International Journal of Business, Human and Social Sciences
ISSN: 2517-9411
Vol:14, No:2, 2020

Evaluating the Understanding of the University Students (Basic Sciences and Engineering) about the Numerical Representation of the Average Rate of Change

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***Abstract*—**The present study aimed to evaluate the understanding of the students in Tehran universities (Iran) about the numerical representation of the average rate of change based on the Structure of Observed Learning Outcomes (SOLO). In the present descriptive-survey research, the statistical population included undergraduate students (basic sciences and engineering) in the universities of Tehran. The samples were 604 students selected by random multi-stage clustering. The measurement tool was a task whose face and content validity was confirmed by math and mathematics education professors. Using Cronbach's Alpha criterion, the reliability coefficient of the task was obtained 0.95, which verified its reliability. The collected data were analyzed by descriptive statistics and inferential statistics (chi-squared and independent t-tests) under SPSS-24 software. According to the SOLO model in the prestructural, unistructural, and multistructural levels, basic science students had a higher percentage of understanding than that of engineering students, although the outcome was inverse at the relational level. However, there was no significant difference in the average understanding of both groups. The results indicated that students failed to have a proper understanding of the numerical representation of the average rate of change, in addition to missconceptions when using physics formulas in solving the problem. In addition, multiple solutions were derived along with their dominant methods during the qualitative analysis. The current research proposed to focus on the context problems with approximate calculations and numerical representation, using software and connection common relations between math and physics in the teaching process of teachers and professors.

***Keywords*—**Average rate of change, context problems, derivative, numerical representation, SOLO taxonomy.

I. Introduction

THE rate of change is one of the most important and applicable topics in the curriculum of most countries. Regarding the Iranian math curriculum [4], the study of the average rate of change related to the slope of a straight line has been presented at the ninth grade [3]. Further, the average rate of change has been completed with the ratio, proportion, and their variation in the tenth grade [2]. Finally, the average and instantaneous rates of change related to slope, speed, velocity, and acceleration in constructing the derivative concept have been discussed in the 12th grade in math and physics courses [1], [6], [20].

Stewart [48] defined the average rate of change as follows: Suppose y is a quantity that depends on another quantity x, thus, y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then, the change in x (also called the increment of x) is $\Delta x = x_1 - x_2$; and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$. The difference quotient ($\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$) is called the average rate of change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$ over the interval $[x_1, x_2]$, and can be interpreted geometrically as the slope of the secant line PQ (Fig. 1).

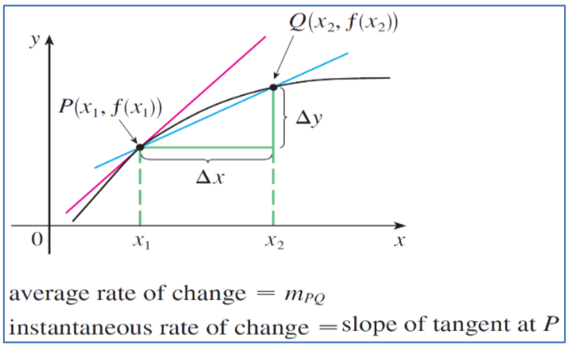

Fig. 1 Defining the average rate of change [27]

Dolores-Flores et al. [22] stated that the rate of change is regarded as an important concept in math education for the following reasons. First, the rate of change is a conceptual powerful link to understand functions and their diagrams and is essential for university-level studies to define the functions and participate in understanding the concept of slope through real-life situations [21], [49]. Second, the slope of a linear function indicates the rate of change of a variable through changing in the other variable and plays an important role in understanding the rate of change. So far, a large body of research has been conducted on the problems and misconceptions of students of the slope. For example, Leinhardt et al. [32] pointed out that students' belief that a greater slope tends to be associated with a greater height or vertical distance. In order to improve the understanding of the slope, Zaslavski et al. [57] suggested that visual slope (slope

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of a straight line) and analytical slope (the rate of change of a function) should be distinguished to better discuss the topics related to the rate of change in non-homogeneous scale. However, they claimed that the rate of change is even difficult for teachers. On the other hand, the rate of change and slope in the math and physics curriculum of Iran are related connected and recognized henceforth with the concept of derivative in the 12th grade [1], [20]. Math connections involve the connection between different mathematical concepts [23]. In fact, math connections are a network of links between definitions, properties, techniques, and procedures, which coordinate the construction of internal concepts. These links create a connection between the representations logically and generally. In the 12th-grade math and physics curriculum of Iran, the concept of derivative, speed, or velocity has been usually used instead of the rate of change and then, a connection has been described between these concepts.

The concept of the derivative with its current form was first introduced in 1666 by Newton and a few years later by Gottfried Leibniz, independent of each other. The two scientists continued their work and independently presented the second part of the mathematical analysis, i.e. the integral calculus, which was based on the integration procedure. Newton examined the derivative with a physical perspective and used it to obtain the instantaneous speed while Leibniz used the derivative to derive the slope of the tangent line in curves with a geometric perspective. Each of these two scientists applied separate symbols to represent the derivative [5], [26], [46]. Research has shown that students do not have a proper understanding of the concept of derivative and have difficulties [55], [56], [45], [37]. However, various frameworks have been presented to examine the students' understanding of the derivative concept, among which Zandieh's framework [56] and its expanded version [45] are the important frameworks, which emphasize the numerical representation of derivative in three layers of ratio, limit, and function. Table I presents the Zandieh's framework of understanding the derivative concept, which consists of three layers of ratio, limit, and function (each layer can be in the form of a process or an object), as well as graphical, verbal, physical, symbolic, and other representations. Zandieh [56] claimed that the three layers and their representations should be examined along with their interconnections, in order to examine the student's understanding of the concept of the derivative at a point. Table III represents the extended (completed) version of the Zandieh’s framework, which includes a numerical representation and a separate line for the instrumental understanding of the derivative (for example, the student calculates the derivative using formulas, although he/ she does not know any concept of derivative, so that it is not placed in the ratio layer). Considering the example given in Table III for physical representation, the derivative $\frac{dV}{dP}$ (describes the compressibility of the air: how easy it is compress) represents the volume of an air-filled piston to the pressure on the piston, which is controlled by a set of weights on the piston. (In this example, a weight by 1 unit is used for the concept of thickness) [27].

Table II is similar to Table I, which Roorda et al. [43] used to investigate students’ understanding of the derivative concept while focusing on the numerical aspect. The present study is based on these frameworks. A number of studies (e.g. [31], [33], [25], [35], [55]) have shown the performance of students who faced difficulties in relation to the symbolic output in layer 4 illustrated by figures or tables, although they could easily use derived rules. Kendal and Stacey [31] found that many students show their problems in numerical representation in the graphical and symbolic representations of layers 2 and 3. Park [37] claimed that some students had misconceptions about the rate of change in a point in layer 3 and the derivative at arbitrary points in layer 4. Other studies reported students' problems in understanding the limit process from layer 2 to layer 3 (e.g. [28], [44]).

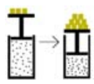
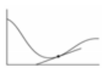
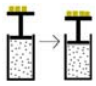
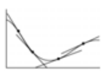
TABLE I
THE FRAMEWORK OF UNDERSTANDING THE CONCEPT OF ZANDIEH’S DERIVATIVE [55], [56]

Contexts / Layers	Graphical	Verbal	Physical	Symbol	Other
Process-object	Slope	Rate	Velocity	Difference quotient	
Ratio					
Limit					
Function					

TABLE II
REPRESENTATIONS AND LAYERS OF DERIVATIVE CONCEPT [43]

	Formulae	Graphical	Numerical
Level1	F1:f: function	G1: graph	N1: table
Level2	F2: $\frac{\Delta f}{\Delta x}$ difference quotient	G2: average slope	N2: average increase
Level3	F3: $\frac{df}{dx}$ differential quotient	G3: slope of a tangent	N3: instantaneous rate of change
Level4	f': derivative	G4: graph of derivative	N4: table with rates of change

TABLE III
THE FRAMEWORK OF STUDENTS’ UNDERSTANDING OF DERIVATIVE [45]

Process-object layer	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of change	Difference quotient	Ratio of changes	Measurement
Ratio		“Average rate of change”	$\frac{f(x+\Delta x)-f}{\Delta x}$	$\frac{y_2-y_1}{x_2-x_1}$ Numerically	
Limit		“Instantaneous…”	$\lim_{\Delta x\to 0}$ …	… with $\triangle$x small	
Function		“… at any point/time”	$f'(x)=\cdots$	…depends on x	Tedious repetition
			Symbolic		
			Instrumental understanding		
Function			Rules to ‘take a derivative’		

The numerical representation of the rate of change implies that students can calculate and approximate the derivative (rate of change) of a function at a single point in numerical order with the calculator, software, or using a table of values.

In this representation, there is no need for the extreme calculation of the formulas and instead, it is important to approximate using the points around the given point. The numerical representation of the rate of change has been studied by various researchers (e.g., [16], [29], [41], [42], [45]).

By presenting a task related to the numerical representation of the average rate of change, the present study seeks to answer the following questions: a) how are the SOLO levels of Tehran's students (basic sciences and engineering) in the task related to the numerical representation of the average rate of change? b) Is there a difference between the mean scores of engineering and basic science students in solving the problem of the numerical representation of the rate of change?

II. Theoretical Background

SOLO Taxonomy is regarded as a tool for classifying students' achievements or the growth level of their specific abilities at a specific time (example: a specific task in a particular area) [47]. This taxonomy was first introduced by Biggs and Collis (1982). SOLO is rooted in the developmental stages of Piaget and the concepts of processing information developed during the 1970 s, and have many common points with Neo-Piagetian theory [40]. It is worth noting that Neo-Piagetian theory is defined as theories, which combine findings, attention, memory, and strategies with Piaget's insights on children's thinking and knowledge creation [12], [24]. According to Neo-Piagetian theories, growth changes depending on the child's ability to process and remember information. In this regard, important variables affecting the quality of response include available active memory, that is, the amount of information that can be kept by the learner and has specific features for learning task [8].

SOLO is a valuable tool in various ways and has the potential to enhance the quality of learning and motivation in deep learning, regarding the assessment, measurement, and flexibility processes [15].

The SOLO taxonomy has two important features of thinking modes and response quality. The first feature is related to nature, the abstract, or the intellectual function, in which the learner should be properly involved with a particular motive. In this way, each of the five modes has its own identity and personality.

A. SOLO Model Thinking Modes

Pegg and Tall [39] stated that the description of the cognitive developmental stages of the SOLO model is as follows: 1) Sensorimotor (soon after birth): the person responds a little to the physical environment. In this mode, the baby acquires motor skills. When the required skills are created to perform different sports, these skills play an important role in future life. This mode is referred to as "tacit knowledge", which is related to motor activities. 2) Iconic (from 2 years old): The person internalizes actions in the form of images. In this case, the child creates words and images in the place of objects and events. However, this mode makes adults to appreciate the art and music, in which knowledge is formed and called "intuitive knowledge", which is related to visualization, imagination, and linguistic development [36]. (3) Concrete Symbolic (from 6 or 7 years old): The person starts to think using a symbolic system such as writing language and numbers. In the final periods of elementary and secondary education, reference to these modes is more common in learning than that of other modes. This mode is referred to as "Declarative knowledge", which is associated with the application and manipulation of symbolic and written systems. 4) Formal (from 15 or 16 years old): The person considers abstract concepts, which can be described in terms of work with principles and theories. In this mode, students are not limited to objective references. The advanced form of this mode includes the creation and development of disciplines, which is called "Theoretical knowledge" and is related to abstract structures that are not limited to real-world resources. 5) Post Formal (probably from 22 years old): The person is able to question or challenge the fundamental structure of scientific theories and fields which is called "theoretical knowledge" and is related to challenging and expanding the structures developed in the intuitive mode [7].

Although the five modes of thinking are distinct and are observed in the above-presented order, the previous modes obtained to support individual perception (like Iconic or sensorimotor modes) are available in the performance of a person in each mode (for example, Concrete Symbolic). This issue is considered as a multi-stage function. An important point in the SOLO cognitive development stages is that all modes are available and are represented throughout life in response to empirical, social, cultural, educational, and genetic motives [13], [19].

B. The Quality of SOLO Model Responses

According to Biggs and Tang [12], the second feature of the SOLO model relies on the personal ability available to the individual, which is dependent on increasing skill. Both features are presented as response levels located throughout the learning cycle and provide a hierarchical description of the structure of the response. The levels of responses, which involve a learning cycle in a particular mode, are as follows: Unistructural: A student can deal with a single aspect and create specific connections. At this level, the student can use the terms, express the cases, execute simple guidance/ algorithms, specify, name, count, and the like (Tables III & IV).

Multistructural: At this level, the student can deal with several aspects, although they are considered independent and not related to each other. Metaphorically, the student will see many trees but not the forest [14], [17], [53]. He/she is able to count, describe, classify, combine, and apply procedures, structures, processes, and the like (Table III).

Relational: At level 4, the student understands the relationships between several aspects and explains how these aspects fit together and form a set. Additionally, the student at this level forms a structural understanding and perceives how trees shape a jungle. Therefore, a student may have the ability to compare, express, study, and apply the theory and explain one phenomenon in terms of cause and effect or capabilities similar to the mentioned cases [10], [11], [30], [51] (Tables IV & V).

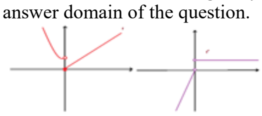
TABLE IV
SOLO TAXONOMY FOR THE LEVEL OF ACHIEVEMENT TO KNOWLEDGE [54]

Target	SOLO LEVEL	Level of knowledge acquisition
I know nothing about the subject.	Pre-structural	
I know one aspect of the subject.	Uni-structural	
I know several aspects of the subject but I am not sure when and why to use them.	Multi-structural	
I can create relations between aspects and know when and why to use them.	Relational	
I can teach the subject to others and apply what I have learned in other fields.	Extended abstract	

Table V provides a sample question of the derivative topic, using which the SOLO levels are identified based on the responses of students. In addition, some of the verbs displayed in Fig. 2 are presented in Table V to explain the SOLO levels. The function $f(x)=\begin{cases} x & x\geq 0 \\ x^2+1 & x<0 \end{cases}$ has been given and the students have been asked to obtain $f'(x)$. If the student provides an irrelevant answer, he/she will be at the pre-structural level. If the student differentiates one of the functions separately, he/she lies at the uni-structural level because he/she only considers one aspect of the question. If the student differentiates from the functions, regardless of whether the function is continuous at zero, he/she will be at the multi-structural level. Finally, if the student observes non-differentiability at zero, he/she is at the relational level. Now, if the student plots the diagram of function and derivative and understands its relation to other mathematical concepts, he is at the extended abstract level. Table V illustrates these topics completely.

TABLE V
A SAMPLE OF AN EXAMPLE BY DETERMINING ITS SOLO LEVELS

Example: if $f(x)=\begin{cases} x & x\geq 0 \\ x^2+1 & x<0 \end{cases}$, then, find $f'(x)$?

SOLO level	Answer	Description	Appropriate verbs
Pre-structural	$f'(x)=x$	Incorrect way or no answer	Lack of understanding the path Misses point
Uni-structural	$f'(x)=2x$ or $f'(x)=1$.	The student considers one aspect.	Recognize/ doing simple work
Multi-structural	$f'(x)=\begin{cases} 1 & x\geq 0 \\ 2x & x<0 \end{cases}$	The student considers several aspects, but without connection. For example, there is no derivative at zero.	Describing/ Listing/ Enumerate
Relational	$f'(x)=\begin{cases} 1 & x>0 \\ 2x & x<0 \end{cases}$	He/she focuses on several aspects or relations between them and knows that the function has no derivative at zero.	Comparing/explaining causes/Apply
Extended abstract	The function is not continuous at zero, and thus, it is not derivative, although the function is integrable. The derivative function is $f'(x)=\begin{cases} 1 & x>0 \\ 2x & x<0 \end{cases}$. The derivative function is ascending for negative xs and both ascending and descending for positive xs (the derivative function is constant). The function has a Riemann integral. The absolute minimum of the function is zero, but it does not have an absolute maximum.	The student has knowledge beyond the answer domain of the question. 	Theorize/ evaluating/ Generalize/Reflect

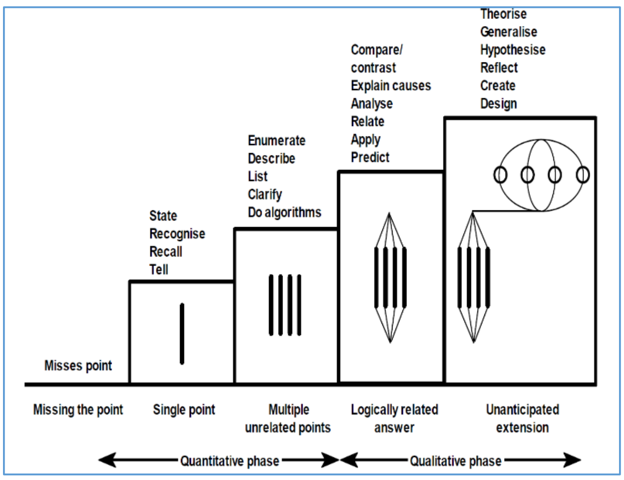


Fig. 2 Verbs corresponding to each level [9]

The three levels of uni-structural, multi-structural, and relational with each other are called UMR Learning Cycle. These levels are developed in a wider context with a pre-structural surface, the answer to a specific problem, which has not even reached the uni-structural level, an incorrect process or data, or a simple way that may lead to an irrelevant outcome and the person fails in the problem. Thus, there are no endings, except a general extended abstract level, in which the quality of the relational levels lies within a larger image that may be the basis of the next construction cycle. Extended abstract responses are structurally similar to relational responses, although data, concepts, and processes are outside the domain of knowledge and experience assumed in the question (Table III). Fig. 2 illustrates the verbs corresponding to each SOLO level in general. Further, Fig. 3 summarizes the thinking modes and response levels. As shown in Fig. 2, verbs are quantitative up to the multi-structural stage and qualitative after the relational stage.

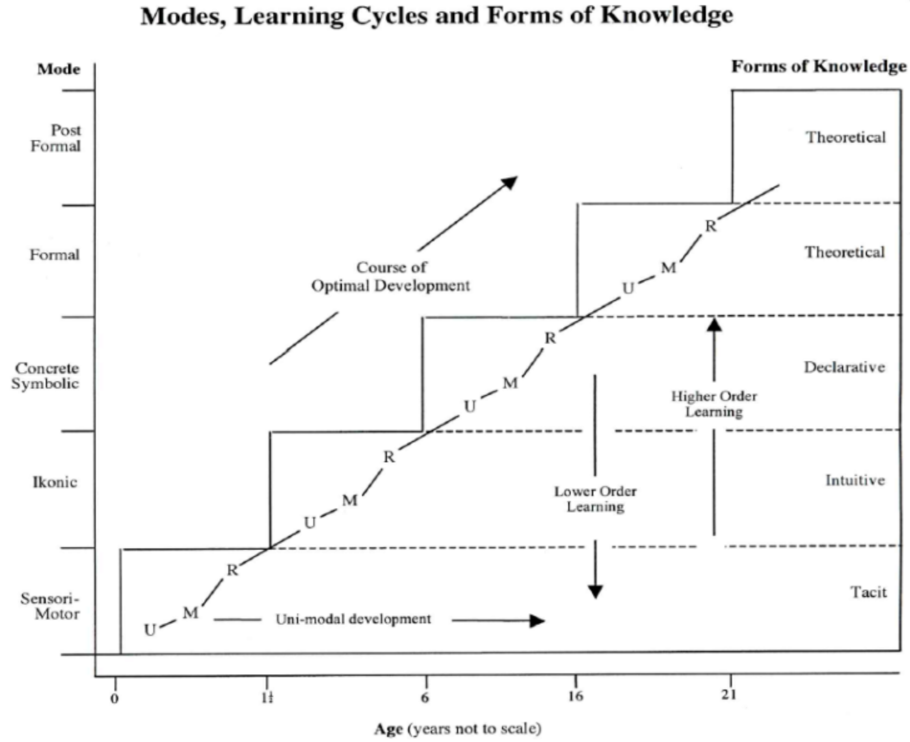


Fig. 3 Modes and levels in SOLO taxonomy [8]

Although U, M, and R levels specify a conceptual understanding cycle in a single mode, some research in the last decade indicated that a single cycle has limitations and problems and fails to fully explain the development of a concept [34], [36], [38], [18], [52]. Subsequently, the two uni-structural/multi-structural/relational cycles (e.g., U_1, M_1, R_1 and U_2, M_2, R_2) in the model are considered for symbolic and intuitive situations, as indicated in Fig. 4.

In addition to studying the students' responses to a numerical representation task about the average rate of change, this research seeks to examine multiple solutions, misconceptions, and different approaches to problem-solving and finally, analyze them based on the SOLO theory.

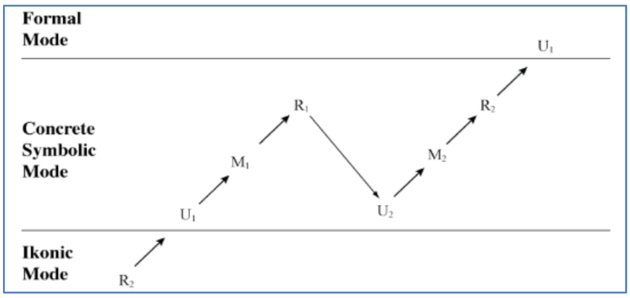


Fig. 4 Two cycles in the concrete symbolic mode [39]

III. Methodology

The research method was descriptive with survey type. The population included all undergraduate students (basic sciences and engineering) in the universities of Tehran, which had passed general mathematics (Calculus 1). The samples were 604 students from the universities of Sharif, Tehran, Shahid Beheshti, Nasir al-Din al-Tusi, Iran Science and Technology, Amir Kabir, Farhangian, Shahid Rajaee Teacher Training, and Islamic Azad University (Central, South, and Technical branches), which were selected in a randomly multi-stage cluster way. The measurement tool of this study was a task whose face and content validity was confirmed by math education teachers and professors. Using Cronbach's Alpha criterion, the reliability coefficient of the task was obtained 0.95, which validated its reliability. In addition, the responses of students were analyzed using the SOLO model. Descriptive statistics (mean, tables, and charts) and inferential statistics (Chi-square and independent t-test) methods were used for analyzing the data, using SPSS 24 software. It should be noted that the maximum grade of the task was determined four.

IV. Findings

The task of numerical representation is as follows:

The analysis of students' responses to the task is discussed below. Table VI provides the descriptive statistics of the levels of students' responses in Tehran's universities based on the SOLO theory. As observed, 178 students surprisingly failed to

International Journal of Business, Human and Social Sciences
ISSN: 2517-9411
Vol:14, No:2, 2020

respond to the task. The highest number of students was at the relational level, although it was about one-third of students. In general, the distribution of SOLO levels among the whole student population was roughly uniform. Fig. 5 illustrates a bar chart of the percentage of the responses of engineering and basic science students to the task.

TABLE VI
THE TASK OF NUMERICAL REPRESENTATION

Task: A ball is thrown into the air from a bridge 11 meters high. $f(t)$ denotes the distance that the ball is from the ground at time t. Some values of $f(t)$ are shown in the table below [41].

$t\ (s)$	0	0.1	0.2	0.3	0.4	0.5	0.6
$f\ (t)\ (m)$	11	12.4	13.8	15.1	16.3	17.4	18.4

Based on the table, at what the approximate speed will the ball be travelling when it reaches a height at $t = 0.4\ s$? Justify your chosen answer.
A. 11.5 m/s B. 1.23 m/s C. 14.91 m/s D. 16.3 m/s E. Others

TABLE VII
DESCRIPTIVE STATISTICS OF THE RESPONSES OF STUDENTS IN THE UNIVERSITIES OF TEHRAN TO THE TASK BASED ON THE SOLO THEORY

Variable	Frequency	Percentage
Pre-structural	106	24.9
Uni-structural	74	17.4
Multi-structural	121	28.4
Relational	125	29.3
Total valid data	426	70.5
Without respond	178	29.5
Total	604	100.0

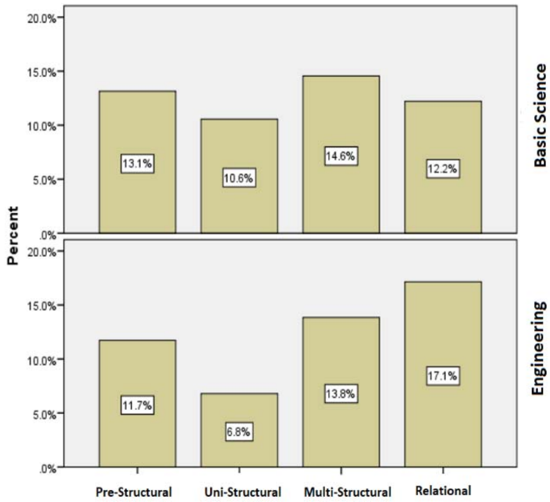


Fig. 5 A bar chart of the SOLO levels of engineering and basic science students to the task

Fig. 6 depicts the percentage of students' general levels based on the SOLO theory. Table VIII reports the students' responses separately in terms of basic science and engineering disciplines using the Chi-square test.

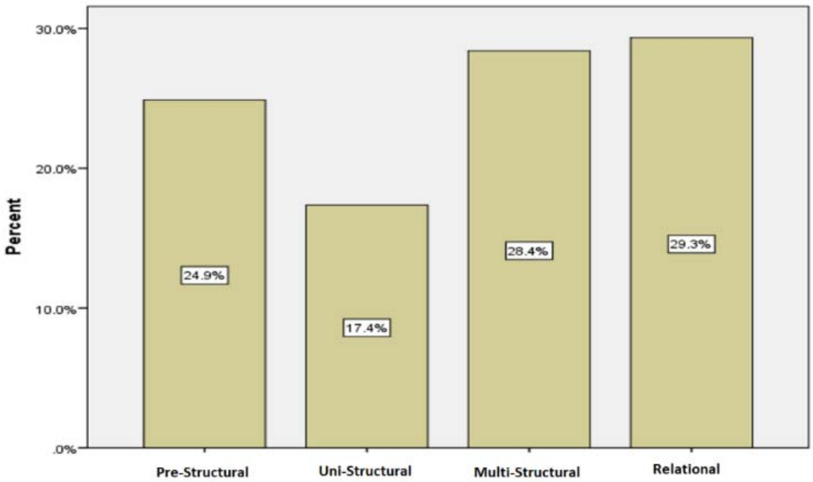


Fig. 6 A bar chart related to the general percentage of SOLO levels of students to the task

TABLE VIII
STUDENTS' RESPONSES TO THE TASK BASED ON ENGINEERING AND BASIC SCIENCE DISCIPLINES

Levels \ Variable	Engineering discipline	Basic science discipline	Degree of freedom	Chi-square value	$P < 0.05$
Pre-structural level	50	56	3	15.11	Yes
Uni-structural level	29	45			
Multi-structural level	59	62			
Relational level	73	52			

Given that the crisis Chi-square is 7.82 at the significant level of 5% and the degree of freedom of 3 and the Chi-square value is 15.11 (Table VIII), it can be concluded that there is a significant difference between the students of engineering and basic sciences in the distribution of SOLO levels.

Table IX represents some of the incorrect answers of student and provides explanations about these responses, which determine the SOLO level of the response.

As shown in Fig. 6, there is a clear dispersion in the responses of engineering and basic science students, although only 29.3% of the students responded to the desired task correctly. According to the answers, most students have been looking for physics formulas to answer this question. The following section presents a qualitative analysis of the students' responses.

TABLE IX
A SAMPLE OF STUDENTS' INCOMPLETE RESPONSES TO THE TASK AND THE SOLO LEVEL OF THE RESPONSES

Incomplete or wrong answers	SOLO level	Explanations
Option E is correct because $x = 16.3 - 11 = 5.3$ $v = \frac{x}{t} = \frac{5.3}{0.4} = 13.25$	Uni-structural	The method is correct, but an appropriate approximation is not selected, so the correct answer is not observed in options and only one aspect is considered.
Option B is correct because the approximate speed is equal to the displacement-to-time ratio: $v = \frac{\Delta x}{\Delta t} = \frac{16.3 - 15.1}{1} = 1.2$	Uni-structural	There is a computational mistake in time calculation and the student has reached the wrong option and considered one aspect.
Option E is correct because: $\frac{12.4 - 11}{0.1} = 14$, $\frac{13.8 - 12.4}{0.1} = 14$ $\rightarrow\ 0.4 \times 14 = 5.6$ $\rightarrow\ 11 - 5.6 = \boxed{6.4}$	Multi-structural	The student used and generalized the linear pattern, and paid attention to several aspects, although he/she failed to recognize their connection.
Option E is correct because: $x = 16.3 - 11 = 5.3$ $v = \frac{x}{t} = \frac{5.3}{0.4} = 14.91$	Uni-structural	There is a computational mistake in the final division and the student has paid attention to one aspect.
Option E is correct because: $16.3 - 11 = 5.3$ $\frac{1}{2} m v^2 = m g h \rightarrow 53 = \frac{1}{2} v^2$ $\rightarrow v = 10.295$	Multi-structural	Student solved the problem for free fall, instead of projectile motion, and used the conservation of energy law, while focusing on several aspects.
Option E is correct because we assume that: $f(t) = a t^2 + b$ $f(0) = b = 11$, $f(0.1) = a(0.01) + 11 = 12.4$ $\rightarrow a = 140 \rightarrow f(t) = 140 t^2 + 11$ $f'(t) = 2 \times 140 \times t \rightarrow f'(0.4) = 112$	Multi-structural	The student assumed that the equation is a second order, but the general state, the role of gravity, the initial velocity, and initial launching angle are ignored and he/she focused on several aspects.
Option E is correct because: $x = \frac{1}{2} a t^2 + v_0 t$ $x = \frac{1}{2}(a)(0.1)^2 + 0 = 12.4 \rightarrow a = 2480$ $v = at + v_0 \rightarrow v = 2480 \times (0.4) = 992$	Multi-structural	The student knew the general solving method, but he/she does not include in the formula that the ball was thrown at a height of 11 m. He/She has paid attention to several aspects and ignored some others.

V. RESULTS

This section first explains the reasons for inserting current options in the response to the task, and then, describes the various methods used by students to achieve these responses [41].

The students chose three approaches to solve this task, which are mathematical approach (32%), physical approach (52%), and the combined approach (16%). Regarding these approaches, students responded correctly to the task and were at the relational level with the SOLO model. Considering the importance of the solutions used in this task, multiple solutions are provided in the following.

A. Mathematical Approach

The mathematical approach means to use the average rate or approximate derivative of the place-to-time changes for two points at $t = 0.4$, or the limit concept of the time-to-place changes or geometric interpretation of the secant line instead of the tangent line (Table XI).

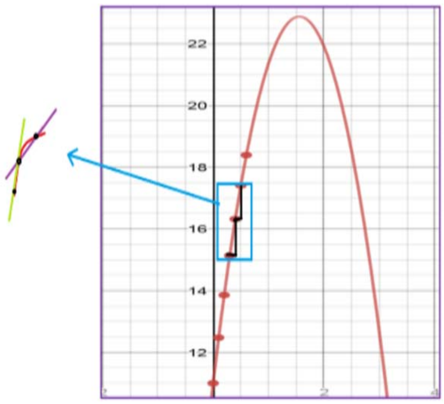


Fig. 7 Option A

B. Physical Approach

The purpose of this approach is to use the concepts of physics and related formulas to answer the task. Fig. 8 displays the formulas used by most students to answer the task.

TABLE X
OPTIONS INTERPRETATION IN RESPONSE TO THE TASK

Option A: $\frac{1}{2}[\frac{f(0.4)-f(0.3)}{0.1}+\frac{f(0.5)-f(0.4)}{0.1}]=\frac{1}{2}[\frac{16.3-15.1+17.4-16.3}{0.1}]=11.5$

This option calculates the average of the two right and left secant slopes (the average rate of change between the left and right intervals of the desired point), which is an appropriate approximation.

Option B: $\frac{1}{6}[(f(0.1)-f(0))+((0.2)-f(0.1))+\ldots+(f(0.6)-f(0.5))]=\frac{1}{6}[18.4-11]=1.23$ This option calculates only the average of rises in the given points and is considerably different from the average rate of change at the requested point. The rate of change is somehow a forward ahead ratio.

Option C: $\frac{1}{7}[f(0)+f(0.1)+\ldots+f(0.6)]=\frac{104.4}{7}=14.91$

This option calculates the average width of the given points and is not related to the average rate of change at point 0.4.

Option D: This option only calculates the width of the given point, which does not relate to the approximate value of the rate of change.

Option E: Students have scored other results in the test.

TABLE XI
MULTIPLE SOLUTIONS OF STUDENTS WITH A MATHEMATICAL APPROACH IN RESPONSE TO THE TASK

Students' mathematical approach to responses to the task	
Multiple solutions	**Explanation**
First method	Approximation with a time before $t = 0.4$: $V=\frac{\Delta y}{\Delta t}=\frac{16.3-15.1}{0.4-0.3}=12$
Second method	Approximation with time after $t = 0.4$: $V=\frac{\Delta y}{\Delta t}=\frac{17.4-16.3}{0.5-0.4}=11$
Third method	Approximation with two times around $t = 0.4$: $V=\frac{\Delta y}{\Delta t}=\frac{17.4-15.1}{0.5-0.3}=11.5$
Fourth method	Approximation with the method of approaching to $t = 0.4$, which should eventually be guessed $V=\frac{\Delta y}{\Delta t}=\frac{16.3-11}{0.4-0}=13.25$ $V=\frac{\Delta y}{\Delta t}=\frac{18.4-16.3}{0.6-0.4}=10.5$ $V=\frac{\Delta y}{\Delta t}=\frac{16.3-12.4}{0.4-0.1}=13$ $V=\frac{\Delta y}{\Delta t}=\frac{17.4-16.3}{0.5-0.4}=11$ $V=\frac{\Delta y}{\Delta t}=\frac{16.3-13.8}{0.4-0.2}=12.5$ $V=\frac{\Delta y}{\Delta t}=\frac{16.3-15.1}{0.4-0.3}=12$ $13.25, 13, 12.5, 12, \rightarrow 11.5 \leftarrow, 11, 10.5$
Fifth method	Approximation with a mean around $t = 0.4$: $v_1=\frac{\Delta y}{\Delta t}=\frac{16.3-15.1}{0.4-0.3}=12$ $v_2=\frac{\Delta y}{\Delta t}=\frac{17.4-16.3}{0.5-0.4}=11$ $v=\frac{v_1+v_2}{2}=11.5$

$$y=y_0+v_0(\sin\theta)t-\frac{1}{2}gt^2 \qquad y=y_0+v_0t-\frac{1}{2}gt^2$$
$$y=v_0(\sin\theta)t-\frac{1}{2}gt^2 \qquad y=v_0t-\frac{1}{2}gt^2$$
$$v=v_0(\sin\theta)-gt \qquad v=v_0-gt$$
$$v^2-(v_0\sin\theta)^2=-2gt(y-y_0) \qquad v^2-v_0^2=-2gt(y-y_0)$$
$$mgh=\frac{1}{2}mv^2$$

Fig. 8 Formulas related to the projectile motion in physics [50]

Given the choice of origin, some relationships may be incorrect. In order to calculate the approximate speed requested in the question, some students considered g equal to 10 or 9.8, although other students ignore the launch angle, or remove it because of approximate calculations and the closeness of the value to one.

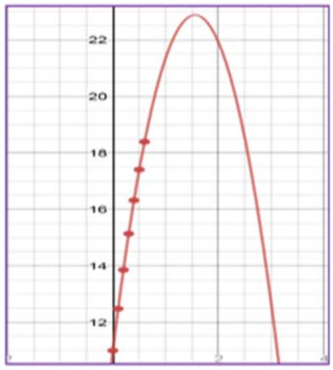


Fig. 9 The approximate chart of the task

TABLE XII
MULTIPLE SOLUTIONS OF STUDENTS WITH A PHYSICAL APPROACH IN RESPONSE TO THE TASK

The physical approach of students in response to the task	
Multiple solutions	**Explanation**
Sixth method	$y=y_0+v_0(\sin\theta)t-\frac{1}{2}gt^2$ $\rightarrow 16.4=11+v_0(0.997)(0.4)-4.9(0.4)^2$ $\rightarrow v_0=15.26$ $v=v_0\sin\theta-gt=(15.26)(0.997)-(9.8)(0.4)=11.29$
Seventh method	$y=y_0+v_0(\sin\theta)t-\frac{1}{2}gt^2$ $\rightarrow 16.4=11+v_0(0.997)(0.4)-4.9(0.4)^2$ $\rightarrow v_0=15.26$ $v^2-(v_0\sin\theta)^2=-2g(y-y_0)$ $\rightarrow v^2-((15.26)(0.997))^2=-2(9.8)(16.3-11)$ $\rightarrow v=11.29$
Eighth method	$y=y_0+v_0t-\frac{1}{2}gt^2$ $\rightarrow 16.4=11+v_0(0.4)-4.9(0.4)^2$ $\rightarrow v_0=15.21$ $v=v_0-gt=15.21-(9.8)(0.4)=11.29$
Ninth method	$y=y_0+v_0t-\frac{1}{2}gt^2$ $\rightarrow 16.4=11+v_0(0.997)(0.4)-4.9(0.4)^2$ $\rightarrow v_0=15.21$ $v^2-v_0{}^2=-2g(y-y_0)$ $\rightarrow v^2-(15.21)^2=-2(9.8)(16.3-11)$ $\rightarrow v=11.29$
Tenth method	If $g = 10$, then, by considering the above relations, we have: $v = 11.25$

As demonstrated in Fig. 9 and using the first two data, $\sin\theta = 0.997$ and the launch angle is approximately 36°. However, $\sin\theta$ can be eliminated in approximate calculations and thus, the approximate speed can be calculated in the form presented in Table XII.

C. Combined Approach

In this part physical formulas are combined with mathematical relations. Table XIII provides a combination of two mathematical and physics approaches in student responses. However, students used other methods to respond to the task, including Lagrange and Newton interpolation, although the basis of all responses was the 12 mentioned methods. Table XIV compares the average of the two groups of engineering and basic sciences with regard to students' understanding of the derivative concept in a numeric way and considering the independent t-test.

TABLE XIII
STUDENTS' MULTIPLE SOLUTIONS WITH A COMBINED APPROACH IN RESPONSE TO THE TASK

The combined approach of students in response to the task	
Multiple solutions	**Explanation**
Eleventh method	Using physics formula and derivative: $y = y_0 + v_0 t - \frac{1}{2} g t^2$ $\to 16.4 = 11 + v_0(0.997)(0.4) - 5(0.4)^2$ $\to v_0 = 15.25$ $y = 11 + 15.25t - 5t^2$ $\to y' = 15.25 - 10t$ $\to y'(0.4) = 11.21$
Twelfth method	A combination of the motion and derivative equation: We know the ball equation is in the form of $y = -5t^2 + v_0 t + 11$. Now, in order to find v_0, we should substitute $t = 0.4$. Then: $v_0 = 15.25 \to y' = -10t + 15.25$ $\to y'(0.4) = 11.25$
Thirteenth method	**Lagrange-Newton Interpolation:** By having points, it is possible to find the unique polynomials corresponding to the points and then, differentiate it with respect to the motion equation, and obtain the answer to the task.

TABLE XIV
INDEPENDENT T-TEST FOR COMPARING THE TWO GROUPS OF ENGINEERING AND BASIC SCIENCES IN THE STUDENT'S UNDERSTANDING OF THE CONCEPT OF THE AVERAGE RATE OF CHANGE

Group	Mean	Standard deviation	t	Degree of freedom	p
Engineering (N = 211)	2.49	1.54	-1.67	424	> 0.05
Basic science (N = 215)	2.25	1.50			

The results of Table XII indicate that there is no significant difference between the two groups of engineering and basic sciences in their understanding of the derivative concept in a numeric way.

VI. DISCUSSION AND CONCLUSION

The present study aimed to evaluate the understanding of the students of Tehran (Iran) about the numerical representation of the average rate of change based on the SOLO Taxonomy. The results indicated that students generally had a modest understanding of the numerical representation of the average rate of change. On the other hand, the use of physics formulas in the given task created misunderstandings to some students while the response to the task could be easily calculated numerically. The result was consistent with that of [16], [41], [45], and [43]. The levels of students were almost uniform in this task based on the SOLO model, in which 29.3% were at the relational level and presented the correct answer. There was no significant difference between the responses of engineering and basic science students in terms of mean scores, although the pre-structural, uni-structural, and multi-structural SOLO levels of basic science students were more than that of engineering students. Nevertheless, this result was reversed at the relational level, highlighting one of the advantages of using SOLO theory.

According to the results of the research and responses analysis, students fail to have a proper understanding about the numerical representation of the average rate of change and solving method, in addition to the incorrect use of the physics formulas to solve the problem. The qualitative analysis of the correct answers of students, who were at the relational level, provided interesting results, among which was multiple solutions. Based on the research, it seems that attention to context and real issues is based on approximate calculation and numerical representations using the software and connections between the common mathematical and physics relations in the teaching of teachers and professors.

ACKNOWLEDGMENT

We would like to thank all students for participating in study.

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