a semiring S .

Proof: Suppose μ is fuzzy ideal of semiring S . For $x, y \in S$, if $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$, then $\mu(x+y) \geq \min \{ \mu(x), \mu(y) \}$, since μ is fuzzy ideal of S .

$$\mu(0) \geq \min \{ \mu(x), \mu(0) \}$$

$$\mu(x) = \mu(0).$$

Thus μ is a k -fuzzy ideal of semiring S .

We have μ_{θ_y} and θ_{μ_y} is the Euclidean level set of μ and θ respectively. Then, for $x, y \in S$, with $0 \neq y$, there exists $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \max \{ \mu(y), \theta(y) \}$ and $\theta(r) \geq \max \{ \mu(y), \theta(y) \}$.

Thus $\max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \}$.

Hence μ is a θ -Euclidean k -fuzzy ideal of a semiring S . \square

Definition 3.7 ([10]): Let $f : S \rightarrow S'$ be a homomorphism of semirings. Let μ be a fuzzy subset of S' . We define a fuzzy subset $f^{-1}\mu$ of S by $f^{-1}\mu(x) = \mu(f(x))$, for all $x \in S$

Theorem 3.7: Let $f : S \rightarrow S'$ be an epimorphism of semirings and μ be a fuzzy ideal of S' . Then μ is a θ -Euclidean k -fuzzy ideal of S' if and only if $f^{-1}(\mu)$

is a $f^{-1}(\theta)$ -Euclidean k -fuzzy ideal of fuzzy ideal of S .

Proof: Suppose μ is a θ -Euclidean k -fuzzy ideal of S' .

(i) For all $x, y \in S'$

$$\begin{aligned} f^{-1}\mu(x+y) &= \mu(f(x+y)) = \mu(f(x)+f(y)) \\ &\geq \min\{\mu(f(x)), \mu(f(y))\} \\ &= \min\{f^{-1}\mu(x), f^{-1}\mu(y)\} \end{aligned}$$

(ii) For all $x, y \in S'$

$$\begin{aligned} f^{-1}\mu(xy) &= \mu(f(xy)) = \mu(f(x)f(y)) \\ &\geq \max\{\mu(f(x)), \mu(f(y))\} \\ &= \max\{f^{-1}\mu(x), f^{-1}\mu(y)\} \end{aligned}$$

(iii) For all $x, y \in S'$, if $f^{-1}\mu(x+y) = f^{-1}\mu(0)$

and $f^{-1}\mu(y) = f^{-1}\mu(0)$ then

$$\begin{aligned} f^{-1}\mu(x) &= \mu(f(x)) = \mu(x) = \mu(0) \\ &= \mu(f(0)) = f^{-1}\mu(0). \end{aligned}$$

iv) We have μ is a θ -Euclidean k -fuzzy ideal of S' , then for

any $x, y \in S$, then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where either $f(r) = 0$ or else

$$\max\{\mu(f(y)), \theta(f(y))\} \geq \max\{\mu(f(r)), \theta(f(r))\}.$$

That is $f(x) = f(yq) + f(r)$ where either $f(r) = 0$ or

$$\text{else } \max\{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max\{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$

Thus $f(x) = f(yq+r)$ where either $f(r) = 0$ or else

$$\max\{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max\{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq+r$ where either $r = 0$ or else

$$\max\{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max\{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$

Conversely, suppose $f^{-1}(\mu)$ is a θ -Euclidean k -fuzzy ideal of S .

(i) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

$$\begin{aligned} \mu(a+b) &= \mu(f(x)+f(y)) = \mu(f(x+y)) \\ &= f^{-1}\mu(x+y) \\ &\geq \min\{f^{-1}\mu(x), f^{-1}\mu(y)\} \\ &= \min\{\mu(f(x)), \mu(f(y))\} \\ &= \max\{\mu(a), \mu(b)\}. \end{aligned}$$

(ii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

$$\begin{aligned} \mu(ab) &= \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}\mu(xy) \\ &\geq \max\{f^{-1}\mu(x), f^{-1}\mu(y)\} \\ &= \max\{\mu(f(x)), \mu(f(y))\} \\ &= \max\{\mu(a), \mu(b)\}. \end{aligned}$$

(iii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$, if $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$ imply

$$\mu(a) = \mu(f(x)) = f^{-1}\mu(x) = f^{-1}\mu(0) = \mu(f(0)) = \mu(0)$$

(iv) For any $x, y, q, r \in S$ then

$$a = f(x), b = f(y), c = f(q), d = f(r) \in S'.$$

We have $f^{-1}(\mu)$ is a θ -Euclidean k -fuzzy ideal of fuzzy ideal of S , then there exists $q, r \in S$ such that $x = yq+r$ either $r = 0$ or else

$$\max\{f^{-1}\mu(y), f^{-1}\theta(y)\} \geq \max\{f^{-1}\mu(r), f^{-1}\theta(r)\}.$$

That is $f(x) = f(yq+r)$ either $f(r) = 0$ or else

$$\max\{\mu(f(y)), \theta(f(y))\} \geq \max\{\mu(f(r)), \theta(f(r))\}.$$

that is $f(x) = f(y)f(q) + f(r)$ either $f(r) = 0$ or else

$$\max\{\mu(f(y)), \theta(f(y))\} \geq \max\{\mu(f(r)), \theta(f(r))\}.$$

Thus there exists $c, d \in S'$ such that $a = bc+d$ either

$$r = 0 \text{ or else } \max\{\mu(c), \theta(c)\} \geq \max\{\mu(d), \theta(d)\}. \quad \square$$

Definition 3.8: Let $f : S \rightarrow S'$ be an homomorphism of the semirings. Let μ be a fuzzy subset of S . we define a fuzzy subset $f(\mu)$ of S' by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(t) \mid t \in R, f(t) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

Theorem 3.9: Let $f : S \rightarrow S'$ epimorphism of semirings. Let μ be a f -invariant θ -Euclidean k -fuzzy ideal of S . Then $f(\mu)$ is $f(\theta)$ -Euclidean k -fuzzy ideal of S' .

Proof: Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$. Then $x+y = f(a)+f(b) = f(a+b)$ and $xy = f(a)f(b) = f(ab)$. Since μ is f -invariant

Thus

$$\begin{aligned} (i) \quad f(\mu)(x+y) &= f(\mu)f(a+b) \\ &= \sup\{\mu(t) \mid t \in S, f(t) = f(a+b)\} \end{aligned}$$

$$\begin{aligned}
 &= \sup \{ \mu(t) \mid t \in S, \mu(t) = \mu(a+b) \} \\
 &= \mu(a+b) \\
 &\geq \min \{ \mu(a), \mu(b) \}, \\
 &\quad \text{since } \mu \text{ is a } k\text{-fuzzy ideal of } S.
 \end{aligned}$$

$$\begin{aligned}
 &= \min \{ \mu(f^{-1}(x)), \mu(f^{-1}(y)) \} \\
 &= \min \{ f(\mu)(x), f(\mu)(y) \}.
 \end{aligned}$$

(ii) $f(\mu)(xy) = f(\mu)f(ab) = \mu(ab)$

$$\begin{aligned}
 &\geq \max \{ \mu(a), \mu(b) \}, \\
 &\quad \text{since } \mu \text{ is a } k\text{-fuzzy ideal of } S.
 \end{aligned}$$

$$\begin{aligned}
 &= \max \{ \mu(f^{-1}(x)), \mu(f^{-1}(y)) \} \\
 &= \max \{ f(\mu)(x), f(\mu)(y) \}.
 \end{aligned}$$

(iii) If $f(\mu)(x+y) = f(\mu)(0)$ and $f(\mu)(y) = f(\mu)(0)$ imply that

$$\begin{aligned}
 f(\mu)(x) &= f(\mu)(f(a)) = \mu(a) \\
 &= \mu(0) = \mu(f^{-1}(0)) = f(\mu)(0).
 \end{aligned}$$

(iv) We have μ is f -invariant θ -Euclidean k -fuzzy ideal of S . If $a, b, c, d \in S$ then $x = f(a)$, $y = f(b)$, $q = f(c)$, $r = f(d)$, for all $x, y, q, r \in S'$. Then for any $a, b \in S$ there exists elements $c, d \in S$, such that $a = bc + d$, where either $d = 0$ or else

$$\max \{ \mu(b), \theta(b) \} \geq \max \{ \mu(d), \theta(d) \}.$$

That is, $f(a) = f(bc + d)$,

$$\text{thus } f(a) = f(b)f(c) + f(d),$$

Thus $x = yq + r$. Let $d = 0$.

Then $f(d) = f(0) = 0$. We get $r = 0$.

Finally, we have

$$\max \{ \mu(b), \theta(b) \} \geq \max \{ \mu(d), \theta(d) \},$$

Since μ is f -invariant.

$$\begin{aligned}
 f(\mu)(y) &= f(\mu)f(b) = \sup \{ \mu(t) \mid t \in R, f(t) = f(b) \} \\
 &= \sup \{ \mu(t) \mid t \in R, \mu(t) = \mu(b) \} \\
 &= \mu(b)
 \end{aligned}$$

so that $\max \{ \mu(b), \theta(b) \} \geq \max \{ \mu(d), \theta(d) \}$ then

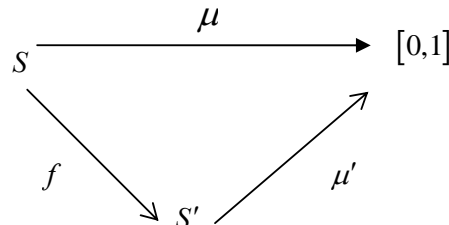
$$\max \{ f(\mu)(y), f(\theta)(y) \} \geq \max \{ f(\mu)(r), f(\theta)(r) \}.$$

Hence $f(\mu)$ is a $f(\theta)$ -Euclidean k -fuzzy ideal of S' .

Theorem 3.10: Let $f : S \rightarrow S'$ be an isomorphism of the semirings and $\mu' : S' \rightarrow [0,1]$ be a θ -Euclidean k -fuzzy ideal of S' . Then $\mu' \circ f : S \rightarrow [0,1]$ is a $(\theta' \circ f)$ -Euclidean

k -fuzzy ideal of S . Here, we mean that $(\mu' \circ f)(x) = \mu'[f(x)]$.

Proof: Let $\mu = \mu' \circ f, \theta = \theta' \circ f$ and also $a, b \in S$ and μ' is an θ -Euclidean k -fuzzy ideal of S' .



It was proved that μ is a fuzzy ideal of S [5] and μ is a θ -Euclidean fuzzy ideal of S [7].

If $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$, then

$$\begin{aligned}
 \mu(a) &= \mu' \circ f(a) = \mu'(f(a)) = \mu'(0). \text{ Since } \mu' \text{ is an } \\
 &\quad \theta\text{-Euclidean } k\text{-fuzzy ideal of } S'. \\
 &= \mu'(f(0)) \\
 &= \mu' \circ f(0) \\
 &= \mu(0)
 \end{aligned}$$

Hence $\mu' \circ f : S \rightarrow [0,1]$ is a $(\theta' \circ f)$ -Euclidean k -fuzzy ideal of S . \square

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