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2^n almost periodic attractors for Cohen-Grossberg neural networks with variable and distribute delays

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Abstract—In this paper, we investigate dynamics of 2^n almost periodic attractors for Cohen-Grossberg neural networks (CGNNs) with variable and distribute time delays. By imposing some new assumptions on activation functions and system parameters, we split invariant basin of CGNNs into 2^n compact convex subsets. Then the existence of 2^n almost periodic solutions lying in compact convex subsets is attained due to employment of the theory of exponential dichotomy and Schauder's fixed point theorem. Meanwhile, we derive some new criteria for the networks to converge toward these 2^n almost periodic solutions and exponential attracting domains are also given correspondingly.

Keywords—CGNNs; Almost periodic solution; Invariant basins; Attracting domains.

I. Introduction

COHEN-Grossberg neural networks (CGNNs) were first introduced by Cohen and Grossberg [1] in 1983, have been successfully applied to pattern recognition, associative memory, combinatorial optimization and so on. Hence, the dynamics and applications of CGNNs have been of interest to a wide range of authors in recent years. Many important results on the existence of a unique equilibrium point and its convergent dynamical behavior have been established so far [2-10].

As we know well, the nonautonomous phenomenon involved in periodic or almost periodic environment often occurs in many realistic systems [11,12]. Hence, in many applications, the property of periodic or almost periodic oscillatory solutions of neural networks is of great interest. Recently, a lot of sufficient conditions have been given for almost periodic oscillation of CGNNs with constant time delays or time-varying delays in the literature, see [13-15] and the references cited therein.

In the applications of neural networks to pattern recognitions, the existence of multiple stable equilibria or almost periodic orbits is an important feature. It is worth to investigate the convergence and coexistence of multiple equilibria or multiple almost periodic solutions of neural networks. However, to the best of our knowledge, most of the reported in the literature focus on dynamical behavior of the unique almost periodic solution. Few papers systematically deal with coexistence of multiple almost periodic solutions of neural networks.

Motivated by the above, in this paper, we consider the following almost periodic Cohen-Grossberg neural networks with variable and distribute time delays:

$$\begin{cases} x_i'(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) \\ \qquad - \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) \\ \qquad - \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma}^{t} k_{ij}(t-s) g_j(x_j(s))\mathrm{d}s \\ \qquad + I_i(t)\Big],\ i\in\Lambda, \\ x_i(s) = \varphi_i(s), \quad -\omega \le s \le 0,\ i \in \Lambda, \end{cases} \tag{1}$$

where $\Lambda = 1,\dots,n$, $n \ge 2$ is the number of neurons in the network, $x_i(t)$ denotes the state of ith neuron at time t, $a_i(x_i)$ represents the amplification function, g_i and f_i are the activation functions which describe the manner in which the neurons respond to each other, $C = (c_{ij})_{n\times n}$ is the feedback matrix which represents the strengthen of the neuron inter connections within the network, while $D = (d_{ij})_{n\times n}$ is the delayed feedback matrix which represents the strengthen of the neuron interconnections within the network with time-varying delay parameter $\tau_{ij}(t)$ which is continuous, $\tau = \max\limits_{t\in[0,\omega]} \tau_{ij}(t)$, the scalar σ is the known distributed delay and $\omega = \max\{\tau,\sigma\}$, k_{ij} denotes continuous kernel function.

Our purpose of this paper is by employing the theory of exponential dichotomy, Schauder's fixed point theorem and inequality technique, we investigate complex dynamics of 2^n almost periodic attractors of CGNNs (1).

Throughout this paper, we assume that:

(H_1) Each $a_i(u)(i \in \Lambda,\ u \in R)$ is positive, continuous and bounded function. $c_{ij}(t), d_{ij}(t), \tau_{ij}(t),\ I_i(t)$ are all almost periodic functions defined on R, where $d_{ii} \ge \inf\limits_{t\in R} d_{ii}(t) > 0$ and derivative $\tau_{ij}'(t)$ is uniformly continuous on R with $\inf\limits_{t\in R}\{1-\tau_{ij}'(t)\} > 0$.

(H_2) Each $b_i(\cdot)$ is continuous with $b_i(0) = 0$ and there exists a constant Θ_i such that

$$\frac{b_i(x)-b_i(y)}{x-y} \ge \Theta_i > 0,\ \forall\, x,y \in R, x \ne y,\ i = 1,2,\cdots,n.$$

(H_3) Each kernel function $k_{ij}(\cdot)$ is positive, continuous and satisfies

$$\int_0^\sigma (k_{ij}(s))^\theta \Delta s = k_{ij}(\theta,\sigma),$$

where $0 < k_{ij}(\theta,\sigma) < +\infty$ is a continuous function on $(0,\hat\theta]$, $\hat\theta > 0$. Furthermore, when $\sigma = +\infty$, $k_{ij}(\theta) = k_{ij}(\theta,+\infty)$ and $k_{ij}(1) = 1$.

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(H_4) The activation functions $f_j(\cdot), g_j(\cdot)$ satisfy $f_j(\cdot), g_j(\cdot) \in C^2(R)$ and

$$\begin{cases} |f_j(x)| \le \gamma_j,\ f_j(0) = 0, \\ f_j'(x) = f_j'(-x) > 0,\ xf_j^{''}(x) < 0, \end{cases} \quad \text{where } x \in R.$$

and

$$|g_j(x)| \le \chi_j,\ g_j(0) = 0,\ \text{where } x \in R.$$

The rest of this paper is organized as follows. In Section 2, we shall make some preparations by giving some definitions and basic lemmas. Meanwhile, we attain an invariant basin Ω of CGNNs (1) and split it into 2^n compact convex subsets. In Section 3, we discuss the existence of 2^n almost periodic solutions of CGNNs (1), some new criteria are derived for the networks to converge exponentially toward to these 2^n almost periodic solutions and exponential attracting domains are also given. Finally, an example is given to illustrate our results.

II. Preliminaries

In this paper, we denote by $C([-\omega, 0], R^n)$ the set of all continuous mappings from $[-\omega, 0]$ to R^n equipped with norm $\|\cdot\|_\omega$ defined by

$$\|\phi\|_\omega = \max_{i\in\Lambda}\{\|\phi^i\|_\omega\}, \quad \Lambda = \{1, 2, \cdots, n\},$$

where $\|\phi^i\|_\omega = \sup\limits_{-\omega\le t\le 0} |\phi^i|$ and $\phi = (\phi^1, \phi^2, \cdots, \phi^n)^T \in C([-\omega, 0], R^n)$. Let $l > 0$, for any $x(\cdot) \in C([-\omega, 0], R^n)$ and $t \in [0, l]$, we define $x_t(s) = x(t + s),\ s \in [-\omega, 0]$, then we have $x_t(\cdot) \in C([-\omega, 0], R^n)$. For any given $\phi \in C([-\omega, 0], R^n)$, we denote by $u(t, \phi)$ the solution of CGNNs (1) with $u_0(s) = \phi(s)$ for all $s \in [-\omega, 0]$.

Definition 2.1([11]) A continuous function $f : R \to R$ is called an almost periodic on R, if the ε-translation set of f:

$$E\{\varepsilon, f\} = \Big\{\tau : |f(t + \tau) - f(t)| < \varepsilon\Big\},\ \forall\ t \in R,$$

for any given $\varepsilon > 0$, there exist a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exist $\tau \in E\{\varepsilon, f\}$, such that the inequality $|f(t+\tau) - f(t)| < \varepsilon,\ \forall\ t \in R$. τ is called the ε-translation number of $f(t)$.

Let $(AP, \|\cdot\|)$ be the Banach space of all real-valued almost periodic functions with commonly used supremum norm $\|\cdot\|$. By Definition 2.1, we know that all almost periodic functions are bounded. For convenience, we denote $\overline{f} = \sup\limits_{t\in R} |f(t)|,\ \underline{f} = \inf\limits_{t\in R} |f(t)|$ for any $f(t) \in AP$.

From (H_1), the antiderivative of $\frac{1}{a\ (x\ )}$ exists. We may choose an antiderivative $F_i(x_i)$ of $\frac{1}{a\ (x\ )}$ with $F_i(0) = 0$. Obviously, $F_i'(x_i) = \frac{1}{a\ (x\ )}$. Due to $a_i(x_i) > 0$, one can imply that $F_i(x_i)$ is increasing about x_i and the inverse function $F_i^{-1}(x_i)$ of $F_i(x_i)$ is existential, continuous and derivative. The composition function $b_i(F_i^{-1}(x_i))$ is differential. We denote by $y_i(t) = F_i(x_i(t))$. It is easy to see that $y_i'(t) = F_i'(x_i)x_i'(t) = \frac{x'(t)}{a\ (x\ (t))}$ and $x_i(t) = F_i^{-1}(y_i(t))$. Substituting these equalities into CGNNs (1), we can get that

$$\begin{cases} y_i'(t) = -b_i(F_i^{-1}(y_i(t))) \\ \qquad + \sum\limits_{j=1}^{n} d_{ij}(t) f_j(F_j^{-1}(y_j(t - \tau_{ij}(t)))) \\ \qquad + \sum\limits_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t - s) g_j(F_j^{-1}(y_j(s)))\mathrm{d}s \\ \qquad + I_i(t),\ i \in \Lambda, \\ x_i(s) = F_i(\varphi_i(s)) = \phi_i(s),\ -\omega \le s \le 0,\ i \in \Lambda, \end{cases} \tag{2}$$

In addition, by the mean value theorem, then

$$b_i(F_i^{-1}(y_i(t))) = [b_i(F_i^{-1}(\hat{\theta} y_i(t)))]' y_i(t) = e_i(y_i(t)) y_i(t),$$

where $e_i(y_i(t)) = [b_i(F_i^{-1}(\hat{\theta} y_i(t)))]',\ 0 < \hat{\theta} < 1$.

Then system (2) can be written as

$$\begin{cases} y_i'(t) = -e_i(y_i(t)) y_i(t) \\ \qquad + \sum\limits_{j=1}^{n} d_{ij}(t) f_j(F_j^{-1}(y_j(t - \tau_{ij}(t)))) \\ \qquad + \sum\limits_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t - s) \times \\ \qquad g_j(F_j^{-1}(y_j(s)))\mathrm{d}s + I_i(t),\ i \in \Lambda, \\ y_i(s) = F_i(\varphi_i(s)) = \phi_i(s), -\omega \le s \le 0,\ i \in \Lambda. \end{cases} \tag{3}$$

It is easy to see that

$$\begin{aligned} |F_j^{-1}(u) - F_j^{-1}(v)| &= |(F_j^{-1})'(v + \hat{\theta}(u - v))(u - v)| \\ &= |a_j(v + \hat{\theta}(u - v))||u - v|, \end{aligned}$$

where $0 < \hat{\theta} < 1$.

By (H_1) we have

$$\underline{a}_j|u - v| < |F_j^{-1}(u) - F_j^{-1}(v)| < \bar{a}_j|u - v|. \tag{4}$$

From (H_2), one can easily obtain that $(H_2)' : b_i'(F_i^{-1}(\cdot)) \ge \Theta_i \underline{a}_i$, where $b_i(\cdot)$ is the derivative of $b_i(\cdot),\ i = 1, 2, \cdots, n$.

Definition 2.2 Let $\hat{\Omega}$ be a subset of $C([-\omega, 0], R^n)$, $\hat{\Omega}$ is said to be an invariant basin of CGNNs (1) if and only if for any $\phi \in \hat{\Omega}$, we have $u_t(\cdot, \phi) \in \hat{\Omega}$ for all $t \ge 0$, where $u_t(t, \phi)$ is the solution of CGNNs (1) with initial condition ϕ.

The following Definition and Lemmas, one can find in [11] and [12].

Definition 2.3 Let $y \in R^n$ and $A(t, y)$ be an $n \times n$ continuous matrix defined on $R \times R^n$. For any continuous function $w(t) : R \to R^n$, the system $y'(t) = A(t, w(t))y(t)$ is said to be an exponential dichotomy on R, if there exist constants $\alpha, \beta > 0$, projection P and the fundamental matrix $Y_w(t)$ satisfying

$$\|Y_w(t) P Y_w^{-1}(s)\| \le \beta \exp(-\alpha(t - s)),\ t \ge s,$$

$$\|Y_w(t)(I - P) Y_w^{-1}(s)\| \le \beta \exp(-\alpha(s - t)),\ s \ge t.$$

Lemma 2.1 If $M[e_i] > 0$, then the linear system $y'(t) = A(t, w(t))y(t)$ has an exponential dichotomy.

Lemma 2.2 If the linear system $y'(t) = A(t, w(t))y(t)$ has an exponential dichotomy, then the almost periodic system

$y'(t) = A(t,w(t))y(t) + f(t,w(t))$ has an almost periodic solution which can be expressed as follows:

$$y(t) = \int_{-\infty}^{t} Y_w(t)PY_w^{-1}(s)f(t,w(t))ds - \int_{t}^{+\infty} Y_w(t)(I-P)Y_w^{-1}(s)f(t,w(t))ds.$$

Definition 2.4 The almost periodic solution $z^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ of CGNNs (1) is said to be exponentially stable, if there exist constants $\gamma > 0$ and $\lambda > 0$ such that

$$\|z - z^*\| \le \gamma \|\varphi - z^*\| e^{-\lambda t}$$

for all $t \ge 0$.

CGNNs (1) has 2^n almost periodic solutions which are exponentially stable, if and only if, system (3) has 2^n almost periodic solutions which are exponentially stable. We only consider the system (3) in the later.

III. 2^n Almost Periodic Attractors for CGNNs

In this section, we should discuss the existence of 2^n almost periodic solutions of system (3) and give an exponential attracting domain for each almost periodic attractor. Before we derive some properties of solutions of system (3), we need to introduce the following lemmas.

Lemma 3.1 Assume that the assumptions $(H_1) - (H_4)$ are satisfied. Any solution $y(t,\phi)$ of system (3) is uniformly bounded with initial condition $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in \Omega$ which be defined as follows:

$$\Omega = \{\phi \in C([-\omega, 0], R^n) \| \phi_i(s)| \le [\overline{I}_i + \sum_{j=1}^{n} (\overline{d}_{ij}\gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)]/\underline{e}_i, \quad s \in [-\omega, 0], i \in \Lambda\}.$$

Moreover, Ω is an invariant basin of system (3).

Proof: From system (3), we have

$$\frac{d^+}{dt}|y_i(t)| \le -\underline{e}_i|y_i(t)| + [\overline{I}_i + \sum_{j=1}^{n}(\overline{d}_{ij}\gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)],$$

where $t \ge 0$ and $\frac{d^+}{dt}(\cdot)$ denotes the upper right Dini derivative operator. Hence it follows that for some $t_0 \ge 0$,

$$|y_i(t)| \le \exp(t_0 - t)\underline{e}_i \Big\{ |y_i(t_0)| - [\overline{I}_i + \sum_{j=1}^{n}(\overline{d}_{ij}\gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)]/\underline{e}_i \Big\} + [\overline{I}_i + \sum_{j=1}^{n}(\overline{d}_{ij}\gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)]/\underline{e}_i, \quad (5)$$

for $t \ge t_0$.

Therefore, given any initial condition $\phi \in \Omega$, we have for all $t \ge 0$,

$$\|y_i(t,\phi)\|_\omega \le [\overline{I}_i + \sum_{j=1}^{n}(\overline{d}_{ij}\gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)]/\underline{e}_i, \quad i \in \Lambda.$$

i.e., $y_t(\cdot,\phi) \in \Omega$ for all $t \ge 0$ and $y(t,\phi)$ is uniformly bounded. The proof is completed. ■

From equation (5), it is easy to see that all almost periodic solutions of system (3) locate themselves in invariant basin Ω. For convenience of investigating the existence of 2^n almost periodic solutions, we should split invariant basin Ω into 2^n compact convex subsets of $\underbrace{Ap \times \cdots \times Ap}_{n}$. Hence we consider the following auxiliary functions:

$$\xi_k(z) = -\overline{e}_i z + \underline{d}_{ii} f_i(z), \ i \in \Lambda.$$

Lemma 3.2 Suppose that the following assumption holds: (A_1) $\underline{d}_{ii} \inf_{\xi \in R} f_i'(\xi) < \overline{e}_i < \underline{d}_{ii} \sup_{\xi \in R} f_i'(\xi)$, where $i \in \Lambda$.
Then there only exist two points z_{i1} and z_{i2} with $z_{i1} < 0 < z_{i2}$ such that $\xi_i'(z_{il}) = 0$ and

$$\xi_i'(z) \cdot \mathrm{sign}\{\frac{z - z_{i1}}{z - z_{i2}}\} < 0,$$

where $z \ne z_{il}(l = 1, 2)$ and $\mathrm{sign}(\cdot)$ denotes a symbolic function.

Proof: We have $\xi_i'(z) = 0$ if and only if $f_i'(z) = \frac{\overline{e}}{\underline{d}}$. For each activation function $f_i(\cdot)$, we know that the graph of positive function $f_i'(\cdot)$ concaves down and has its maximal value at zero. By (H_1), there only exist two points z_{i1} and z_{i2} with $z_{i1} < 0 < z_{i2}$ such that $f_i'(z_{il}) = \frac{\overline{e}}{\underline{d}}$; that is, $\xi_k'(z_{il}) = 0(i = 1, 2)$. Since $f_k'(z)$ is strictly increasing on $(-\infty, z_{il}]$ and is strictly decreasing on $[z_{i2}, +\infty)$, we get that

$$(-\overline{e}_i + \underline{d}_{ii} f_i'(z)) \cdot \mathrm{sign}\{\frac{z - z_{i1}}{z - z_{i2}}\} < 0,$$

that is

$$\xi_i'(z) \cdot \mathrm{sign}\{\frac{z - z_{i1}}{z - z_{i2}}\} < 0,$$

where $z \ne z_{il}(l = 1, 2)$. The proof is completed. ■

With the basic property of $\xi_i(z)$ given in Lemma 3.2, we consider the following additional assumption:

(A_2) $(-1)^i \cdot \{\xi_i(z_{il}) + I_i(t)\} > \sum_{j=1}^{n}(\bar{d}_{ij}\ \gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j)$

for all $t \in R$, $l = 1, 2$, and $i \in \Lambda$, where $\bar{d}_{ij} \ := \bar{d}_{ij}$ when $j^o \ne i$, otherwise $\bar{d}_{ij} \ := 0$.

Let $l = 1$ in (A_2), it is easy for us to get that

$$\xi_i(z_{il}) + \sum_{j=1}^{n}(\bar{d}_{ij}\ \gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j) + \sup_{t \in R} I_i(t) < 0. \quad (6)$$

From Lemma 3.2, we know that $\xi_i(z)$ is strictly decreasing on $(-\infty, z_{il}]$. Noting that $\xi_i(z) \to +\infty$ as $z \to -\infty$, we know that there exists a unique $\hat{z}_{il}$ with $\hat{z}_{il} < z_{il} < 0$ such that

$$\xi_i(\hat{z}_{il}) + \sum_{j=1}^{n}(\bar{d}_{ij}\ \gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j) + \sup_{t \in R} I_i(t) = 0. \quad (7)$$

Let $l = 2$ in (A_2), by the similar argument, we derive that there exists a unique $\hat{z}_{i2}$ with $0 < z_{i2} < \hat{z}_{i2}$ such that

$$\xi_i(\hat{z}_{i2}) + \sum_{j=1}^{n}(\bar{d}_{ij}\ \gamma_j + \overline{c}_{ij}k_{ij}(1,\sigma)\chi_j) + \inf_{t \in R} I_i(t) = 0. \quad (8)$$

Take the following notations:

$$\begin{cases} d_{i1} := -\dfrac{\bar{I} + \sum_{=1} (\overline{d} \quad \gamma + \overline{c} \quad k \quad (1,\sigma)\chi \quad)}{\underline{e}}, \quad c_{i1} := \hat{z}_{i1}, \\ d_{i2} := \hat{z}_{i2}, \quad c_{i2} := \dfrac{\bar{I} + \sum_{=1} (\overline{d} \quad \gamma + \overline{c} \quad k \quad (1,\sigma)\chi \quad)}{\underline{e}}, \end{cases}$$

where $i \in \Lambda$. By equations (7) and (8), it is easy for us to check that $d_{i1} < c_{i1} < 0 < d_{i2} < c_{i2}$ for each $i \in \Lambda$. Then we define the following sets:

$$\begin{aligned} &H_{i1} := \{\psi \in C([-\omega, 0], R) | \psi(s) \leq c_{i1}, \forall\ s \in [-\omega, 0]\}, \\ &H_{i2} := \{\psi \in C([-\omega, 0], R) | \psi(s) \geq d_{i2}, \forall\ s \in [-\omega, 0]\}, \\ &K_{i1} := \{z \in R | z \leq c_{i1}\}, \quad K_{i2} := \{z \in R | z \geq d_{i2}\}, \\ &\Gamma_{il} := \{\psi \in AP | d_{il} \leq \psi(t) \leq c_{il},\ \forall\ t \in R\}, \end{aligned}$$

where $i \in \Lambda,\ l = 1, 2$. Let

$$H^\alpha := \underbrace{H_{1\alpha_1} \times H_{2\alpha_2} \cdots \times H_{n\alpha}}_{n} \subset C([-\omega, 0]_R, R^n),$$

$$K_i^\alpha := K_{i\alpha}\ ,\ K^\alpha := \underbrace{K_{1\alpha_1} \times K_{2\alpha_2} \cdots \times K_{n\alpha}}_{n} \subset R^n,$$

$$\Gamma^\alpha := \underbrace{\Gamma_{1\alpha_1} \times \Gamma_{2\alpha_2} \cdots \times \Gamma_{n\alpha}}_{n},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_i = 1, 2,\ i \in \Lambda$. It is obvious that $\Gamma_{il} (i \in \Lambda,\ l = 1, 2)$ are compact convex subsets of AP. With above notations, we split invariant basin Ω into 2^n compact convex subsets Γ^α of $\underbrace{Ap \times \cdots \times Ap}_{n}$. In this paper, without otherwise statement, we always designate $\alpha \in \underbrace{\{1, 2\} \times \cdots \times \{1, 2\}}_{n}$.

Theorem 3.1 Under the basic assumptions (H_1)-(H_4) and (A_1)-(A_2), for each α, there exists at least one almost periodic solution $u_\alpha(t)$ of system (3) in Γ^α.

Proof: Since $M[e_i] > 0$, then by Lemma 2.1, the following linear system

$$y'(t) = -\text{diag}(e_1(y_1(t)), e_2(y_2(t)), \cdots, e_n(y_n(t)))y(t)$$

admits an exponential dichotomy on R. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and any $\phi = (\phi_1, \phi_2, \cdots, \phi_n) \in \Gamma^\alpha$, from Lemma 2.2, we know that the following almost periodic system:

$$\begin{aligned} y_i'(t) \;\; = \;\; & -e_i(y_i(t))y_i(t) \\ & + \sum_{j=1}^{n} d_{ij}(t) f_j(F_j^{-1}(\phi_j(t - \tau_{ij}(t)))) \\ & + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s) f_j(F_j^{-1}(\phi_j(s))) ds \\ & + I_i(t),\ i \in \Lambda \end{aligned} \tag{9}$$

has an almost periodic solution defined by $G^\alpha \phi = (G_1^\alpha \phi, G_2^\alpha \phi, \cdots, G_n^\alpha \phi)$, where

$$\begin{aligned} (G_i^\alpha \phi)(t) \;\; = \;\; & \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} e_i(y_i(w)) dw \right) \times \\ & \Big[\sum_{j=1}^{n} d_{ij}(s) f_j(F_j^{-1}(\phi_j(s - \tau_{ij}(s)))) \\ & + \sum_{j=1}^{n} c_{ij}(s) \int_{s-\sigma}^{s} k_{ij}(s - v) \times \\ & f_j(F_j^{-1}(\phi_j(v))) dv + I_i(t) \Big] ds, \end{aligned} \tag{10}$$

Next we need two steps to complete our proof.

Step 1: For each $i \in \Lambda$, we should prove that $d_{i\alpha} \leq (G_i^\alpha \phi)(t) \leq c_{i\alpha}$ for all $t \in R$. Fix $i \in \Lambda$. From (H_1)-(H_3) and equation (10), one obtain that

$$|(G_i^\alpha \phi)(t)| \leq \frac{\bar{I}_i + \sum_{j=1}^{n} (\overline{d}_{ij} \gamma_j + \overline{c}_{ij} k_{ij}(1, \sigma) \chi_j)}{\underline{e}_i} = c_{i2}. \tag{11}$$

If $\alpha_i = 2$, then $\phi^i(t) \geq d_{i2}$ for all $t \in R$. From equation (10), (A_2) and equation (8), we get

$$\begin{aligned} & (G_i^\alpha \phi)(t) \\ = \;\; & \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} e_i(y_i(w)) dw \right) \times \\ & \Big[d_{ii}(s) f_i(F_i^{-1}(\phi_j(s - \tau_{ij}(s))) + I_i(s) \Big] ds \\ & + \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} e_i(y_i(w)) dw \right) \times \\ & \Big[\sum_{j=1, j \neq i}^{n} d_{ij}(s) f_j(F_j^{-1}(\phi_j(s - \tau_{ij}(s)))) \\ & + \sum_{j=1}^{n} c_{ij}(s) \int_{s-\sigma}^{s} k_{ij}(s - v) g_j(F_j^{-1}(\phi_j(v))) dv \Big] ds \\ \geq \;\; & \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} e_i(y_i(w)) dw \right) ds \Big[\bar{d}_{ii} f_i(d_{i2}) \\ & + \inf_{t \in R} I_i(t) \Big] \\ & - \int_{-\infty}^{t} \exp\left(-\int_{s}^{t} e_i(y_i(w)) dw \right) ds \\ & \times \sum_{j=1}^{n} (\overline{d}_{ij} \gamma_j + \overline{c}_{ij} k_{ij}(1, \sigma) \chi_j) \\ \geq \;\; & \frac{1}{\bar{e}_i} \Big[\underline{d}_{ii} f_i(d_{i2}) + \inf_{t \in R} I_i(t) \\ & - \sum_{j=1}^{n} (\overline{d}_{ij} \gamma_j + \overline{c}_{ij} k_{ij}(1, \sigma) \chi_j) \Big] = d_{i2} \end{aligned} \tag{12}$$

for all $t \in R$. By equations (11) and (12), we have $d_{i2} \leq (G_i^\alpha \phi)(t) \leq c_{i2}$, if $\alpha_i = 1$, from similar argument, we can prove that $d_{i1} \leq (G_i^\alpha \phi)(t) \leq c_{i1}$ for all $t \in R$. Hence, we have $d_{i\alpha} \leq (G_i^\alpha \phi)(t) \leq c_{i\alpha}$ for each $i \in \Lambda$ and all $t \in R$.

Step 2: We should prove that $G^\alpha : \Gamma^\alpha \to \Gamma^\alpha$ is continuous. Take any two initial conditions $\phi, \tilde{\phi} \in \Gamma^\alpha$ with $\phi =$

$(\phi_1, \phi_2, \cdots, \phi_n)$ and $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \cdots, \tilde{\phi}_n)$. From equation (10) and the mean value theorem, we have

$$\begin{aligned} & |(G_i^\alpha \phi)(t) - (G_i^\alpha \tilde{\phi})(t)| \\ \leq & \int_{-\infty}^{t} \exp\Big(- \int_s^t e_i(y_i(w))dw \Big) \times \\ & \Big[\sum_{j=1}^{n} \bar{d}_{ij} \times |f_j(F_j^{-1}(\phi_j(s-\tau_{ij}(s)))) \\ & -f_j(F_j^{-1}(\tilde{\phi}_j(s-\tau_{ij}(s))))| \\ & + \sum_{j=1}^{n} \bar{c}_{ij}(s) \int_{s-\sigma}^{s} k_{ij}(s-v)|g_j(F_j^{-1}(\phi_j(v))) \\ & -g_j(F_j^{-1}(\tilde{\phi}_j(v)))|dv \Big] ds \\ \leq & \frac{\sum_{j=1}^{n} (\bar{d}_{ij} \sup_{\zeta \in R} f_j'(\zeta) + \bar{c}_{ij} k_{ij}(1,\sigma) \sup_{\zeta \in R} g_j'(\zeta)) \bar{a}_j}{\underline{e}_i} \\ & \times \|\phi - \tilde{\phi}\|_\omega, \end{aligned} \tag{13}$$

which leads to

$$\begin{aligned} & |(G^\alpha \phi)(t) - (G^\alpha \tilde{\phi})(t)| \\ \leq & \max_{i\in\Lambda} \left\{ \frac{\sum_{j=1}^{n} (\bar{d}_{ij} \sup_{\zeta \in R} f_j'(\zeta) + \bar{c}_{ij} k_{ij}(1,\sigma) \sup_{\zeta \in R} g_j'(\zeta)) \bar{a}_j}{\underline{e}_i} \right\} \\ & \times \|\phi - \tilde{\phi}\|_\omega, \end{aligned}$$

This implies that $G^\alpha(\cdot)$ is continuous with respect to $\phi \in G^\alpha$.

Since each G^α is compact and $G^\alpha : \Gamma^\alpha \to \Gamma^\alpha$ is continuous, by Schauder fixed point theorem, there exists at least one $u_\alpha(t) \in G^\alpha$ such that $G^\alpha u_\alpha = u_\alpha$. Hence $u_\alpha(t)$ is an almost periodic solution of system (3) in Γ^α. The proof is completed. ■

From Theorem 3.1, there exist 2^n almost periodic solutions of system (3) in these Γ^α. Now we shall discuss their convergent dynamics.

Theorem 3.2 Assume that (H_1)-(H_4) and (A_1)-(A_2) hold, then each H^α is an invariant basin of system (3).

Proof: For any initial condition $\phi \in H^\alpha$, we should prove that the solution $y(t,\phi)$ of system (3) satisfies $y(t,\phi) \in H^\alpha$ for all $t \geq 0$. For any given $i \in \Lambda$, we only consider the case $\alpha_i = 1$, i.e., $\phi_i(s) \leq c_{i1}$ for all $s \in [-\omega, 0]$. We assert that, for any sufficiently small $\varepsilon > 0 (\varepsilon \ll z_{i1} - c_{i1})$, the solution $y_i(t,\phi) < c_{k1} + \varepsilon$ holds for all $t \geq 0$. If this is not true, there exists a $t^* > 0$ such that $y_i(t^*) = c_{k1} + \varepsilon$, $y_i'(t^*) \geq 0$ and $y_i(t) < c_{k1} + \varepsilon$ for $t \in [-\omega, t^*]$. Due to $d_{ii}(t) > 0$, $c_{i1} + \varepsilon < 0$ and the monotonicity of $f_i(\cdot)$, we derive from system (3) that

$$\begin{aligned} y_i'(t^*) &= -e_i(y_i(t^*))y_i(t^*) \\ & + \sum_{j=1}^{n} d_{ij}(t^*) f_j(F_j^{-1}(y_j(t^* - \tau_{ij}(t^*)))) \\ & + \sum_{j=1}^{n} c_{ij}(t^*) \int_{t^*-\sigma}^{t^*} k_{ij}(t^*-s) g_j(F_j^{-1}(y_j(s)))ds \\ & + I_i(t^*) \\ &\leq -e_i(y_i(t^*))y_i(t^*) \\ & + d_{ii}(t^*) f_i(F_i^{-1}(y_i(t^* - \tau_{ii}(t^*)))) + I_i(t^*) \\ & + \sum_{j=1}^{n} (\bar{d}_{ij} \ )\gamma_j \\ & + \sum_{j=1}^{n} \bar{c}_{ij} \Big(\int_{t^*-\sigma}^{t^*} k_{ij}(t^*-s)ds \Big) \chi_j \\ &\leq -\bar{e}_i(c_{i1}+\varepsilon) + \underline{d}_{ii} f_i(c_{i1}+\varepsilon) + \sum_{j=1}^{n} (\bar{d}_{ij} \ \gamma_j \\ & + \bar{c}_{ij} k_{ij}(1,\sigma)\chi_j) + \sup_{t\in R} I_i(t) \\ &\leq \xi_k(c_{i1}+\varepsilon) + \sum_{j=1}^{n} (\bar{d}_{ij} \ \gamma_j + \bar{c}_{ij} k_{ij}(1,\sigma)\chi_j) \\ & + \sup_{t\in R} I_i(t). \end{aligned} \tag{14}$$

From Lemma 3.2, we know that $\xi_i(z)$ is strictly decreasing on $(-\infty, z_{i1}]$. By using equations (7) and (14), we get $u_i'(t^*) < 0$ which leads to a contradiction. Since the choice of ε is arbitrary, for each $i \in \Lambda$, if $\phi_i(s) \leq c_{i1}$ for all $s \in [-\omega, 0]$, then $u_i(t,\phi) \leq c_{i1}$ for all $t \geq 0$. When $\alpha_i = 2$, similar argument can be performed to show that if $\phi_i(s) \geq d_{i2}$ for all $s \in [-\omega, 0]$, then $u_i(t,\phi) \geq c_{i2}$ for all $t \geq 0$. Hence, for any $\phi \in H^\alpha$, we have that $u(t,\phi) \in H^\alpha$ for all $t \geq 0$. That is, each H^α is an invariant basin of system (3). ■

Theorem 3.3 Assume that $(H_1) - (H_4)$ and $(A_1) - (A_2)$ hold, suppose further that

(H_5) The activation functions $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_n)$ are Lipschitz functions, that is, there exist positive numbers λ_i, η_i such that $|f_i(x) - f_i(y)| \leq \lambda_i |x-y|$, $|g_i(x) - g_i(y)| \leq \eta_i |x-y|$, $i = 1, \ldots, n$.

(H_6) there exist n positive constants $\omega_i > 0$, $i = 1, 2, \ldots, n$, such that

$$-\Theta_i \underline{a}_i \omega_i + \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i \omega_j + \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \omega_j k_{ij}(1,\sigma) < 0,$$

for $i = 1, 2, \ldots, n$.

Then the following affirmations are true.

(1) There exists a unique almost periodic solution $u_\alpha(t)$ of system (3) in each Γ^α.

(2) H^α is an exponential attracting domain of almost periodic solution $u_\alpha(t)$.

Proof: Considering the function

$$\begin{aligned} \Upsilon_i(h) &= \Theta_i \underline{a}_i - h - \omega_i^{-1} \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i \omega_j e^{h\tau} \\ & -\omega_i^{-1} \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \omega_j \int_0^\sigma k_{ij}(s) e^{hs} ds \end{aligned}$$

From (H_3) and (H_6), we have

$$\begin{aligned} \Upsilon_i(0) &= \Theta_i \underline{a}_i - \omega_i^{-1} \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i \omega_j \\ & -\omega_i^{-1} \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \omega_j k_{ij}(1,\sigma) > 0 \end{aligned}$$

and $\Upsilon_i(h)$ is continuous, $\Upsilon_i(h) \to -\infty$ as $h \to +\infty$. So, there exists a $h_i > 0$ such that $\Upsilon_i(h_i) = 0$. Without loss of generally, set $\gamma_i = \min\{h > 0 | \Upsilon_i(h) = 0\}$, so $\Upsilon_i(\beta) > 0$ when $\beta \in (0, \gamma_i)$. Now, let $\gamma = \min\{\gamma_i,\ i = 1, 2, \cdots, n\}$, when $\mu \in (0, \gamma)$, we have $\Upsilon_i(\mu) > 0,\ i = 1, 2, \cdots, n$, i.e.

$$\begin{aligned}\Upsilon_i(\mu) &= \Theta_i \underline{a}_i - \mu - \omega_i^{-1} \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i \omega_j e^{\mu\tau} \\ &\quad -\omega_i^{-1} \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \omega_j \int_0^{\sigma} k_{ij}(s) e^{\mu s} ds \\ &> 0. \end{aligned} \tag{15}$$

According to Theorem 3.1, for each α, there exists at least one almost periodic solution $u_\alpha(t)$ of system (3) in Γ^α. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of system (3) and $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ be an almost periodic solution of system (3). Then

$$\begin{cases} y_i'(t) = -b_i(F_i^{-1}(y_i(t))) \\ \qquad + \sum_{j=1}^{n} d_{ij}(t) f_j(F_j^{-1}(y_j(t-\tau_{ij}(t)))) \\ \qquad + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s)\times \\ \qquad g_j(F_j^{-1}(y_j(s)))\mathrm{d}s + I_i(t),\ i \in \Lambda, \\ y_i(s) = F_i(\varphi_i(s)) = \phi_i(s), -\omega \le s \le 0,\ i \in \Lambda, \end{cases} \tag{16}$$

$$\begin{cases} y_i^{*\prime}(t) = -b_i(F_i^{-1}(y_i^*(t))) \\ \qquad + \sum_{j=1}^{n} d_{ij}(t) f_j(F_j^{-1}(y_j^*(t-\tau_{ij}(t)))) \\ \qquad + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s)\times \\ \qquad g_j(F_j^{-1}(y_j^*(s)))\mathrm{d}s + I_i(t),\ i \in \Lambda, \\ y_i^*(s) = F_i(\varphi_i^*(s)) = \phi_i^*(s), -\omega \le s \le 0,\ i \in \Lambda, \end{cases} \tag{17}$$

Let $z(t) = y(t) - y^*(t)$, then we can get the following system

$$\begin{cases} z_i'(t) = -[b_i(F_i^{-1}(z_i(t) + y_i^*(t))) \\ \qquad -b_i(F_i^{-1}(y_i^*(t)))] \\ \qquad + \sum_{j=1}^{n} d_{ij}(t)[f_j(F_j^{-1}(z_j(t-\tau_{ij}(t)) \\ \qquad +y_j^*(t-\tau_{ij}(t)))) \\ \qquad -f_j(F_j^{-1}(y_j^*(t-\tau_{ij}(t))))] \\ \qquad + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s)[g_j(F_j^{-1}(z_j(t) \\ \qquad +y_j^*(t))) - g_j(F_j^{-1}(y_j^*(s)))]\mathrm{d}s,\ t > 0, \\ z_i(s) = \phi_i(s) - \phi_i^*(s) = \Phi_i(t),\ t \le 0. \end{cases} \tag{18}$$

Similarly, by the mean value theorem, then

$$\begin{aligned} & b_i(F_i^{-1}(z_i(t) + y_i^*(t))) - b_i(F_i^{-1}(y_i^*(t))) \\ = & [b_i(F_i^{-1}(y_i^*(t) + \hat{\theta} z_i(t)))]' z_i(t) = \hat{e}_i(z_i(t)) z_i(t), \end{aligned}$$

where $\hat{e}_i(z_i(t)) = [b_i(F_i^{-1}(y_i^*(t) + \hat{\theta} z_i(t)))]',\ 0 < \hat{\theta} < 1$.

In the light of $(H_2)'$ and (4), we have $\hat{e}_i(F_i^{-1}(\cdot)) \ge \Theta_i \underline{a}_i > 0$.

Now, define a Lyapunov function

$$V = (V_1, V_2, \ldots, V_n)^T,$$

where $V_i = \omega_i^{-1} e^{\mu t} |z_i(t)|,\ \mu \in (0, \gamma),\ i = 1, 2, \ldots, n$.

$$\begin{aligned} & D^+ V_i(t) \\ = \ & \omega_i^{-1} \mu e^{\mu t} |z_i(t)| + \omega_i^{-1} e^{\mu t} \mathrm{sign} z_i \Big\{ - \hat{e}_i(z_i(t)) z_i(t) \\ & + \sum_{j=1}^{n} d_{ij}(t)[f_j(F_j^{-1}(z_j(t-\tau_{ij}(t)) \\ & + y_j^*(t-\tau_{ij}(t)))) - f_j(F_j^{-1}(y_j^*(t-\tau_{ij}(t))))] \\ & + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s)[g_j(F_j^{-1}(z_j(t) + y_j^*(t))) \\ & - g_j(F_j^{-1}(y_j^*(s)))]\mathrm{d}s \Big\} \\ \le \ & \omega_i^{-1} \mu e^{\mu t} \Big\{ \mu |z_i(t)| - \Theta_i \underline{a}_i |z_i(t)| \\ & + \sum_{j=1}^{n} d_{ij}(t) |f_j(F_j^{-1}(z_j(t-\tau_{ij}(t)) \\ & + y_j^*(t-\tau_{ij}(t)))) - f_j(F_j^{-1}(y_j^*(t-\tau_{ij}(t))))| \\ & + \sum_{j=1}^{n} c_{ij}(t) \int_{t-\sigma}^{t} k_{ij}(t-s) |g_j(F_j^{-1}(z_j(t) + y_j^*(t))) \\ & - g_j(F_j^{-1}(y_j^*(s)))| \mathrm{d}s \Big\} \\ \le \ & \omega_i^{-1} \mu e^{\mu t} \Big\{ \mu |z_i(t)| - \Theta_i \underline{a}_i |z_i(t)| \\ & + \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i |z_j(t-\tau_{ij}(t))| \\ & + \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \int_{t-\sigma}^{t} k_{ij}(t-s) |z_j(s)| ds \Big\} \\ \le \ & -(\Theta_i \underline{a}_i - \mu) V_i(t) + \omega_i^{-1} \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i e^{\mu\tau} V_j(t-\tau_{ij}(t)) \\ & + \omega_i^{-1} \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \int_{t-\sigma}^{t} k_{ij}(t-s) e^{\mu(t-s)} V_j(s) ds \\ \le \ & -(\Theta_i \underline{a}_i - \mu) V_i(t) + \omega_i^{-1} \sum_{j=1}^{n} \bar{c}_{ij} \eta_j \bar{a}_i e^{\mu\tau} \sup_{t-\tau \le s \le t} V_j(s) \\ & + \omega_i^{-1} \sum_{j=1}^{n} \bar{d}_{ij} \bar{a}_i \lambda_j \int_0^{\sigma} k_{ij}(t-s) e^{\mu s} ds \sup_{s \le t} V_j(s). \end{aligned}$$

Let $M = \sup_{t \le 0} \max_i \{\omega_i^{-1} |z_i(t)|\}$ and $\tilde{\xi} > 1$ is an arbitrary real number. Then, we have

$$\begin{aligned} V_i(t) &= \omega_i^{-1} e^{\mu t} |z_i(t)| \le \omega_i^{-1} |z_i(t)| \le M < \tilde{\xi} M,\ t \le 0, \\ & \quad i = 1, 2, \cdots, n. \end{aligned} \tag{19}$$

In the following, we shall show that

$$V_i(t) < \tilde{\xi} M,\ t > 0,\ i = 1, 2, \cdots, n. \tag{20}$$

if (20) is not true, without loss of generality, then there exist an $\hat{k}$ and a first time $t_1 > 0$ such that

$$V_i(t) < \tilde{\xi}M,\ i \neq \hat{k},\ t \in (-\infty, t_1];$$
$$V_{\hat{k}}(t) < \tilde{\xi}M,\ t \in (-\infty, t_1],\ V_{\hat{k}}(t_1) = \tilde{\xi}M,\ \frac{dV_{\hat{k}}(t_1)}{dt} \geq 0.$$

Combining with (15), we have

$$\begin{aligned}
& D^+V_i(t) \\
\leq\ & -(\Theta_i\underline{a}_i - \mu)V_i(t) + \omega_i^{-1}\sum_{j=1}^{n}\bar{c}_{ij}\eta_j\bar{a}_i e^{\mu\tau}\sup_{t-\tau\leq s\leq t}V_j(s) \\
& +\omega_i^{-1}\sum_{j=1}^{n}\bar{d}_{ij}\bar{a}_i\lambda_j\int_0^{\sigma}k_{ij}(t-s)e^{\mu s}ds\sup_{s\leq t}V_j(s) \\
\leq\ & \Big\{-(\Theta_i\underline{a}_i - \mu) + \omega_i^{-1}\sum_{j=1}^{n}\bar{c}_{ij}\eta_j\bar{a}_i e^{\mu\tau} \\
& +\omega_i^{-1}\sum_{j=1}^{n}\bar{d}_{ij}\bar{a}_i\lambda_j\int_0^{\sigma}k_{ij}(t-s)e^{\mu s}ds\Big\}\tilde{\xi}M \\
=\ & -\Big\{\Theta_i\underline{a}_i - \mu - \omega_i^{-1}\sum_{j=1}^{n}\bar{c}_{ij}\eta_j\bar{a}_i e^{\mu\tau} \\
& -\omega_i^{-1}\sum_{j=1}^{n}\bar{d}_{ij}\bar{a}_i\lambda_j\int_0^{\sigma}k_{ij}(t-s)e^{\mu s}ds\Big\}\tilde{\xi}M \\
=\ & -\Upsilon_{\hat{k}}(\mu)\tilde{\xi}M < 0.
\end{aligned}$$

This is a contradiction, hence (20) holds. Let $\tilde{\xi} \to 1$, then

$$V_i(t) < M,\ t > 0,\ i = 1, 2, \cdots, n. \tag{21}$$

Together (19) with (21), we have

$$V_i(t) < M,\ t \in R,\ i = 1, 2, \cdots, n.$$

that is $\omega_i^{-1}e^{\mu t}|z_i(t)| \leq \sup_{t\leq 0}\max_i\{\omega_i^{-1}|z_i(t)|$.

So, we have

$$|z_i(t)| = |y_i(t) - y_i^*(t)| \leq \omega_i e^{-\mu t}\sup_{t\leq 0}\max_i\{\omega_i^{-1}|y_i(t) - y_i^*(t)|\},$$

which implies that all other solutions converge exponentially to its almost periodic solution. This completes the proof. ■

IV. An example

Consider the following system

$$\begin{cases}
x_i'(t) = -a_i(x_i(t))\Big(b_i(x_i(t)) \\
\quad - \sum_{j=1}^{n} c_{ij}(t)\int_{t-\sigma}^{t}k_{ij}(t-s)g_j(x_j(s))ds \\
\quad - \sum_{j=1}^{n} d_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t)\Big), \\
\quad t \in R, t > 0,\ i \in \Lambda, \\
x_i(s) = \varphi_i(s), -\omega \leq s \leq 0,\ i \in \Lambda,
\end{cases} \tag{22}$$

where

$$\begin{aligned}
& a_i(x_i(t)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
& b_i(x_i(t)) = \begin{bmatrix} 1.5x_1(t) & 0 \\ 0 & 3x_2(t) \end{bmatrix}, \\
& c_{ij}(t) = \begin{bmatrix} -1 + 0.1\sin 2t & 0 \\ 0 & -1 + 0.2\cos\sqrt{5}t \end{bmatrix}, \\
& d_{ij}(t) = \begin{bmatrix} 8 + 2\sin\sqrt{2}t & 0 \\ 0 & 9 + \cos\sqrt{3}t \end{bmatrix}, \\
& I_i(t) = \begin{bmatrix} 3\sin\sqrt{5}t & 0 \\ 0 & 3\cos 2t \end{bmatrix}, \\
& \tau_{ij}(t) = \begin{bmatrix} 1 - 0.5\sin t & 0 \\ 0 & 1 + 0.5\cos t \end{bmatrix}, \\
& k_{ij}(s) = \begin{bmatrix} 0 & e^{-s} \\ 2se^{-s^2} & 0 \end{bmatrix}, \\
& f(x) = f_i(x) = \tanh x,\ g_i(x) = 10\tanh x, \\
& \sigma = +\infty,\ i, j = 1, 2.
\end{aligned}$$

Then we have the following equalities:

$$\begin{aligned}
& \xi_1(z) = -1.5z + 6f_1(z),\ \tau_{12}(1) = \int_0^{\infty}k_{12}(s)ds = 1, \\
& \gamma_1 = 1,\ \chi_1 = 10,\ \inf_{t\in\mathbb{R}} f'(z) = 0, \\
& \xi_2(z) = -3z + 10f_2(z),\ \tau_{21}(1) = \int_0^{\infty}k_{21}(s)ds = 1, \\
& \gamma_2 = 1,\ \chi_2 = 10,\ \sup_{t\in\mathbb{R}} f'(z) = 1.
\end{aligned}$$

It is easy for us to check that $(H_1) - (H_4)$, (A_1) hold and any solution of CGNNs (22) is uniformly bounded in Ω which be defined as follows:

$$\begin{aligned}
\Omega = \{\phi \in C([-\infty, 0], \mathbb{R}^2)||\phi_1(s)| \leq 24.0667, \\
|\phi_2(s)| \leq 18.1333,\ s \in [-\infty, 0]\}.
\end{aligned}$$

From some computations, we get that $z_{11} = -1.3169$, $z_{12} = 1.3169$, $z_{21} = -1.2099$, $z_{22} = 1.2099$ such that $\xi_i'(z_{il}) = 0$, then $\xi_1(z_{1l}) = (-1)^l 3.2207$, $\xi_2(z_{1l}) = (-1)^l 1.9962$, $l = 1, 2$. With these numerical results, we can check (A_2)

$$\begin{aligned}
& (-1)^l \cdot \{\xi_1(z_{il}) + I_1(t)\} \\
=\ & (-1)^l[(-1)^l 3.2207 + 3\sin\sqrt{5}t] > -0.9 = \bar{c}_{12}, \\
& (-1)^l \cdot \{\xi_2(z_{il}) + I_2(t)\} \\
=\ & (-1)^l[(-1)^l 11.9962 + 0.3\cos 2t] > -0.8 = \bar{c}_{21}
\end{aligned}$$

hold. and

$$\begin{aligned}
& d_{11} = -24.0667,\ d_{21} = -18.1333,\ c_{12} = 24.0667, \\
& c_{22} = 18.1333,\ c_{11} = -6.9999,\ c_{21} = -4.9997, \\
& d_{12} = 6.2999,\ d_{22} = 4.9997.
\end{aligned}$$

Let $\lambda_i = 1, \eta_i = 10$, it is easy to check that $(H_5) - (H_6)$ hold. From Theorem 3.1 and Theorem 3.3, we know that there exists only four exponentially stable almost periodic solutions of CGNNs (22) in each Γ^{α}.

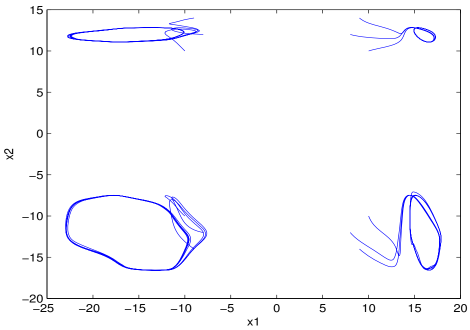

Fig. 1. Convergence dynamics of four almost periodic solutions of CGNNs (22).

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