# Convergence Analysis of Training Two-Hidden-Layer Partially Over-Parameterized ReLU Networks via Gradient Descent 

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#### Abstract

Over-parameterized neural networks have attracted a great deal of attention in recent deep learning theory research, as they challenge the classic perspective of over-fitting when the model has excessive parameters and have gained empirical success in various settings. While a number of theoretical works have been presented to demystify properties of such models, the convergence properties of such models are still far from being thoroughly understood. In this work, we study the convergence properties of training two-hidden-layer partially over-parameterized fully connected networks with the Rectified Linear Unit activation via gradient descent. To our knowledge, this is the first theoretical work to understand convergence properties of deep over-parameterized networks without the equally-wide-hidden-layer assumption and other unrealistic assumptions. We provide a probabilistic lower bound of the widths of hidden layers and proved linear convergence rate of gradient descent. We also conducted experiments on synthetic and real-world datasets to validate our theory.


Keywords-Over-parameterization, Rectified Linear Units (ReLU), convergence, gradient descent, neural networks.

## I. Introduction

T1 HERE is a recent breakthrough in deep neural networks theory: the expressivity and generalization of neural networks become surprisingly good as the number of parameters exceeds the number of training samples [1]. This setting, namely over-parameterization, challenges the classic view that excessive number of parameters leads to over-fitting. This observation was further supported by empirical results in various settings [2]-[4].

Despite its empirical success, the theoretical properties of over-parameterized models remain ill-understood. As one of the first steps in understanding over-parameterized models, the training process of fully connected over-parameterized neural networks with the Rectified Linear Unit (ReLU) activation and (stochastic) gradient descent optimizer is only partially understood. For shallow networks where there is only one hidden layer, the convergence of the training process has been proven by a few prior works [5]-[9]. For deep networks there there are $\geq 2$ hidden layers, the convergence was endorsed under the assumption that all hidden layers are equally wide [10]-[13]. We denote it as the equally-wide-hidden-layer assumption.

In this work, we provide convergence analysis to training two-hidden-layer partially over-parameterized ReLU networks using gradient descent with infinitely small learning rates.

[^0]Different from prior works on deep architectures, we do not require the equally-wide-hidden-layer assumption in this work. Particularly, we show convergence when the second hidden layer is over-parameterized and much wider than the first hidden layer. Therefore, previous analysis on deep ReLU networks do not apply to our setting.

The technique in this work is to generalize the minimum-eigenvalue framework in [6] to the two-hidden-layer case. We follow the intuition from [5] and [6] that during training process, the weights and activation patterns do not change much, and the least eigenvalue $\lambda_{0}$ of the Gram matrix is lower bounded. However, it is not trivial to extend the one-hidden-layer analysis because the expressions are much more complicated for two hidden layers. To resolve this problem, we applied a more fine-grained statistical and numerical analysis to achieve applicable probabilistic bounds for certain parameters, and finally provided a lower bound for the widths of hidden layers.

We make two contributions in this work. First, we show that when the second hidden layer is over-parameterized and much wider than the first hidden layer (which could be either over-parameterized or $\Theta(1)$ ), gradient descent with infinitely small learning rates converges with high probability. We provided a lower bound on the widths of hidden layers that leads to convergence. Second, we conducted experiments on synthetic, MNIST [14] and Fashion-MNIST [15] datasets. Experimental results on the loss, $\lambda_{0}$, weight change, and pattern change rates support our theoretical analysis.

## II. Related Works

It was addressed that expressivity and generalization of neural networks become surprisingly good under the over-parameterized setting [1]. As the first fatal property to understand over-parameterized neural networks, convergence was empirically studied in various settings [2]-[4].

Specifically, since the ReLU activation by [16] gained huge success in deep learning, many previous works were presented to understand the convergence properties of fully connected over-parameterized ReLU networks trained by (stochastic) gradient descent. These works can be classified into two groups based on their network architecture: shallow networks (where there is only one hidden layer) and deep networks (where there are multiple hidden layers).

Shallow networks. Widely believed, understanding the one-hidden-layer (namely two-layer in many papers)
over-parameterized architecture is crucial to understanding deeper networks: if there is only one hidden layer, then it is possible to examine every parameter and every node, so that the dynamics during training process can be explicitly analyzed. An early work under this setting by [17] showed that when data is generated by a linearly separable function, then stochastic gradient descent finds a global minimum of the Hinge loss. Later on, a surprising phenomenon was discovered by [5]: the activation patterns do not change much during training process. Based on this observation, they proved that for cross-entropy loss, gradient descent converges to $\epsilon$ loss if the width of the hidden layer is polynomial in $1 / \epsilon$. [6], the closest to our work, proved that gradient descent converges to zero squared error loss under a more practical setting where the width is independent of the accuracy. The core idea is that the convergence rate is controlled by the least eigenvalue of the Gram matrix [18]-[20] when we analyze the output dynamics (instead of the weight dynamics). This matrix form was further analyzed by [7] and proved to approximate an integral operator. If the target function has a low-rank approximation w.r.t. the eigenspaces of this integral operator, then gradient descent approximately applies the powers of this integral operator on the target function, which yields a linear convergence rate. In other directions, [8] generalized the convergence results to adaptive gradient methods, and [9] improved the $\mathcal{O}\left(n^{6}\right)$ bound on width in [6] to $\mathcal{O}\left(n^{2}\right)$, where $n$ is the number of training samples.

Deep networks: As a first step to analyze deep over-parameterized ReLU networks, deep linear networks were proved to enjoy the global-minima convergence property under various settings [21]-[26]. Then, a number of theoretical works were developed to demystify the optimization landscape of deep over-parameterized ReLU networks trained with (stochastic) gradient descent [10]-[13]. In detail, [10] and [12] proved convergence of (stochastic) gradient descent trained on deep over-parameterized ReLU networks for classification and regression problems respectively. In both papers, global minima is found by (stochastic) gradient descent in polynomial time with a polynomial bound on width. Their approaches are different in that [10] followed the observation in [5] and showed that the parameters are closed to their Gaussian initialization, while [12] proved equivalence between deep over-parameterized neural networks and finite-width neural tangent kernel (NTK) in [18]. Then, [13] improved the $\tilde{\Omega}\left(n^{24}\right)$ bound on width and $\mathcal{O}\left(n^{6}\right)$ iteration complexity in [12] to $\tilde{\Omega}\left(n^{8}\right)$ and $\mathcal{O}\left(n^{2}\right)$. Besides analysis on general deep networks, [11] made a specific analysis to three-layer (that is, two-hidden-layer) over-parameterized ReLU network, which was proved able to learn three-layer neural networks equipped with smooth activation functions efficiently via stochastic gradient descent.

Interestingly, in [6], an approach based on the Gram matrices was discussed to generalize the convergence analysis to deep over-parameterized ReLU networks. However, due to the Hoeffding's inequality, every hidden layer has to be equally wide (or at least over-parameterized) to guarantee that the Gram matrices are closed to initial. In fact, such equally-wide-hidden-layer assumption exists in all the works
mentioned above regarding convergence properties of training deep ReLU networks with (stochastic) gradient descent and squared error loss (see Assumption 3.6 in [10], Theorem 2 in [11], Section 2 and Section 3 in [12], and Section 2 in [13]). However, there is one recent work without the equally-wide-hidden-layer assumption. Instead of proving specifically for deep neural networks, [27] showed that for a general class of over-parameterized nonlinear learning problems, (stochastic) gradient descent takes a surprisingly short path i.e. the parameters follow an almost straight route towards optima. As a cost to its generality, a few strong assumptions (Jacobian is Lipschitz with bounded least singular value and spectrum norm) are made, which rules out the popular deep ReLU networks. In addition, the bounds in the assumptions could be extreme for deep neural networks, so their conclusions may be weakened.

In this paper, we follow the idea from [6] and generalized the minimum-eigenvalue framework to the two-hidden-layer architecture. By applying a more fine grained analysis, we do not need the equally-wide-hidden-layer assumption any more: we proved that if the second hidden layer is over-parameterized and the much wider than the first hidden layer (which could be either over-parameterized or $\Theta(1)$ ), gradient descent with infinitely small step size converges to zero training loss with high probability. To our knowledge, this is the first work that proves the convergence of gradient descent on a kind of deep ReLU network without the equally-wide-hidden-layer assumption and other unrealistic assumptions.

## III. Preliminaries

In this paper, we consider the class of over-parameterized neural networks with two hidden layers and ReLU activation. The network is fully connected, taking inputs in $\mathbb{R}^{d}$ and outputs in $\mathbb{R}$. Particularly, we consider the case when the network is partially over-parameterized: the second hidden layer has a large amount of neurons and is much wider than the first hidden layer. We do not constrain the width of the first hidden layer; it could be either over-parameterized or $\Theta(1)$.

## A. Notations

First, we formally define our network structure. The widths of the input layer, first hidden layer, and the second hidden layer are $d, h_{1}$, and $h_{2}$, respectively. The output is a real value. The weight matrix connecting the input layer and the first hidden layer is $A=\left(A_{1}^{\top}, \cdots, A_{h_{1}}^{\top}\right)^{\top} \in \mathbb{R}^{h_{1} \times d}$, and the weight matrix connecting the hidden layers is $B=$ $\left(B_{1}^{\top}, \cdots, B_{h_{2}}^{\top}\right)^{\top} \in \mathbb{R}^{h_{2} \times h_{1}}$. Each hidden layer has activation function $\sigma(x)=\operatorname{Re} L U(x)=\max (x, 0)$. The output is computed by the dot product of weight vector $w \in\{-1,1\}^{h_{2}}$ and the output of the second hidden layer with a normalizing factor $1 / \sqrt{h_{1} h_{2}}$. Therefore, given an input $x \in \mathbb{R}^{d}$, the output is

$$
\begin{equation*}
f(x ; A, B, w)=\frac{1}{\sqrt{h_{1} h_{2}}} w^{\top} \sigma(B \sigma(A x)) \tag{1}
\end{equation*}
$$

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Suppose we have $n$ different training samples $\left\{\left(x_{i}, y_{i}\right)\right\}_{i=1}^{n}$. Then, the prediction of each sample $x_{i}$ is

$$
\begin{equation*}
u_{i}=f\left(x_{i} ; A, B, w\right), 1 \leq i \leq n \tag{2}
\end{equation*}
$$

We initialize the parameters in the following way: each element in $w$ is drawn $i . i . d$. from the uniform distribution over $\{-1,+1\}$, and each element in $A$ and $B$ is drawn i.i.d. from the standard normal distribution $\mathcal{N}(0,1)$. Then, each row of $A$ and $B$ are normalized to have $\ell_{2}$ norm $\leq 1$. For the training data, we assume that each sample is normalized and its label is bounded by some constant $C$ : for any $1 \leq i \leq n,\left\|x_{i}\right\|_{2}=1$, $\left|y_{i}\right| \leq C$. We use the $\ell_{2}$ norm as our loss function, defined by

$$
\begin{equation*}
L=\frac{1}{2} \sum_{i=1}^{n}\left(u_{i}-y_{i}\right)^{2}=\frac{1}{2}\|u-y\|_{2}^{2} \tag{3}
\end{equation*}
$$

To minimize this loss, we apply gradient descent over $A$ and $B$ and constantly update these two weight matrices. In order to make rigorous theoretical analysis, we assume that the learning rate is infinitely small so that we can formulate the update rule in a differential form:

$$
\begin{align*}
& \frac{d A_{r}}{d t}=-\frac{\partial L}{\partial A_{r}}, 1 \leq r \leq h_{1}  \tag{4}\\
& \frac{d B_{p}}{d t}=-\frac{\partial L}{\partial B_{p}}, 1 \leq p \leq h_{2} \tag{5}
\end{align*}
$$

We define 1 \{statement to be 1 if the statement is true and 0 if it is not true. Finally, the following variables take time $t$ as the parameter since they update with $t$ :

$$
\begin{equation*}
A=A(t), B=B(t), u=u(t), L=L(t) \tag{6}
\end{equation*}
$$

In the rest of the paper, we omit the time parameter $t$ for conciseness when there is no ambiguity.

## B. Problem Definition

The dynamics of the output vector $u(t)$ at time $t$ can be calculated through the chain rule in the multivariable case:

$$
\begin{align*}
\frac{d}{d t} u_{i}(t)= & \sum_{p=1}^{h_{2}}\left\langle\frac{\partial u_{i}}{\partial B_{p}}, \frac{d B_{p}}{d t}\right\rangle+\sum_{r=1}^{h_{1}}\left\langle\frac{\partial u_{i}}{\partial A_{r}}, \frac{d A_{r}}{d t}\right\rangle \\
= & \sum_{p=1}^{h_{2}}\left\langle\frac{\partial u_{i}}{\partial B_{p}},-\frac{\partial L}{\partial B_{p}}\right\rangle+\sum_{r=1}^{h_{1}}\left\langle\frac{\partial u_{i}}{\partial A_{r}},-\frac{\partial L}{\partial A_{r}}\right\rangle \\
= & \sum_{j=1}^{n}\left(\sum_{p=1}^{h_{2}}\left\langle\frac{\partial u_{i}}{\partial B_{p}}, \frac{\partial u_{j}}{\partial B_{p}}\right\rangle+\sum_{r=1}^{h_{1}}\left\langle\frac{\partial u_{i}}{\partial A_{r}}, \frac{\partial u_{j}}{\partial A_{r}}\right\rangle\right) \\
& \times\left(y_{j}-u_{j}\right) \tag{7}
\end{align*}
$$

Therefore, the dynamics of $u(t)$ can be expressed in the following closed form:

$$
\begin{equation*}
\frac{d}{d t} u(t)=H(t)(y-u(t)) \tag{8}
\end{equation*}
$$

where $H$ is an $n \times n$ matrix with the $(i, j)$-th element

$$
\begin{equation*}
H_{i j}=\sum_{p=1}^{h_{2}}\left\langle\frac{\partial u_{i}}{\partial B_{p}}, \frac{\partial u_{j}}{\partial B_{p}}\right\rangle+\sum_{r=1}^{h_{1}}\left\langle\frac{\partial u_{i}}{\partial A_{r}}, \frac{\partial u_{j}}{\partial A_{r}}\right\rangle \tag{9}
\end{equation*}
$$

In Section IV, we prove convergence of the dynamics (8) under the following partial over-parameterization assumption (Assumption 1) and the minimal eigenvalue assumption (Assumption 2).

## C. The Partial Over-Parameterization Assumption

Assumption 1. The network is partially over-parameterized: $h_{2} \gg h_{1}$.

## D. The Minimal Eigenvalue Assumption

In order to design a vital tool to analyze $H$, we first compute the explicit expression of $H$.

Proposition 1. $\forall i, j \in\{1, \cdots, n\}, h_{1} h_{2} H_{i j}$ is equal to

$$
\begin{align*}
& \sum_{p=1}^{h_{2}} \sum_{r=1}^{h_{1}}\left(x_{i}^{\top} A_{r} A_{r}^{\top} x_{j}+x_{i}^{\top} x_{j} B_{p r}^{2}\right) \\
& \quad \times 1\left\{B_{p}^{\top} y_{i}>0, B_{p}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\} \\
& +x_{i}^{\top} x_{j} \cdot \sum_{r=1}^{h_{1}} \sum_{p=1}^{h_{2}} \sum_{\substack{h_{2} \\
q \neq 1}}^{h_{2}} w_{p} w_{q} B_{p r} B_{q r} \\
& \quad \times 1\left\{B_{p}^{\top} y_{i}>0, B_{q}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\} \tag{10}
\end{align*}
$$

where $y_{i}=\sigma\left(A x_{i}\right)$.
Then, we define $H^{\infty}=\mathbb{E}_{A(0)} \mathbb{E}_{B(0)} \mathbb{E}_{w} H(0)$, a fatal bridge to analyze $H$. This is because the $\sum_{r} \sum_{p \neq q}$ term in $H_{i j}$ is intractable but will vanish as we take the expectation over $w$, and terms are easily bounded as we take the expectation over $A(0)$ and $B(0)$. Given the concise and symmetric expression of $H^{\infty}$, it is not hard to prove that $H^{\infty}$ is positive semi-definite (PSD) in Proposition 2.

## Proposition 2. $H^{\infty}$ is PSD.

Therefore, it is reasonable for us to make the following assumption in this paper:

Assumption 2. $\lambda_{0}=\lambda_{\text {min }}\left(H^{\infty}\right)>0$.
In the main result (Theorem 3), $\lambda_{0}$ appears to be a key factor in the convergence rate, and $\lambda_{0} / 2$ is used to lower bound the smallest eigenvalue of $H(t)$ throughout the proof in Section IV-B.

## IV. Main Result

In this section, we prove convergence of the dynamics (8). In detail, we prove that if we use gradient descent to train a two-hidden-layer ReLU network with an infinitely small learning rate, and if the second hidden layer is over-parameterized and much wider than the first one, then we can expect the training loss converges to zero exponentially with high probability. This result is a theoretical extension to the convergence analysis in [6], where the one-hidden-layer setting was proved. In addition, our result does not overlap with analysis for the multiple-layer setting in [10]-[13] because they require each hidden layer should be equally wide (or lower bounded), which does not include situations where one hidden layer might be not wide enough. Our result does provide a positive answer to such situations. Therefore, our result is a complimentary work of their results on the convergence properties of training general two-hidden-layer over-parameterized ReLU networks with gradient descent.

## A. Theorem of Convergence

Our formal result is shown in the following theorem.
Theorem 3. Under Assumption 1 and Assumption 2, if the number of nodes of the second hidden layer satisfies $h_{2}=\Omega\left(\frac{n^{9 / 2} L(0)^{3 / 2} h_{1}^{3 / 2}}{\lambda_{0}^{{ }^{\circ} \delta^{3}}}\right)$, then with probability $\geq 1-\delta$ over random initialization, the training error is bounded by $L(t) \leq \exp \left(-\lambda_{0} t\right) L(0)$.

In short, a lower bounded convergence rate is provided in Theorem 3 given the partially over-parameterized architecture. To keep our conclusion concise, we use the smallest eigenvalue to bound the convergence rate. Intuitively, a tighter bound induced by the mean of eigenvalues of $H^{\infty}$ is likely to exist, at a cost of having a much more complicated probabilistic analysis.

## B. Steps of Proof

In this section, we present four steps to prove Theorem 3. In detail, we use the minimum-eigenvalue framework, which was initially introduced in [6]. In Lemma 4, we show that $H(0)$ is closed to $H^{\infty}$ and the least eigenvalue of $H(0)$ is bounded. In Lemma 5, we prove that the minimum eigenvalue of $H$ at time $t$ is also lower bounded. In Lemma 6, we first demonstrate that the minimum eigenvalue of $H$ provides a convergence guarantee to the gradient descent algorithm, and then show that the change of parameters $A$ and $B$ is bounded. Finally, Lemma 7 ties the previous lemmas into a cycle, showing that the conditions always hold and the probability that the cycle breaks is tiny, thus finalizing the proof. The proofs of these lemmas are given in Appendix A. Compared to the analysis in [6], their bounds (especially $R$ and $R^{\prime}$ in their paper) cannot be naturally extended to the problem in this work, we use various mathematical tools to provide applicable probabilistic bounds in the analysis.
Lemma 4. If $h_{2}=\Omega\left(\frac{n^{2}}{\lambda_{0}^{2}} \log \frac{2 n^{2}}{\delta}\right)$, then we have

$$
\begin{equation*}
\lambda_{\min }(H) \geq \frac{3}{4} \lambda_{0} \tag{11}
\end{equation*}
$$

with probability at least $1-\delta$ at time $t=0$.
In the Lemma 4, we show that with high probability, the minimum eigenvalue of $H=H(0)$ at time 0 is bounded by a constant times $\lambda_{0}$. In the proof, we first bound $\mid H_{i j}-$ $H_{i j}^{\infty} \mid$ using the Hoeffding's inequality and the Chebyshev's inequality. Then, we are able to bound the Frobenius norm of $H-H^{\infty}$. Finally, the difference between their least eigenvalues can be bounded by this amount.

Lemma 5. Suppose at time $t$, there exists a small constant $c>$ 0 such that $\left\|A_{r}(t)-A_{r}(0)\right\|_{2} \leq R_{A} \forall r,\left\|B_{p}(t)-B_{p}(0)\right\|_{2} \leq$ $R_{B} \forall p$, where $R_{A}=\mathcal{O}\left(\frac{c \delta \lambda_{0}}{n h_{1}}\right), R_{B}=\mathcal{O}\left(\frac{c \delta \lambda_{0}}{n}\right)$. Then with probability at least $1-\delta$ over initialization, $\lambda_{\min }(H(t)) \geq$ $\lambda_{0} / 2$.

In Lemma 5, we show that if at time $t$, the weight matrices $A$ and $B$ are close to initialization, then the minimum eigenvalue of $H=H(t)$ is also bounded by a constant times $\lambda_{0}$. The core
idea in the proof is that when we calculate the expectation of $\left|H_{i j}(0)-H_{i j}(t)\right|$, we divide it into two cases: one with activation pattern change and the other one with no activation pattern change. If no activation pattern changes, then slight modification to $A$ and $B$ must yield slight change of $H_{i j}$. On the other side, the probability that some activation pattern changes can be bounded by a small constant. Therefore, we obtain our desired result.
Lemma 6. Suppose $\lambda_{\text {min }}(H(s)) \geq \lambda_{0} / 2 \forall s \in[0, t]$. Then the training error is bounded as $L(t) \leq \exp \left(-\lambda_{0} t\right) L(0)$. The modification of $A$ and $B$ from initialization can be bounded as $\left\|A_{r}(t)-A_{r}(0)\right\|_{2} \leq R_{A}^{\prime} \forall 1 \leq r \leq h_{1}$ and $\| B_{p}(t)-$ $B_{p}(0) \|_{2} \leq R_{B}^{\prime} \forall 1 \leq p \leq h_{2}$, where $R_{A}^{\prime}=\frac{2}{\lambda_{0}} \sqrt{\frac{2 n L(0)}{h_{1} h_{2}^{2 / 3}}}$ and $R_{B}^{\prime}=\frac{2}{\lambda_{0}} \sqrt{\frac{2 n L(0)}{h_{2}}}$.

In the Lemma 6, we show that if at time $s$, the minimum eigenvalue of $H=H(s)$ is bounded, then the training loss converges to 0 with an exponential rate, and the weight matrices are closed to initialization with high probability. In the proof, we first calculate the derivative of the loss function w.r.t. time $t$, and then bound the loss with an exponential rate. To prove that $A$ and $B$ are closed to initialization, we also bound their derivatives first, and then do the integration.
Lemma 7. If $h_{2}=\Omega\left(\frac{n^{9 / 2} L(0)^{3 / 2} h_{1}^{3 / 2}}{\lambda_{0}^{6} \delta^{3}}\right)$, then with high probability the following statements hold: for any $t>0$, $\left\|A_{r}(t)-A_{r}(0)\right\|_{2} \leq R_{A}^{\prime} \forall 1 \leq r \leq h_{1},\left\|B_{p}(t)-B_{p}(0)\right\|_{2} \leq$ $R_{B}^{\prime} \forall 1 \leq p \leq h_{2}$, and $L(t) \leq \exp \left(-\lambda_{0} t\right) L(0)$.

In the Lemma 7, we show that if $R_{A}^{\prime}<R_{A}, R_{B}^{\prime}<R_{B}$, then Lemma 5 and Lemma 6 are able to form a cycle: the conditions consistently hold as time $t$ goes to infinity. The proof is finished by drawing a contradiction.

## C. Finalizing the Proof

Proof: According to Lemmas 4-7, the theorem holds if $R_{A}^{\prime} \leq R_{A}$ and $R_{B}^{\prime} \leq R_{B}$. This is equivalent to $h_{2}^{2 / 3}=$ $\Omega\left(\frac{n^{3} L(0) h_{1}}{\lambda_{0}^{4} \delta^{2}}\right)$. Finally, notice that

$$
\begin{equation*}
\mathbb{E} L(0)=\|y\|_{2}^{2}+n=\Theta(n) \tag{12}
\end{equation*}
$$

Thus, by Markov's inequality, with probability $\geq 1-\delta, L(0)=$ $\mathcal{O}(n / \delta)$. Therefore, the conclusion holds with probability at least $1-2 \delta-h_{2}^{-1 / 3}=1-2 \delta-\mathcal{O}(\delta)$. Since there is only a constant in front of $\delta$, we finish the proof.

## V. EXPERIMENTS

In this section, we present experiments to validate our theory.

## A. Settings

We conducted experiments on several datasets including synthetic datasets, the MNIST dataset [14], and the Fashion-MNIST dataset [15]. To construct the synthetic dataset, we randomly select $50 / 100$ points on the unit ball in $\mathbb{R}^{100}$ and generate their labels in $[0,1]$ by random. These
two datasets are denoted as synthetic-50 and synthetic-100, respectively. To construct real-world datasets, we randomly select 50 samples with label $c_{1}$ and 50 samples with label $c_{2}$ from the MNIST dataset or the Fashion-MNIST dataset, denoted as MNIST/Fashion- $c_{1} c_{2}$. In this paper, we select $\left(c_{1}, c_{2}\right)=(0,1),(1,7)$ for MNIST and $\left(c_{1}, c_{2}\right)=(0,1),(7,9)$ for Fashion-MNIST. For both MNIST and Fashion-MNIST datasets, samples with label 0 and label 1 are visually quite distinct, so we also look at other cases where training samples with different labels also share some similarities. This encourages us to conduct experiments on MNIST-17 (where both 1 and 7 have a long straight line in the main body) and Fashion-MNIST-79 (where both sneakers and ankle boots are in the shoe-shape). Some samples are shown in Fig. 1.


Fig. 1 Samples of training data. Each column shows three samples from the selected training datasets. The datasets are (from left to right,) MNIST-0, MNIST-1, MNIST-7, Fashion-0 (T-shirt/top), Fashion-1 (Trouser), Fashion-7 (Sneaker), Fashion-9 (Ankle boot).

We also surveyed various network structures. The width of the second hidden layer is selected from $\mathcal{W}=$ $\{100,500,1000,2000,4000\}$, and the first hidden layer is set to have five neurons. Therefore, for each dataset, we did five sub-experiments on networks with $h_{1}=5, h_{2} \in \mathcal{W}$. We set the learning rate as small as 0.0002 to simulate the continuous dynamics. At each iteration $t(t \leq 1000)$, we recorded the following values:

- Weight change: the Frobenius norm of the difference between the weight matrices at time $t$ and the initial ones, computed as $\|A(t)-A(0)\|_{F}$ (first hidden layer) and $\|B(t)-B(0)\|_{F}$ (second hidden layer);
- Pattern change rate: the average percentage of activation pattern changes on the whole training set for each hidden layer. This is calculated by \#total activation pattern changes for $i$-th hidden layer divided by $n \times h_{i}, i=1,2$;
- The minimum eigenvalue $\lambda_{\min }(H(t))$;
- The $\ell_{2} \operatorname{loss} L(t)$.


## B. Results

In Fig. 2, we present the $\ell_{2}$ losses on the six datasets. The full results can be found in Appendix B.

These results are consistent with our theory according to the following observations. First, the logarithm of loss decreases at a bounded rate as $t$ increases, showing that the gradient descent algorithm converges at a bounded rate. This directly substantiates our main result in Theorem 3. In addition, the smallest eigenvalue $\lambda_{\min }(H(t))$ remains stable during the training process, and a larger $\lambda_{\min }(H(t))$ typically yields a faster convergence rate. This is especially shown on real-world datasets. Finally, both the weight change rate and pattern


Fig. 2 The logarithm of loss function $\log L(t)$ with 10 network structures on different training datasets.
change rate increase as $t$ increases, but are still below a low threshold. As the networks become wider the threshold generally gets lower. These observations comply with the minimum-eigenvalue framework in our proof.

## VI. CONCLUSION

In this paper, we surveyed the convergence properties of two-hidden-layer ReLU networks trained by gradient descent with $\ell_{2}$ loss, where the network is partially over-parameterized. In Section IV, we proved that if the width of the second hidden layer is large enough, then with high probability the gradient descent algorithm with infinitely small step size converges to an optimal solution that achieves zero training loss. The main part of the proof is divided into 4 lemmas in Section IV-B, which form a cycle and reveal that the minimum eigenvalue of $H$ is lower bounded both initially and during the training process. Since the $\ell_{2}$ loss can be written as a quadratic form in $H$, this minimum eigenvalue thus provides guarantee of the convergence rate. We also conducted experiments on synthetic dataset and real-world datasets in Section V. The experimental results are consistent with our theory in that convergence of dynamics, stability of the minimum eigenvalues, small weight change and small pattern change rates are verified.

Based our theory and experiments, there are several further directions worth deeper exploitation.

- In this paper we proved convergence for the continuous dynamics, while it is widely acknowledged in the area of dynamical systems [28] that discrete dynamics are generally more complicated and thus harder to analyze. Therefore, it is significant to prove the convergence theory with a constant step size.


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- Unlike the one-hidden-layer over-parameterized ReLU network in [6], in which $\lambda_{0}$ can be easily calculated since it has an explicit form, $\lambda_{0}$ for the two-hidden-layer network remains mysterious. Since $\lambda_{0}$ is a vital term in the convergence rate, its strict positiveness and relationship to data as well as network structure are crucial in understanding the convergence properties. To achieve this, a method that either provides an analytic form or an approximation algorithm of $\lambda_{0}$ is needed.
- It is observed that the logarithm of loss does not decrease at a constant rate in experiments. Instead, it decreases much faster during the initial training status, and achieves a stable rate after many iterations. However, the minimum eigenvalue $\lambda_{\min }(H(t))$ appears to be stable during the whole training process. Therefore, it is likely that the dynamics are controlled by some other indicators that dominate the training procedure at the very first beginning.
- Finally, we believe that using more advanced inequalities in our proof will help decrease the bound of the width in our main theorem. We used classical bounds in most of our proof, where it is possible to introduce more refined concentration and tail inequalities, especially when the dimension of data is high.


## Appendix A. Omitted Proofs

Since we have many sums in this paper, we make some shorthand here:

$$
\sum_{r}=\sum_{r=1}^{h_{1}} ; \sum_{p(q)=1}=\sum_{p(q)=1}^{h_{2}} ; \sum_{q: q \neq p}=\sum_{\substack{q=1 \\ q \neq p}}^{h_{2}} ; \sum_{p \neq q}=\sum_{p=1}^{h_{2}} \sum_{q: q \neq p} ; \sum_{i(j)}=\sum_{i(j)=1}^{n}(13)
$$

Also, the activation patterns of each neuron have to be decided, so we make the following shorthand:

$$
\begin{equation*}
1_{i}\{r\}=1\left\{A_{r}^{\top} x_{i}>0\right\} ; 1_{i}\{p(q)\}=1\left\{B_{p(q)}^{\top} \sigma\left(A x_{i}\right)>0\right\} \tag{14}
\end{equation*}
$$

## Proof of Proposition 1

Proof: First, we specify the calculations of $\frac{\partial f(x)}{\partial B_{p}}, \frac{\partial f(x)}{\partial A_{r}}$ :

$$
\begin{aligned}
\frac{\partial f(x)}{\partial B_{p}} & =\frac{1}{\sqrt{h_{1} h_{2}}} \cdot \frac{\partial}{\partial B_{p}}\left(w_{p} \sigma\left(B_{p}^{\top} \sigma(A x)\right)\right) \\
& =\frac{1}{\sqrt{h_{1} h_{2}}} \cdot w_{p} \sigma(A x) \cdot 1\left\{B_{p}^{\top} \sigma(A x)>0\right\} \\
\frac{\partial f(x)}{\partial A_{r}} & =\frac{1}{\sqrt{h_{1} h_{2}}} \cdot \frac{\partial}{\partial A_{r}} \sum_{p} w_{p} \sigma\left(\sum_{r} B_{p r} \sigma\left(A_{r}^{\top} x\right)\right) \\
& =\frac{1}{\sqrt{h_{1} h_{2}}} \cdot \sum_{p} w_{p} \cdot 1\left\{B_{p}^{\top} \sigma(A x)>0\right\} \cdot \frac{\partial}{\partial A_{r}}\left(B_{p r} \sigma\left(A_{r}^{\top} x\right)\right) \\
& =\frac{1}{\sqrt{h_{1} h_{2}}} \cdot \sum_{p} w_{p} B_{p r} x \cdot 1\left\{B_{p}^{\top} \sigma(A x)>0, A_{r}^{\top} x>0\right\}
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
& h_{1} h_{2} \cdot H_{i j}=y_{i}^{\top} y_{j} \cdot \sum_{p} 1\left\{B_{p}^{\top} y_{i}>0, B_{p}^{\top} y_{j}>0\right\} \\
& +x_{i}^{\top} x_{j} \cdot \sum_{r}\left(\sum_{p} w_{p} B_{p r} \cdot 1\left\{B_{p}^{\top} y_{i}>0, A_{r}^{\top} x_{i}>0\right\}\right) \\
& =u_{i}^{\top} \cdot\left(\sum_{p} w_{p} B_{p r} \cdot 1\left\{B_{p}^{\top} y_{j}>0, A_{r}^{\top} x_{j}>0\right\}\right) \\
& =y_{i}^{\top} y_{j} \cdot \sum_{p} 1\left\{B_{p}^{\top} y_{i}>0, B_{p}^{\top} y_{j}>0\right\} \\
& \sum_{p, r}^{+x_{i}^{\top} x_{p r}^{\top} \cdot 1}\left\{B_{p}^{\top} y_{i}>0, B_{p}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{r}^{+x_{i} \sum_{p} \sum_{p} w_{p} w_{q} B_{p r} B_{q r} \cdot 1\left\{B_{p}^{\top} y_{i}>0, B_{q}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\}} \\
& \sum_{p, r}^{r p}\left(x_{i}^{q} A_{r} A_{r}^{\top} x_{j}+x_{i}^{\top} x_{j} B_{p r}^{2}\right) \\
& { }_{1}^{D_{p}, r}\left\{B_{p}^{\top} y_{i}>0, B_{p}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\} \\
& +x_{i}^{\top} x_{j} \cdot \sum_{r} \sum_{p \neq q} w_{p} w_{q} B_{p r} B_{q r} . \\
& 1\left\{B_{p}^{\top} y_{i}>0, B_{q}^{\top} y_{j}>0, A_{r}^{\top} x_{i}>0, A_{r}^{\top} x_{j}>0\right\}
\end{aligned}
$$

where $y_{i}=\sigma\left(A x_{i}\right)$.

## Proof of Proposition 2

Proof: Since $\mathbb{E}_{w} w_{p} w_{q}=0$, we have

$$
\begin{equation*}
h_{1} h_{2} \cdot H_{i j}^{\infty}=\mathbb{E} \sum_{p, r}\left(x_{i}^{\top} A_{r} A_{r}^{\top} x_{j}+x_{i}^{\top} x_{j} B_{p r}^{2}\right) \cdot 1_{i, j}\{p, r\} \tag{17}
\end{equation*}
$$

Therefore, we only need to prove that for any non-zero vector $v \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{p, r} v_{i} v_{j}\left(x_{i}^{\top} A_{r} A_{r}^{\top} x_{j}+x_{i}^{\top} x_{j} B_{p r}^{2}\right) \cdot 1_{i, j}\{p, r\} \geq 0 \tag{18}
\end{equation*}
$$

We change the summing order into $\sum_{p, r} \sum_{i, j}$ and for each pair ( $p, r$ ), we show the sums of two parts (the $x_{i}^{\top} A_{r} A_{r}^{\top} x_{j}$ term and the $x_{i}^{\top} x_{j} B_{p r}^{2}$ term) over $i, j$ are non-negative. For the first part,
$\sum_{i, j} v_{i} v_{j} x_{i}^{\top} A_{r} A_{r}^{\top} x_{j} \cdot 1_{i, j}\{p, r\}=\left(\sum_{i} v_{i} A_{r}^{\top} x_{i} \cdot 1_{i}\{p, r\}\right)^{2}$
For the second part,

$$
\sum_{i, j} v_{i} v_{j} x_{i}^{\top} x_{j} B_{p r}^{2} \cdot 1_{i, j}\{p, r\}=B_{p r}^{2}\left\|\sum_{i} v_{i} x_{i} \cdot 1_{i}\{p, r\}\right\|_{2}^{2}
$$

As a result, $H^{\infty}$ is PSD.

## Proof of Lemma 4

Proof: We rewrite $H_{i j}$ in the following way:

$$
\begin{equation*}
H_{i j}=\frac{1}{h_{1} h_{2}} \sum_{p, r} X_{p r} \tag{21}
\end{equation*}
$$

where
$X_{p r}=\left(x_{i}^{\top} A_{r} A_{r}^{\top} x_{j}+x_{i}^{\top} x_{j} B_{p r}^{2}\right) \cdot 1_{i, j}\{p, r\}+\sum_{q: q \neq p} x_{i}^{\top} x_{j} w_{p} w_{q} B_{p r} B_{q r} \cdot 1_{i}\{p, r\} 1_{j}\{q, r\}$
Let $\bar{B}_{:, r}=\sum_{q} w_{q} B_{q r}$. By the Chebyshev's inequality, for any $\tau>0$,

$$
\begin{equation*}
P\left(\left|\bar{B}_{:, r}\right| \geq \tau\right) \leq \frac{\sum_{q} B_{q r}^{2}}{\tau^{2}} \leq \frac{1}{\tau^{2}} \tag{23}
\end{equation*}
$$

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Thus, with probability at least $1-\frac{h_{1}}{\tau^{2}}$, we have $\left|X_{p r}\right| \leq \tau+2$ for any $p, r$ and $\tau>0$. Applying this bound to Hoeffding's inequality, we have

$$
\begin{equation*}
P\left(\left|H_{i j}-H_{i j}^{\infty}\right| \geq \delta\right) \leq 2\left(1-\frac{h_{1}}{\tau^{2}}\right) \exp \left(-\frac{h_{1} h_{2} \delta^{2}}{(t+2)^{2}}\right) \tag{24}
\end{equation*}
$$

If we let $\tau+2=h_{1}^{1 / 2} h_{2}^{k / 2}$ with $k \in(0,1)$, we can simplify the inequality as

$$
\begin{equation*}
P\left(\left|H_{i j}-H_{i j}^{\infty}\right| \geq \delta\right) \leq 2\left(1-\frac{1}{h_{2}^{k}}\right) \exp \left(-\frac{h_{2}^{1-k} \delta^{2}}{2}\right) \tag{25}
\end{equation*}
$$

This is equivalent to: with probability $\geq 1-\delta$, for any $0<$ $k<1$,

$$
\begin{equation*}
\left|H_{i j}-H_{i j}^{\infty}\right| \leq \sqrt{\frac{2}{h_{2}^{1-k}} \log \frac{2\left(h_{2}^{k}-1\right)}{\delta h_{2}^{k}}} \tag{26}
\end{equation*}
$$

Now, we set $k=\left(\log \frac{\sqrt{2}}{\sqrt{2}-\sqrt{\delta}}\right) / \log h_{2}$ and $\delta \leq 1 / 2$. The above inequality can be simplified as

$$
\begin{equation*}
\left|H_{i j}-H_{i j}^{\infty}\right| \leq \sqrt{\frac{2}{h_{2}} \frac{\sqrt{2}}{\sqrt{2}-\sqrt{\delta}} \log \sqrt{2 / \delta}} \leq \sqrt{\frac{2}{h_{2}} \log \frac{2}{\delta}} \tag{27}
\end{equation*}
$$

Then, we apply this bound for all $(i, j)$ pairs, and obtain that with probability at least $1-\delta$

$$
\begin{equation*}
\left|H_{i j}-H_{i j}^{\infty}\right| \leq \sqrt{\frac{2}{h_{2}} \log \frac{2 n^{2}}{\delta}} \tag{28}
\end{equation*}
$$

which gives that

$$
\begin{equation*}
\left\|H-H^{\infty}\right\|_{2}^{2} \leq \frac{2 n^{2}}{h_{2}} \log \frac{2 n^{2}}{\delta} \tag{29}
\end{equation*}
$$

Similar to the previous analysis, if $h_{2} \geq \frac{32 n^{2}}{\lambda_{0}^{2}} \log \frac{2 n^{2}}{\delta}$, we have the desired result.

## Proof of Lemma 5

Proof: By the definition of $H$, we have

$$
\begin{array}{r}
\left|H_{i j}(t)-H_{i j}(0)\right| \leq \frac{1}{h_{1} h_{2}} \\
\left\{\sum_{p, r}\left|x_{i}^{\top} A_{r}(t) A_{r}(t)^{\top} x_{j} \cdot 1_{i j}^{(t)}\{p, r\}-x_{i}^{\top} A_{r}(0) A_{r}(0)^{\top} x_{j} \cdot 1_{i j}^{(0)}\{p, r\}\right|\right. \\
+\left|x_{i}^{\top} x_{j}\right| \sum_{p, r}\left|B_{p r}(t)^{2} \cdot 1_{i j}^{(t)}\{p, r\}-B_{p r}(0)^{2} \cdot 1_{i j}^{(0)}\{p, r\}\right| \\
+\left|x_{i}^{\top} x_{j}\right| \sum_{p, r}\left|w_{p}\right| \mid \sum_{q: q \neq p} w_{q} B_{p r}(t) B_{q r}(t) \cdot 1_{i}^{(t)}\{p, r\} 1_{j}^{(t)}\{q, r\} \\
\left.-\sum_{q: q \neq p} w_{q} B_{p r}(0) B_{q r}(0) \cdot 1_{i}^{(0)}\{p, r\} 1_{j}^{(0)}\{q, r\} \mid\right\}
\end{array}
$$

(30)

According to Lemma 3.2 in [6], we have

$$
\begin{equation*}
P\left(1_{i}^{(t)}\{r\} \neq 1_{i}^{(0)}\{r\}\right) \leq \frac{2 R_{A}}{\sqrt{2 \pi}} \tag{31}
\end{equation*}
$$

and

$$
\begin{aligned}
P\left(1_{j}^{(t)}\{p\}=1_{j}^{(0)}\{p\}\right) \geq & \\
& P\left(1_{j}^{(t)}\{p\}=1_{j}^{(0)}\{p\} \mid 1_{i}^{(t)}\{r\}=1_{i}^{(0)}\{r\} \forall i\right) \\
& P\left(1_{i}^{(t)}\{r\}=1_{i}^{(0)}\{r\}\right)^{h_{1}} \\
\geq & \left(1-\frac{2 R_{A}}{\sqrt{2 \pi}}\right)^{h_{1}}\left(1-\frac{2 R_{B}}{\sqrt{2 \pi}}\right)
\end{aligned}
$$

which indicates that

$$
\begin{equation*}
P\left(1_{j}^{(t)}\{p\} \neq 1_{j}^{(0)}\{p\}\right) \leq \frac{2\left(h_{1} R_{A}+R_{B}\right)}{\sqrt{2 \pi}} \tag{33}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
P\left(1_{i}^{(t)}\{p, r\} \neq 1_{i}^{(0)}\{p, r\}\right) \leq \frac{2\left(h_{1} R_{A}+R_{A}+R_{B}\right)}{\sqrt{2 \pi}} \tag{34}
\end{equation*}
$$

and as a result,

$$
\begin{equation*}
P\left(1_{i j}^{(t)}\{p, r\} \neq 1_{i j}^{(0)}\{p, r\}\right) \leq \frac{4\left(h_{1} R_{A}+R_{A}+R_{B}\right)}{\sqrt{2 \pi}} \tag{35}
\end{equation*}
$$

Therefore,
$\mathbb{E}\left|H_{i j}(t)-H_{i j}(0)\right|=\quad P$ (activation does not change) $\cdot \mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation does not change $)$
$+P($ activation changes $) \cdot \mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation changes $)$
$\leq \mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation does not change $)$
$+\frac{4\left(h_{1} R_{A}+R_{A}+R_{B}\right)}{\sqrt{2 \pi}} \cdot \mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation changes $)$
When there some activation pattern changes, we apply the simple bound
$\mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation changes $) \leq \mathbb{E}\left|H_{i j}(t)\right|+\mathbb{E}\left|H_{i j}(0)\right|$ where

$$
\begin{align*}
\mathbb{E}\left|H_{i j}(t)\right| & \leq \frac{1}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left(\left|x_{i}^{\top} A_{r}(t) A_{r}^{\top}(t) x_{j}\right|+\left|x_{i}^{\top} x_{j} B_{p r}(t)^{2}\right|\right)  \tag{37}\\
& \leq \frac{1}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left(\left\|A_{r}(t)\right\|_{2}^{2}+B_{p r}^{2}(t)\right) \\
& \leq \frac{2}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left(R_{A}^{2}+\left\|A_{r}(0)\right\|_{2}^{2}+B_{p r}^{2}(0)+\left(B_{p r}(t)-B_{p r}(0)\right)^{2}\right) \\
& \leq 2\left(1+R_{A}^{2}+\frac{1}{h_{1} h_{2}} \sum_{r}\left(\left\|B_{p}(0)\right\|_{2}^{2}+R_{B}^{2}\right)\right) \\
& \leq 2\left(1+R_{A}^{2}+\frac{1+R_{B}^{2}}{h_{2}}\right)
\end{align*}
$$

and similarly,

$$
\begin{equation*}
\mathbb{E}\left|H_{i j}(0)\right| \leq 2\left(1+\frac{1}{h_{2}}\right) \tag{39}
\end{equation*}
$$

Since $\delta$ can be arbitrarily small, both $R_{A}$ and $R_{B}$ can be $\mathcal{O}(1)$. Therefore,
$\mathbb{E}\left|H_{i j}(t)-H_{i j}(0)\right|=\mathcal{O}\left(\max \left(\mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.\right.\right.$ activation does not change $\left.\left.), h_{1} R_{A}, R_{B}\right)\right)$
Now, we bound $\mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \mid\right.$ activation does not change), the expected difference between $H_{i j}(t)$ and $H_{i j}(0)$ when no activation pattern changes. This expected difference is upper bounded by the sum of three terms:

$$
\begin{aligned}
& \mathbb{E}\left(\left|H_{i j}(t)-H_{i j}(0)\right| \text { activation does not change }\right) \\
\leq & \frac{1}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left|x_{i}^{\top}\left(A_{r}(t) A_{r}^{\top}(t)-A_{r}(0) A_{r}^{\top}(0)\right) x_{j}\right| \\
& +\frac{1}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left|x_{i}^{\top} x_{j}\left(B_{p r}^{2}(t)-B_{p r}^{2}(0)\right)\right| \\
& +\frac{1}{h_{1} h_{2}} \sum_{r} \sum_{p \neq q} \mathbb{E}\left|B_{p r}(t) B_{q r}(t)-B_{p r}(0) B_{q r}(0)\right| \\
\leq & \frac{1}{h_{1} h_{2}} \sum_{p, r} \mathbb{E}\left(\left|x_{i}^{\top}\left(A_{r}(t)-A_{r}(0)\right)\right|\right) \\
& +\left(\left|x_{j}^{\top}\left(A_{r}(t)-A_{r}(0)\right)\right|+\left|x_{i}^{\top}\left(A_{r}(t)-A_{r}(0)\right) x_{j}^{\top}\left(A_{r}(t)-A_{r}(0)\right)\right|\right) \\
& +\frac{1}{h_{1} h_{2}} \sum_{p} \mathbb{E} \sum_{r}\left(2\left|B_{p r}(0)\left(B_{p r}(t)-B_{p r}(0)\right)\right|+\left(B_{p r}(t)-B_{p r}(0)\right)^{2}\right) \\
& +\frac{1}{h_{1} h_{2}} \sum_{p, q} \mathbb{E} \sum_{r}\left(\left|B_{p r}(t)\left(B_{q r}(t)-B_{q r}(0)\right)\right|+\left|B_{q r}(0)\left(B_{p r}(t)-B_{p r}(0)\right)\right|\right) \\
\leq & \frac{1}{h_{1} h_{2}} \sum_{p, r}\left\|A_{r}(t)-A_{r}(0)\right\|_{2}+\left\|A_{r}(t)-A_{r}(0)\right\|_{2}+\left\|A_{r}(t)-A_{r}(0)\right\|_{2}^{2} \\
& +\frac{1}{h_{1} h_{2}} \sum_{p} 2\left\|B_{p}(0)\right\|_{2}\left\|B_{p}(t)-B_{p}(0)\right\|_{2}+\left\|B_{p}(t)-B_{p}(0)\right\|_{2}^{2} \\
& +\frac{1}{h_{1} h_{2}} \sum_{p, q}\left(\left\|B_{p}(0)\right\|_{2}+\left\|B_{q}(0)\right\|_{2}+\left\|B_{q}(t)-B_{q}(0)\right\|_{2}\right)\left\|B_{p}(t)-B_{p}(0)\right\|_{2} \\
\leq & \frac{1}{h_{1} h_{2}}\left(\sum_{p, r}\left(2 R_{A}+R_{A}^{2}\right)+\sum_{p}\left(2 R_{B}+R_{B}^{2}\right)+\sum_{p, q}\left(2 R_{B}+R_{B}^{2}\right)\right) \\
= & \mathcal{O}\left(\max \left(R_{A}, R_{B}\right)\right)
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
\mathbb{E}\left|H_{i j}(t)-H_{i j}(0)\right|=\mathcal{O}\left(\max \left(h_{1} R_{A}, R_{B}\right)\right) \tag{42}
\end{equation*}
$$

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In order to bound $\|H(t)-H(0)\|_{2} \leq \frac{\lambda_{0}}{4}$, we only need to set $\|H(t)-H(0)\|_{F} \leq \frac{\lambda_{0}}{4}$. By Markov inequality, this is satisfied with probability $\geq 1-\delta$ when

$$
\begin{equation*}
\max \left(h_{1} R_{A}, R_{B}\right)=\mathcal{O}\left(\frac{c \delta \lambda_{0}}{n}\right) \tag{43}
\end{equation*}
$$

where $c$ is a small positive number. Finally, we obtain that when

$$
\begin{equation*}
R_{A}=\mathcal{O}\left(\frac{c \delta \lambda_{0}}{n h_{1}}\right), R_{B}=\mathcal{O}\left(\frac{c \delta \lambda_{0}}{n}\right) \tag{44}
\end{equation*}
$$

we have with high probability over initialization, $\lambda_{\text {min }}(H(t)) \geq \lambda_{0} / 2$.

## Proof of Lemma 6

Proof: The derivative of loss is bounded by

$$
\begin{equation*}
L^{\prime}(t)=-(y-u(t))^{\top} \frac{d u(t)}{d t}=-(y-u(t))^{\top} H(t)(y-u(t)) \leq-\lambda_{0} L(t) \tag{45}
\end{equation*}
$$

Therefore, we know $\frac{d}{d t}\left(\exp \left(\lambda_{0} t\right) L(t)\right) \leq 0$, indicating that $L(t) \leq \exp \left(-\lambda_{0} t\right) L(0)$.

Next, we bound the difference between $A, B$ and their initialization. This is achieved by integrating the gradient of $A$ and $B$. Therefore, we bound their gradient first.

We set $\tau=h_{2}^{1 / 6}$ in the proof of Lemma 4. Then, we obtain with probability at least $1-1 / h_{2}^{1 / 3}$, the sum $\left|\sum_{q} w_{q} B_{q r}\right|$ can be up bounded by $h_{2}^{1 / 6}$. Then, for $s \in[0, t]$ and $\forall 1 \leq r \leq h_{1}$,

$$
\begin{align*}
\left\|\frac{d}{d s} A_{r}(s)\right\|_{2} & =\left\|\sum_{i=1}^{n}\left(y_{i}-u_{i}(s)\right) \frac{\partial f\left(x_{i}\right)}{\partial A_{r}(s)}\right\|_{2} \\
& =\left\|\sum_{i=1}^{n}\left(y_{i}-u_{i}(s)\right) \sum_{p} w_{p} B_{p r}(s) x_{i} \cdot 1_{i}^{(s)}\{p, r\}\right\|_{2} / \sqrt{h_{1} h_{2}} \\
& \leq \frac{\|y-u(s)\|_{1} h_{2}^{1 / 6}}{\sqrt{h_{1} h_{2}}} \\
& \leq\|y-u(s)\|_{2} \sqrt{\frac{n}{h_{1} h_{2}^{2 / 3}}} \\
& \leq \sqrt{\frac{2 n \exp \left(-\lambda_{0} s\right) L(0)}{h_{1} h_{2}^{2 / 3}}}
\end{align*}
$$

Similarly, $\forall 1 \leq p \leq h_{2}$,

$$
\begin{align*}
\left\|\frac{d}{d s} B_{p}(s)\right\|_{2} & =\left\|\sum_{i=1}^{n}\left(y_{i}-u_{i}(s)\right) \frac{\partial f\left(x_{i}\right)}{\partial B_{p}(s)}\right\|_{2} \\
& =\left\|\sum_{i=1}^{n}\left(y_{i}-u_{i}(s)\right) w_{p} \sigma\left(A(s) x_{i}\right) \cdot 1_{i}^{(s)}\{p\}\right\|_{2} / \sqrt{h_{1} h_{2}} \\
& =\left\|\sum_{i=1}^{n}\left(y_{i}-u_{i}(s)\right) w_{p} A(s) x_{i} \cdot 1_{i}^{(s)}\{p, r\}\right\|_{2} / \sqrt{h_{1} h_{2}} \\
& \leq\|y-u(s)\|_{1}\|A(s)\|_{2} / \sqrt{h_{1} h_{2}} \\
& \leq\|y-u(s)\|_{1}\|A(s)\|_{F} / \sqrt{h_{1} h_{2}} \\
& \leq \sqrt{n / h_{2}\|y-u(s)\|_{2}}  \tag{47}\\
& \leq \sqrt{2 n \exp \left(-\lambda_{0} s\right) L(0) / h_{2}}
\end{align*}
$$

Then, we are able to bound the modification through integration

$$
\begin{align*}
& \left\|A_{r}(t)-A_{r}(0)\right\|_{2} \leq \int_{0}^{t}\left\|\frac{d}{d s} A_{r}(s)\right\|_{2} d s \leq \frac{2}{\lambda_{0}} \sqrt{\frac{2 n L(0)}{h_{1} h_{2}^{2 / 3}}} \\
& \left\|B_{p}(t)-B_{p}(0)\right\|_{2} \leq \int_{0}^{t}\left\|\frac{d}{d s} B_{p}(s)\right\|_{2} d s \leq \frac{2}{\lambda_{0}} \sqrt{\frac{2 n L(0)}{h_{2}}} \tag{49}
\end{align*}
$$

## Proof of Lemma 7

Proof: Given the bounds of $h_{1}$ or $h_{2}$ above, we are able to show that $R_{A}^{\prime}<R_{A}$ and $R_{B}^{\prime}<R_{B}$ using basic arithmetic. Suppose the conclusion does not hold at time $t$. According to Lemma 6, with high probability there exists $s \leq t$ such that $\lambda_{\text {min }}(H(s))<\lambda_{0} / 2$. Then, according to Lemma 5 , with high probability there exists $r$ such that $\left\|A_{r}(t)-A_{r}(0)\right\|_{2}>R_{A}$, or there exists $p$ such that $\left\|B_{p}(t)-B_{p}(0)\right\|_{2}>R_{B}$. This indicates that the following infimum exists:
$t_{0}=\inf _{t>0}\left\{\exists r:\left\|A_{r}(t)-A_{r}(0)\right\|_{2} \geq R_{A}\right.$ or $\left.\exists p:\left\|B_{p}(t)-B_{p}(0)\right\|_{2} \geq R_{B}\right\}$
(50)

Since $A_{r}(t)$ and $B_{p}(t)$ change continuously, we have that $\left\|A_{r}\left(t_{0}\right)-A_{r}(0)\right\|_{2} \leq R_{A},\left\|B_{p}\left(t_{0}\right)-B_{p}(0)\right\|_{2} \leq R_{B}$, and equality holds for at least one of them. Then by Lemma 5, with high probability $\lambda_{\min }\left(H\left(t^{\prime}\right)\right) \geq \lambda_{0} / 2 \forall t^{\prime} \leq t_{0}$, and thus $\left\|A_{r}\left(t_{0}\right)-A_{r}(0)\right\|_{2} \leq R_{A}^{\prime} \forall r$ and $\left\|B_{p}\left(t_{0}\right)-B_{p}(0)\right\|_{2} \leq R_{B}^{\prime} \forall p$ by Lemma 6. Recall that $R_{A}^{\prime}<R_{A}$ and $R_{B}^{\prime}<R_{B}$, we draw a contradiction here.

## Appendix B. Omitted Experiments



Fig. 3 Results on synthetic- 50 data


Fig. 4 Results on synthetic-100 data


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