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$1-$skeleton resolution of free simplicial algebras with given $CW-$basis

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Abstract—In this paper we use the definition of CW basis of a free simplicial algebra. Using the free simplicial algebra, it is shown to construct free or totally free $2-$crossed modules on suitable construction data with given a $CW-$basis of the free simplicial algebra. We give applications free crossed squares, free squared complexes and free $2-$crossed complexes by using of 1(one) skeleton resolution of a step by step construction of the free simplicial algebra with a given $CW-$basis.

Keywords—Free crossed square, Free $2-$crossed modules, Free simplicial algebra, Free square complexes, Free $2-$crossed complexes $CW-$basis, $1-$skeleton.
A. M. S.Classification:[2000] 18D35, 18G30,18G50, 18G55, 55Q05, 55Q20.

I. INTRODUCTION

Simplicial commutative algebras play an important role in homological algebra, homotopy theory and algebraic $K-$theory. The present article intends to study the 1(one) skeleton resolution of a step by step construction of a free simplicial algebra via André method's, using with a given $CW-$basis.

This study will be appear using step by step of Andre's method for a free simplicial algebra with a given $CW-$basis in the $1-$skeleton. Brown [2] and Ellis [3] presented the definitions of coproduct and tensor product of groups. Coproduct structure of algebras was introduced by Shammu in [8]. Brown, Ellis and Shammu did not introduce to free simplicial algebra with a given $CW-$basis and also they never used these definitions in their researches.

The step by step construction was used by Arvasi and Porter in [1], but they did not build up this construction for the free simplicial algebra with a given $CW-$basis. For this reason, our method is completely different from their method and the soonest finding results show that likeness on free crossed square, squared complex and $2-$crossed complex are basically structure of algebraic topology.

We reach some results on these structures of algebraic topology using the free simplicial algebra with a given $CW-$basis. Therefore our method is more trustworthy which is easily verified.

In this article we firstly introduce to the free simplicial algebra with a given $CW-$basis. In addition, we find some results using the free simplicial algebra with a given $CW-$basis. Secondly, we give $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis $\mathbb{A}^{(1)}$. Furthermore, we present the details of calculation concerning with in the applications of coproduct and tensor product of crossed square, square complex and $2-$crossed complex in the section 3 and 4.

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II. THE MOORE COMPLEX OF SIMPLICIAL ALGEBRA

Let us given a simplicial algebra $\mathcal{A}$, then we may definite the *Moore complex* of simplicial algebra as follows: The *Moore complex* $(\mathbf{N}\mathcal{A},\partial)$ *of* $\mathcal{A}$ is the chain complex defined by $(\mathbf{N}\mathcal{A})_n = \bigcap\limits_{i=0}^{n-1} \mathrm{Ker} d_i^n$ with $\partial_n : N\mathcal{A}_n \to N\mathcal{A}_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(\mathcal{A})$ of $\mathcal{A}$ is the n^{th} homology of the Moore complex of $\mathcal{A}$, i.e., $\pi_n(\mathcal{A}) \cong H_n(\mathbf{N}\mathcal{A},\partial) = \bigcap\limits_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}(\bigcap\limits_{i=0}^{n} \mathrm{Ker} d_i^{n+1})$.

III. FREE SIMPLICIAL ALGEBRA WITH GIVEN CW-BASIS

We recall that two definitions in [6] as follows in this section.

Definition 3.1: A simplicial algebra $\mathbb{A}$ is called free if
(i) $\mathcal{A}_n$ is a free algebra with a given basis, for every integer $n \geq 0$,
(ii) the basis are stable under all degeneracy operators, i.e., for every pair of integers (i,n) with $0 \leq i \leq n$ and every given generator $x \in \mathcal{A}_n$ and the element $s_i(x)$ is a given generator of $\mathcal{A}_{n+1}$.

Definition 3.2: Let $\mathbb{A}$ be a free simplicial algebra (as above). A subset $\mathfrak{A} \subset \mathbb{A}$ will be called a $CW-basis$ for $\mathbb{A}$ if
(a) $\mathfrak{A}_\mathfrak{n} = \mathfrak{A} \cap \mathcal{A}_n$ freely generates $\mathcal{A}_n$ for all $n \geq 0$,
(b) $\mathfrak{A}$ is closed under degeneracies, i.e., $x \in \mathfrak{A}_\mathfrak{n}$ implies $s_i(x) \in \mathfrak{A}_{n+1}$ for all $0 \leq i \leq n$,
(c) if $x \in \mathfrak{A}_n$ is non-degenerate, then $d_i(x) = e_{n-1}$, (e_{n-1}, the identity element of $\mathcal{A}_{n-1}$) for all $0 \leq i < n$.
Let $\mathbb{A}$ be a free simplicial algebra with a given $CW-$basis, $\mathfrak{A}$, then $X_0 = \mathfrak{A}_0$ freely generates $\mathcal{A}_0$, that is, $\mathcal{A}_0 = \mathcal{A}[X_0]$. If we continue process of Definition 3.2, then $\mathfrak{A}_1$ freely generates $\mathcal{A}_1$ and let $s_0(X_0) \subseteq \mathfrak{A}_1$ and also if $Y_1 = \mathfrak{A}_1 \setminus s_0(X_0)$, then $d_i(y) = 0$ for $0 \leq i < 1$ where $y \in Y_1$. Therefore let $\mathcal{A}_1 = \mathcal{A}[s_0(X_0) \cup Y_1] \cong \mathcal{A}[s_0(X_0)] * \mathcal{A}[Y_1]$, where "$*$" is free product of $\mathcal{A}[s_0(X_0)]$ and $\mathcal{A}[Y_1]$. For $\mathcal{A}_2$, let $s_0(\mathfrak{A}_1) \cup s_1(\mathfrak{A}_1) \subseteq \mathfrak{A}_2$ and also if $y \in Y_2 = \mathfrak{A}_2 \setminus \bigcup\limits_{i=0}^{1} s_i(\mathfrak{A}_i)$, then $d_0(y) = d_1(y) = 0$ and additionally y is in $N\mathcal{A}_2$. In general, note that if $Y_n = \mathfrak{A}_n \setminus \bigcup\limits_{i=0}^{n-1} s_i(\mathfrak{A}_i)$ then $Y_n \subseteq N\mathcal{A}_n$ and Y_n normally generates $N\mathcal{A}_n$.

Thus we can give $1-$skeleton resolution of $\mathbb{A}^1$ of the free simplicial algebra with a given $CW-$basis. The $1-$skeleton $\mathbb{A}^{(1)}$ of the free simplicial resolution of an algebra A can be built by adding new indeterminate for instance, if there is an one to one correspondence with Ω^0 as of generator for $\pi_1(\mathbb{A})$, $\mathcal{A}_1^{(1)} = \mathcal{A}_1^{(0)}[X_0] = \mathcal{A}[s_0(X_0) \cup Y_1] \cong \mathcal{A}[s_0(X_0)] * \mathcal{A}[Y_1]$, where with the face maps and degeneracy map

$$\mathcal{A}[s_0(X_0) \cup Y_1] \underset{s_0}{\overset{d_0,d_1}{\rightleftarrows}} \mathcal{A}[X_0] \xrightarrow{d_0^0} A$$

here $\mathcal{A}(X_0) \xrightarrow{d_0^0} A$ is an augmentation map and s_0, d_0^1 and d_1^1 are given by

$$d_1^1(y_1) = b_1 \in \mathrm{Ker} d_0^0, \qquad d_0^1(y_1) = 0, y_1 \in Y_1,$$
$$x_0 = s_0(x_0) \quad \text{for} \quad x_0 \in X_0$$

Thus $1-$skeleton resolution $\mathbb{A}^{(1)}$ looks like:

$$\mathbb{A}^{(1)} : \cdots \ \mathcal{A}_2 \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightleftarrows}} \mathcal{A}_1 \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} \mathcal{A}_0 \xrightarrow{f} A/I.$$

where $\mathcal{A}_2^{((1),2)} = \mathcal{A}[s_1s_0(X_0) \cup s_1(Y_1) \cup s_0(Y_1)]$, $\mathcal{A}_1^{((1),1)} = \mathcal{A}[s_0(X_0) \cup Y_1]$ and $\mathcal{A}_0^{((1),0)} = \mathcal{A}[X_0]$.

A. Applications of Coproduct and Tensor Product of Crossed Square

In the background of $CW-$complexes, Ellis [3] presented an extraordinary definition of the top group in (totally) the free crossed square using topological methods. A free simplicial algebra with a given $CW-$basis is the algebraic analogue of a $CW-$complex. Therefore one expect that the similar result to hold in that setting. Now we give to application of tensor product for crossed square of $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis. The functor from the category of simplicial algebras to the crossed cubes is defined as

$$\mathfrak{M}(-,2) : SimpAlg \to Crn^n$$

in [1]. If for $n = 2$ we apply this functor to the $1-$skeleton resolution $\mathbb{A}^{(1)}$ of the free simplicial algebra with a given $CW-$basis $\mathcal{A}$, we get $\mathfrak{M}(\mathbb{A}^{(1)}, 2)$ which is the free crossed square is

$$\mathfrak{M}(\mathbb{A}^{(1)},2) \approxeq \begin{array}{ccc} N\mathcal{A}_2^{((1),2)}/\partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3) & \xrightarrow{\partial_2} & \mathrm{Ker}\ d_0^{((1),2)} \\ \downarrow{\partial_2'} & & \downarrow{\mu} \\ \mathrm{Ker}\ d_1^{((1),2)} & \xrightarrow[\mu']{} & \mathcal{A}_1^{((1),1)} \end{array}$$

together with the $h-$map

$$h{:}\mathrm{Ker} d_1{}^{((1),2)} \times \mathrm{Ker} d_0{}^{((1),2)} \longrightarrow N\mathcal{A}_2^{((1),2)}/\partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3)$$

given by

$$x \otimes y = h(x,y) = s_1y_0(s_1y_1 - s_0y_1) \mod \partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3)$$

where " $\otimes$ " is defined as coproduct or tensor product, and let I_n be the ideal generated by the degenerate elements of $\mathcal{A}_n$ (for $n = 3$). Then

$$\mathcal{A}_2^{((1),2)} = \mathcal{A}[s_1s_0(X_0) \cup s_1(Y_1)],$$

$$N\mathcal{A}_2^{((1),2)} = \bigcap_{i=0}^{1} \mathrm{Ker} d_i^1 = [s_1(Y_1)]^+ \cap Z',$$

$$\mathrm{Ker} d_0^1 = [s_1(Y_1)]^+,$$

$$\mathrm{Ker} d_1^1 = Z' = \{s_1(y_1) - s_0(y_1) : \quad y_1 \in Y_1\};$$

$$\mathcal{A}_3^{((1),3)} = \mathcal{A}[s_2s_1s_0(X_0) \cup s_2s_1(Y_1) \cup s_2s_0(Y_1) \cup s_1s_0(Y_1)],$$

$$N\mathcal{A}_3^{((1),3)} = \bigcap_{i=0}^{2} \mathrm{Ker} d_i^{((1),3)} = [s_2s_1(Y_1)]^+ \cap Z^+ \cap Z_1^+,$$

$$\mathrm{Ker} d_0^{((1),3)} = [s_2s_1(Y_1)]^+,$$

$$\mathrm{Ker} d_1^{((1),3)} = Z^+, \quad \mathrm{Ker} d_2^{((1),3)} = Z_1^+,$$

$Z = \{s_2s_1(x) - s_2s_0(x) \quad y_1 \in Y_1\}$ and
$Z_1 = \{s_2s_0(y_1) - s_1s_0(y_1) : \quad y_1 \in Y_1\}$.

Now we can conclude that if we use a $CW-$basis of the free simplicial algebra, we calculate the $1-$skeleton resolution of a $CW-$basis of the free simplicial algebra. Thus we get the result as follows.

Corollary 3.3: Let $\mathbb{A}^{(1)}$ be the $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis. If the free crossed square $\mathfrak{M}(\mathbb{A}^{(1)}, 2)$ as described above, then

$$N\mathcal{A}_2^{((1),2)}/\partial_3(N\mathcal{A}_3^{((1),3)} \cap I_3) \cong \mathrm{Ker} d_1{}^{((1),2)} \otimes_{\mathcal{A}_1^{((1),1)}} \mathrm{Ker} d_0{}^{((1),2)}.$$

Proof: The proof is clear since
$x \otimes y = h(x,y) = s_1y_0(s_1y_1 - s_0y_1)$. See [1] in details. □

Corollary 3.4: If the simplicial algebra $\mathbb{A}$ equals to its $1-$skeleton, so that $\mathbb{A} = \mathbb{A}^{(1)}$, then

$$\pi_2(\mathbb{A}) = \mathrm{Ker}(\mathrm{Ker} d_1^{((1),1)} \otimes_{\mathcal{A}_1^{((1),1)}} \mathrm{Ker} d_0^{((1),1)} \to \mathcal{A}_0^{((1),0)}).$$

Proof: We prove for $k \geq 1$. If $\mathbb{A}^{(k)}$ is $k-$skeleton resolution of the free simplicial algebra with a given $CW-$basis, $\mathbb{A}$, for $k \geq 1$, then

$$\pi_k(\mathbb{A}^{(k)}) = \mathrm{Ker}(N\mathcal{A}_k^{((k),k)}/\partial_{k+1}(N\mathcal{A}_{k+1}^{((k),k+1)}) \longrightarrow \mathcal{A}_{k+1}^{(k)}).$$

By taking $k = 1$ and the previous corollary, we get the result. □

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IV. Applications of Coproduct and Tensor Product of Squared Complex and 2-Crossed Complex

Now we can give some of applications of $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis such as square complex and $2-$crossed complex in some structure of algebraic topology. Let $\mathbb{A}^{(1)}$ be the $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis of an algebra. By Corollary 3.3, we know that

$$\begin{array}{ccc} \mathrm{Ker}d_1^{((1),1)} \otimes_{\mathcal{A}_1^{(1)}} \mathrm{Ker}d_0^{((1),1)} & \longrightarrow & \mathrm{Ker}d_0^{((1),1)} \\ \downarrow & & \downarrow \\ \mathrm{Ker}d_1^{((1),1)} & \longrightarrow & \mathcal{A}_1^{((1),1)} \end{array}$$

is free square. Thus the free squared complex is

$$\cdots \to \mathcal{C} \to \overline{N\mathcal{A}_1^{((1),1)}} \otimes_{\mathcal{A}_1^{((1),1)}} N\mathcal{A}_1^{((1),1)} \begin{array}{c} \xrightarrow{\lambda'} \overline{N\mathcal{A}_1^{((1),1)}} \xrightarrow{\mu'} \\ \xrightarrow{\lambda} N\mathcal{A}_1^{((1),1)} \xrightarrow{\mu} \end{array} \mathcal{A}_1^{((1),1)} \quad (*_1)$$

where $\mathcal{A}_2^{((1),2)} = \mathcal{A}[s_1 s_0(X_0) \cup s_1(Y_1) \cup s_0(Y_1)]$, $\mathcal{A}_1^{((1),1)} = \mathcal{A}[s_0(X_0) \cup Y_1]$ and $\mathcal{A}_0^{((1),0)} = \mathcal{A}[X_0]$, where$N\mathcal{A}_1^{((1),1)} = \mathrm{Ker}d_0^1 = [Y_1]^+$, $\overline{N\mathcal{A}_1^{((1),1)}} = \mathrm{Ker}d_1^{((1),1)} = [s_1(y_1) - s_0(y_1)] = Z'$,

$$\mathcal{C} = \frac{N\mathcal{A}_3^{((1),3)}}{(N\mathcal{A}_3^{((1),3)} \cap I_3) + d_4(N\mathcal{A}_4^{((1),4)} \cap I_4)},$$

$\mathcal{A}_3^{((1),3)} = \mathcal{A}[s_2 s_1 s_0(X_0) \cup s_2 s_1(Y_1) \cup s_2 s_0(Y_1) \cup s_1 s_0(Y_1)]$,
$N\mathcal{A}_3^{((1),3)} = \bigcap_{i=0}^{2} \mathrm{Ker}d_i^1 = [s_2 s_1(Y_1)]^+ \cap Z^+ \cap Z_1^+$
$\mathrm{Ker}d_0^{((1),3)} = [s_2 s_1(Y_1)]^+$,
$\mathrm{Ker}d_1^{((1),3)} = Z^+$, $Z = \{s_2 s_1(y_1) - s_2 s_0(y_1), y_1 \in Y_1\}$
and $\mathrm{Ker}d_2^{((1),3)} = Z_1^+$, $Z_1 = \{s_2 s_0(y_1) - s_1 s_0(y_1) : \ y_1 \in Y_1\}$.
$\mathcal{A}_4^{((1),4)} = \mathcal{A}[s_3 s_2 s_1 s_0(X_0) \cup s_3 s_2 s_1(Y_1) \cup s_3 s_2 s_0(Y_1) \cup s_3 s_1 s_0(Y_1) \cup s_2 s_1 s_0(Y_1)]$,
$N\mathcal{A}_4^{((1),4)} = \bigcap_{i=0}^{3} \mathrm{Ker}d_i^{((1),4)} = [s_3 s_2 s_1(Y_1)]^+ \cap Z_2^+ \cap Z_3^+ \cap Z_4^+$,
$\mathrm{Ker}d_0^{((1),4)} = [s_3 s_2 s_1(Y_1)]^+$ $Z_2^+ = \mathrm{Ker}d_1^{((1),4)}$, $Z_2 = \{s_3 s_2 s_1(y_1) - s_3 s_2 s_0(y_1), \ y_1 \in Y_1\}$, $Z_3^+ = \mathrm{Ker}d_2^{((1),4)}$, $Z_3 = \{s_3 s_2 s_0(y_1) - s_3 s_1 s_0(y_1), \ y_1 \in Y_1\}$, $Z_4^+ = \mathrm{Ker}d_3^{((1),4)}$,
since $Z_4 = \{s_3 s_1 s_0(y_1) - s_2 s_1 s_0(y_1) : \ y_1 \in Y_1\}$ by a *free square complex*, we mean one in which the crossed square is free, and in which each C_n is free algebra with a given $CW-$basis for $n \geq 3$.

Crossed modules techniques give a very efficient way to get the information on a homotopy type. If they clearly correspond to $1-$types (and hence topological $2-$types) then this is called as the algebraic model. We have recalled from Grandjeán and Vale's work [4], in 1986 that $2-$crossed modules correspond to $2-$types (and hence topological $3-$types) is also defined as algebraic model.

If $C_1 \to C_0$ is a crossed module in a crossed complex $(\cdots \to C_n \to \cdots \to C_2 \to C_1 \to C_0)$, then we say that a crossed complex is called a $1-$crossed complex and still if $C_2 \to C_1 \to C_0$ is a $2-$crossed module in a crossed complex, then we say that a crossed complex is also called a $2-$crossed complex. Additionally $C_1 \to C_0$ and $C_2 \to C_1 \to C_0$ are said to be "tail" of crossed complex. Furthermore, totally free $2-$crossed complex of group was defined by A. Mutlu and T. Porter in [7], and $2-$crossed complex of algebra was defined by A. Mutlu in [5].

Now we can give the definition in [5] as follows.

Definition 4.1: A $2-$crossed complex of algebras is a sequence of algebras

$$C: \qquad \cdots \to C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots \to C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$$

in which
(i) C_0 acts on C_n, $n \geq 1$, the action of ∂C_1 is trivial on C_n for $n \geq 3$;
(ii) for each ∂_n this is a C_0-algebra homomorphism and $\partial_i \partial_{i+1} = 0$ for all $i \geq 1$;
and
(iii) $C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$ is a $2-$crossed module.
A $2-$crossed complex C will be said to be *free* if for $n \geq 3$, the $C_0/\partial C_1$-module, C_n are free and the $2-$crossed complex at the base is also free $2-$crossed module. Additionally it will be *totally free* if the base $2-$crossed module is totally free (see [5]).

Now we can examine a free $2-$crossed complex of the free simplicial algebra. Firstly $\mathbb{A}^{(1)}$ be the $1-$skeleton resolution of the free simplicial algebra with a given $CW-$basis. Then free $2-$crossed complex is

$$\cdots \xrightarrow{\partial_5} C_5 \xrightarrow{\partial_4} C_4 \xrightarrow{\partial_3} \mathcal{C}_3 \xrightarrow{\partial_2} N\mathcal{A}_1^{((1),1)} \xrightarrow{\partial_1} N\mathcal{A}_0^{((1),0)}$$

where $\mathcal{C}_3 = \frac{N\mathcal{A}_2^{((1),2)}}{(N\mathcal{A}_2^{((1),2)} \cap I_2) + d_3(N\mathcal{A}_3^{((1),3)} \cap I_3)}$, and for each C_n is the free simplicial algebra with a given $CW-$basis, then

$$C_n = \begin{cases} N\mathcal{A}_n & \text{for } n = 0, 1 \\ N\mathcal{A}_2 / d_3(N\mathcal{A}_3 \cap I_3) & \text{for } n = 2 \\ N\mathcal{A}_n / (N\mathcal{A}_n \cap I_n) + d_{n+1}(N\mathcal{A}_{n+1} \cap I_{n+1}) & \text{for } n \geq 3 \end{cases}$$

is induced by the differential of $N\mathcal{A}$ with ∂_n (see [7], [1]).

We thus have a functor

$$\rho : \mathfrak{SkSimpAlg}_{\leq 1} \longrightarrow \mathfrak{FreeSqComp}$$

where $\mathfrak{SkSimpAlg}_{\leq 1}$ is the category of the $1-$skeleton of the free simplicial algebra with a given $CW-$basis and $\mathfrak{FreeSqComp}$ is the category of the free squared complex.

The *homotopy module* $\pi_n(\rho)$, for $n \geq 1$, the squared complex $(*_1)$ is defined to be the homology module of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \rtimes N \xrightarrow{\partial_2} \mathcal{A} \longrightarrow 0$$

with $\partial_3(l) = (-\lambda' l, \lambda l)$ and $\partial_2(m, n) = \mu(m) + \mu'(n)$. The axioms of a crossed square guarantee that ∂_3 and ∂_2 are homomorphism with $\partial_4(C_4)$ ideal in $\mathrm{Ker}(\partial_3)$, $\partial_3(L)$ ideal in $\mathrm{Ker}(\partial_2)$, and $\partial_2(M \rtimes N)$ ideal in $\mathcal{A}$ respectively. Clearly $\pi_n(\rho) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$.

Note that the homotopy module $\pi_n(\rho(\mathbb{A}^{(1)}))$ of the squared complex

$$\rho\left(\mathbb{A}^{(1)}\right) = \left(C_n, \left(\begin{array}{ccc} \overline{N\mathcal{A}_1^{((1),1)}} \otimes_{\mathcal{A}_1^{(1)}} N\mathcal{A}_1^{((1),1)} & \longrightarrow & N\mathcal{A}_1^{((1),1)} \\ \downarrow & & \downarrow \\ \overline{N\mathcal{A}_1^{((1),1)}} & \longrightarrow & \mathcal{A}_1^{(1)} \end{array} \right) \right)$$

is the homology module of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} \mathcal{C}_3 \xrightarrow{\lambda} \overline{N\mathcal{A}_1^{((1),1)}} \xrightarrow{\partial} \mathcal{A}_1^{((1),1)} \longrightarrow 0.$$

This is also the $2-$crossed complex where $\mathcal{C}_3 = \overline{N\mathcal{A}_1^{((1),1)}} \otimes_{\mathcal{A}_1^{((1),1)}} N\mathcal{A}_1^{((1),1)}$.

By the *(totally) free square complex*, we mean that the crossed square is (totally) free, and also each C_n is the free simplicial algebra with a given $CW-$basis for $n \geq 3$.

Now we can give some theorems, such as the following theorem and corollary obviously use the definition of a $CW-$basis.

Moreover we give some results concerning with the free simplicial algebra with a given $CW-$basis by using in the $1-$skeleton resolution for an algebra.

Theorem 4.2: The category of squared complexes of algebras is equivalent to the category of the $2-$crossed complexes of algebras.

Proof: The category of crossed square of algebra is equivalent to the category of $2-$crossed module of algebra. □

Thus we may obtain the result of the free version of Theorem 4.2 where the proof of its is trivial between equivalent two categories as follows.

Corollary 4.3: The category of the free squared complexes of algebras is equivalent to the category of the free $2-$crossed complexes of algebras. □

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