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The Martingale Options Price Valuation for European Puts Using Stochastic Differential Equation Models

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***Abstract*—**In modern financial mathematics, valuing derivatives such as options is often a tedious task. This is simply because their fair and correct prices in the future are often probabilistic. This paper examines three different Stochastic Differential Equation (SDE) models in finance; the Constant Elasticity of Variance (CEV) model, the Balck-Karasinski model, and the Heston model. The various Martingales option price valuation formulas for these three models were obtained using the replicating portfolio method. Also, the numerical solution of the derived Martingales options price valuation equations for the SDEs models was carried out using the Monte Carlo method which was implemented using MATLAB. Furthermore, results from the numerical examples using published data from the Nigeria Stock Exchange (NSE), all share index data show the effect of increase in the underlying asset value (stock price) on the value of the European Put Option for these models. From the results obtained, we see that an increase in the stock price yields a decrease in the value of the European put option price. Hence, this guides the option holder in making a quality decision by not exercising his right on the option.

***Keywords*—**Equivalent Martingale Measure, European Put Option, Girsanov Theorem, Martingales, Monte Carlo method, option price valuation, option price valuation formula.

I. INTRODUCTION

FINANCIAL derivatives are financial contracts that are linked to an underlying asset and through which specific financial risks can be traded in a typical financial market. The value of a financial derivative is a function of the underlying asset and time from whence its price is derived. Since the future reference price of the derivative is not known with certainty, its value at maturity can only be anticipated or estimated. Options which are a type of financial derivative are used for several purposes which include risk management, hedging, etc. [1].

Options allow parties to trade peculiar financial risks to other investors that are more willing and ready to accommodate such risk. The risk involved in option pricing contract can either be traded as itself or by initiating a new contract that bears the burden of risks involved in the contract [1]. There are basically four types of financial derivatives which are swaps, forwards, futures, and options.

There are many justifications why investors opt for trading in options instead of trading in stocks. One major reason to this decision is that it aids mitigation of risks and saves transaction costs.

To value or price financial derivative products such as options is one of the most common problems in mathematical finance. In order to value an option using the Martingales approach, a replicating portfolio is constructed from trade-able assets, and the replicating portfolio is assumed to be driven by a financing strategy that is self-financing. The portfolio replicates the payoff of the financial derivative at expiry, and because of no arbitrage, it also replicates the value of the financial derivative at every instant before expiry. We then use the fact that the numeraire that turns the trade-able assets into martingales also turns the replicating portfolio into a martingale. The value of the financial derivative is the expected value of the payoff at expiry, discounted by the numeraire [2]. However, this paper will focus on the use of Martingale approach in the valuation of options price which is a type of financial derivatives.

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II. THEORETICAL BACKGROUND

Definition 1 (Risk-neutral measure). A probability measure $\mathbb{P}^*$ on Ω is called a risk-neu.tral measure if it satisfies

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{r(t-u)}S_u, \quad 0 \le u \le t$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

Definition 2 (Self-financing portfolio). A portfolio allocation $(\xi_t, \eta_t)_{t\epsilon\mathbb{R}_+}$ with price (value) V_t given by $V_t = \xi_t S_t + \eta_t A_t, t \epsilon \mathbb{R}_+$I s self-financing if and only if the relation $dV_t = \eta_t dA_t + \xi_t dS_t$ holds, where ξ_t is the number of shares in S_t (could be any real number) and η_t is the amount in the bank.

Definition 3 (Numeraires). A ***numeraire*** is an asset with positive price, namely $N_t > 0$ for all t. Any asset with this property can serve as a numeraire. The relative price $\widetilde{S_t}$ of an asset is its price S_t divided by the numeraire price, so that $\tilde{S}_t = \frac{S_t}{N_t}$ and S is measured in units of N.

Definition 4 (Martingales). An integrable process $(X_t)_{t\epsilon R_+}$ is said to be a martingale with respect to the filtration $(\mathcal{F}_t)_{t\epsilon R_+}$if

$$E[X_t|\mathcal{F}_s] = X_s, \quad 0 \le s \le t.$$

Theorem 1 (Fundamental theorem of arbitrage). The Fundamental Theorem of Arbitrage asserts that if the market is complete, then for each numeraire N_t, there exists a unique Equivalent Martingale Measure $\mathbb{N}$ such that the relative price of the assets (and consequently, of the replicating portfolio) using that numeraire is a martingale. In the other words, $\frac{S_t}{N_t}$ is a martingale under $\mathbb{N}$. Hence,

$$E^{\mathbb{N}}\left[\frac{S_T}{N_T}|\mathcal{F}_t\right] = \frac{S_t}{N_t} \tag{1}$$

III. Methods

A. Preliminaries for the Model Formulation

Proposition 1 [3]**.** The measure $\mathbb{P}^*$ is risk-neutral if and only if the discounted price process $(X_t)_{t\,\epsilon\,\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$.

Proof. If $\mathbb{P}^*$ is a risk-neutral measure, we have

$$\mathbb{E}^*[X_t|\mathcal{F}_u] = \mathbb{E}^*[e^{-rt}S_t|\mathcal{F}_u] = e^{-rt}\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{-ru}S_u = X_u, 0 \le u \le t$$

hence $(X_t)_{t\,\epsilon\,\mathbb{R}_+}$ is a martingale. Conversely, if $(X_t)_{t\,\epsilon\,\mathbb{R}_+}$ is a martingale, then

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{rt}\mathbb{E}^*[X_t|\mathcal{F}_u] = e^{rt}X_u = e^{r(t-u)}S_u \quad 0 \le u \le t$$

hence the measure $\mathbb{P}^*$ is risk-neutral according to Definition 1.6.

Theorem 2. (Girsanov Theorem; [4]). The process $\widetilde{W_t} = W_t - \int_0^t \theta_s\, ds$ is Brownian motion under the measure $\mathbb{Q}$.

Theorem 3. Let $(\phi_t)_{t\epsilon[0,T]}$ be an adapted process satisfying the Novikov integrability condition

$$E\left[exp\left(\frac{1}{2}\int_0^T |\phi|^2 dt\right)\right] < \infty$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = exp\left(-\int_0^T \phi_s\, dW_s - \frac{1}{2}\int_0^T \phi_s^2\, ds\right)$$

then

$$\widetilde{W}_t = W_t + \int_0^t \phi_s\, ds, \quad t\,\epsilon\,[0,T],$$

is a standard Brownian motion under $\mathbb{Q}$.

Theorem 4. (Martingale Representation Theorem; [4]). Suppose that M_t is an $\mathcal{F}_t$ −martingale where $\{\mathcal{F}_t\}_{t\ge 0}$ is the filteration generated by the n −dimensional standard Brownian motion, $W_t = \left(W_t^{(1)}, \dots, W_t^{(n)}\right)$. If $E[M_t^2] < \infty$ for all t then there exists a unique n −dimensional adapted stochastic process, ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_s^T dW_t \quad \text{for all } t \ge 0$$

where ϕ_s^T denotes the transpose of the vector, ϕ_s.

B. The Martingale Approach

In this approach, options are not part of the traded assets $S_t(t), \dots, S_N(t)$, so cannot be priced directly. However, we can form a replicating portfolio $\prod(t) = \sum_{i=1}^N a_i(t)S_i(t)$ that replicates the price of the option at every time, so that $V_t = \Pi_t$ for every $t > 0$ and $V_T = \Pi_T$. Moreover, the portfolio is traded since each asset is traded. The Fundamental Theorem of Arbitrage (Theorem 1) guarantees that given a numeraire N_t, each relative asset will be a martingale under the corresponding measure $\mathbb{N}$, and consequently, so will V_t/N_t since it is a linear combination of martingales. The martingale property of V_t/N_t implies that

$$E^{\mathbb{N}}\left[\frac{V_T}{N_T}\middle|\mathcal{F}_t\right] = \frac{V_t}{N_t} \tag{2}$$

from which the time − t price of the derivative, V_t, is

$$V_t = N_t E^{\mathbb{N}}\left[\frac{V_T}{N_T}\middle|\mathcal{F}_t\right] \tag{3}$$

In the Black-Scholes economy, we have two assets, a stock S_t that follows the SDE

$$dS = rSdt + \sigma SdW \tag{4}$$

and a fixed bond B

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{5}$$

$$dB_t = rB_t dt$$

We apply Girsanov's theorem so that the process for dS_t becomes

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{B}} \tag{6}$$

where $dW_t^{\mathbb{B}} = dW_t + \frac{\mu - r}{\sigma}dt$ and $B_t = \exp\left(\int_0^t r du\right) = e^{rt}$.

We use B_t as the numeraire so that $\tilde{S}_t = \frac{S_t}{B_t}$ is a martingale under $\mathbb{B}$. The European Put option has payoff $V_T = (K - S_T)^+$, so in accordance with (3), the time-t price of the Put is

$$V_t = B_t E^{\mathbb{B}}\left[\frac{(K - S_T)^+}{B_T}\middle|\mathcal{F}_t\right] = e^{-r(T-t)}E^{\mathbb{B}}[(K - S_T)^+|\mathcal{F}_t] \tag{7}$$

We can use the choice of another numeraire. The choice of the numeraire, B_t, is arbitrary, and S_t can be used instead. In the previous section, we saw that $\frac{S_t}{B_t}$ is a martingale under an EMM $\mathbb{B}$. Now, we have that $\frac{B_t}{S_t}$ is a martingale, but under a different measure $\mathbb{S}$. In (7), the value V_t of the Put is derived from

$$\frac{V_t}{B_t} = E^{\mathbb{B}}\left[\frac{V_T}{B_T}\middle|\mathcal{F}_t\right].$$

Equivalently, using S_t as the numeraire, the same value V_t can be derived from

$$\frac{V_t}{S_t} = E^{\mathbb{S}}\left[\frac{V_T}{S_T}\middle|\mathcal{F}_t\right]$$

The European Put has payoff, $V_T = (K - S_T)^+$, so the time-t price of the Put is

$$V_t = S_t E^{\mathbb{S}}\left[\left(\frac{K}{S_T}-1\right)^+ \middle| \mathcal{F}_t\right] = S_t E^{\mathbb{S}}\left[\frac{(K-S_T)^+}{S_T}\middle| \mathcal{F}_t\right] \quad (8)$$

Even though the expression in (7) and (8) are different, they both produce the same solution [5].

C. Derivation of the Model Equations

1. The CEV Model

We consider the pricing of European put option in the CEV model introduced by [6]. There are two assets; the bank account B is given by $B_t = e^{rt}$ and the stock X follow the SDE

$$dX_u = rX_u du + \sigma X_u^\alpha dW_u, \qquad X_0 > 0 \quad (9)$$

with constants $\sigma > 0$ and $\alpha > 0$. We want to obtain the Martingale and PDE formula for the function u.

To derive the Martingale option price valuation formula for the above SDE, we let $\phi = (\eta_t, \xi_t)_{t\,\epsilon[0,T]}$ be portfolio strategy with price

$$V_t(\phi) = \eta_t B_t + \xi_t X_t$$

and that it satisfies the self-financing condition

$$dV_t(\phi) = \eta_t dB_t + \xi_t dX_t = re^{rt}\eta_t dt + \xi_t dX_t$$

or equivalently

$$V_t(\phi) = V_0(\phi) + \int_0^t \eta_t dB_s + \int_0^t \xi_t dX_s$$

The next is to apply the change in Numeraire. Since the price processes are strictly positive, in particular $B_t > 0$, one can always normalize the market by considering

$$\tilde{B}_t = B_t^{-1}B_t = 1$$

and

$$\tilde{X}_t = B_t^{-1}X_t = e^{-rt}X_t$$

Hence, we consider the discounted portfolio

$$\tilde{V}_t(\emptyset) = B_t^{-1}V_t(\phi) = e^{-rt}(\eta_t B_t + \xi_t X_t) = \eta_t + \xi_t\tilde{X}_t$$

and applying integration by parts, we have

$$d\tilde{V}_t(\phi) = B_t^{-1}dV_t(\phi) - re^{-rt}V_t(\phi)dt + \underbrace{(dB_t^{-1})(dV_t(\phi))}_{0} = B_t^{-1}dV_t(\phi) - re^{-rt}V_t(\phi)dt \quad (10)$$

If we assume that

$$dV_t(\phi) = \eta_t dB_t + \xi_t dX_t,$$

i.e., ϕ is self –financing, then

$$\begin{aligned} d\tilde{V}_t(\phi) &= e^{-rt}\{r\eta_t e^{rt}dt + \xi_t dX_t\} - r(\eta_t + \xi_t\tilde{X}_t)dt \\ &= re^{-rt}e^{rt}\eta_t dt + e^{-rt}\xi_t dX_t - r\eta_t dt - r\xi_t e^{-rt}X_t dt \\ &= r\eta_t dt + e^{-rt}\xi_t dX_t - r\eta_t dt - r\xi_t e^{-rt}X_t dt \end{aligned}$$

$$\begin{aligned} d\tilde{V}_t(\phi) &= \xi_t e^{-rt}dX_t - r\xi_t e^{-rt}X_t dt = \xi_t\{e^{-rt}dX_t + d(e^{-rt})X_t\} \\ &= \xi_t d\tilde{X}_t \end{aligned}$$

which yields that $\tilde{V}_t(\phi)$ is self-financing. Note that, in the discounted market, a self-financing portfolio is written in integral form as

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \xi_s\, d\tilde{X}_s, \qquad t\,\epsilon\,\mathbb{R}_+ \quad (11)$$

Then, we need to show that (11) is a martingale under $\mathbb{B}$. So, by Girsanov's Theorem, we can define a probability measure $\mathbb{B}$ and the process

$$\widetilde{W}_t = \frac{\mu - r}{\sigma}t + W_t,$$

is a Brownian motion under $\mathbb{B}$. Now given

$$dX_t = rX_t dt + \sigma X_t^\alpha d\widetilde{W}_t$$

if we now compute $d\tilde{X}_t$, we get

$$\begin{aligned} d\tilde{X}_t &= d(e^{-rt}X_t) = -re^{-rt}X_t dt + e^{-rt}dX_t = -r\tilde{X}_t dt + \\ & e^{-rt}[rX_t dt + \sigma X_t^\alpha d\widetilde{W}_t] = -r\tilde{X}_t dt + re^{-rt}X_t dt + \sigma X_t^\alpha e^{-rt}d\widetilde{W}_t = \\ & \underbrace{-r\tilde{X}_t dt + r\tilde{X}_t dt}_{=0} + \sigma\tilde{X}_t^\alpha e^{-rt}d\widetilde{W}_t \end{aligned}$$

$$d\tilde{X}_t = \sigma\tilde{X}_t^\alpha d\widetilde{W}_t$$

or in explicit form, let

$$\ln\left(\frac{\tilde{X}_t}{\tilde{X}_0}\right)\sigma\tilde{X}_t^\alpha d\widetilde{W}_t$$

Next, we now seek the solution of the above by applying the Ito's formula. Setting,

$$Y(t) = \ln\tilde{X}_t^\alpha, \quad f_t = \frac{\partial(\ln\tilde{X}_t^\alpha)}{\partial t} = 0\,, \quad f_x = \frac{\partial(\ln\tilde{X}_t^\alpha)}{\partial\tilde{X}_t^\alpha} = \frac{1}{\tilde{X}_t^\alpha}, \quad f_{xx} = -\frac{1}{\tilde{X}_t^{2\alpha}}$$

Noting that, $u(t) = \mu X_t$, $v(t) = \sigma\tilde{X}_t$ and so we have

$$d(\ln\tilde{X}_t^\alpha) = \left[0 + 0\times\frac{1}{\tilde{X}_t^\alpha} + \frac{1}{2}(\sigma\tilde{X}_t^\alpha)^2\left(-\frac{1}{\tilde{X}_t^{2\alpha}}\right)\right]dt + \frac{1}{\tilde{X}_t^\alpha}\sigma\tilde{X}_t^\alpha d\widetilde{W}_t \Rightarrow$$

$$d(\ln\tilde{X}_t^\alpha) = \frac{1}{2}\left(\frac{-\sigma^2\tilde{X}_t^{2\alpha}}{\tilde{X}_t^{2\alpha}}\right)dt + \frac{\sigma\tilde{X}_t^\alpha d\widetilde{W}_t}{\tilde{X}_t^\alpha}$$

$$d(\ln\tilde{X}_t^\alpha) = -\frac{1}{2}\sigma^2 dt + \sigma d\widetilde{W}_t.$$

Integrating both side of the above equation, we have

$$\begin{aligned} \ln\tilde{X}_t^\alpha - \ln\tilde{X}_0^\alpha &= \int_0^t -\frac{1}{2}\sigma^2 dt + \int_0^t \sigma d\widetilde{W}_t \\ \ln\tilde{X}_t^\alpha &= \ln\tilde{X}_0^\alpha - \frac{1}{2}\sigma^2 t + \sigma\widetilde{W}_t \end{aligned}$$

since $W_0 = 0$. Taking exponential of both sides, we have

$$e^{\ln\tilde{X}_t^\alpha} = e^{\left[\ln\tilde{X}_0^\alpha - \frac{\sigma^2}{t}t + \sigma\widetilde{W}_t\right]}$$

$$\tilde{X}_t^\alpha = e^{\ln\tilde{X}_0^\alpha}.e^{\left[-\frac{\sigma^2}{t}t + \sigma\widetilde{W}_t\right]}$$

$$\therefore \ \tilde{X}_t^{\alpha} = \tilde{X}_0^{\alpha} exp\left[\sigma\widetilde{W}_t - \frac{\sigma^2}{2}t\right] \quad (12)$$

We now go ahead to show that the above solution in (12) is a Martingale under $\mathbb{B}$. We want to find a measure $\mathbb{B}$ such that under $\mathbb{B}$, the discounted stock price that uses B_t as the numeraire is a martingale.

We write

$$dX_t = r_t X_t dt + \sigma X_t^{\alpha} dW_t^{\mathbb{B}}$$

where,$W_t^{\mathbb{B}} = W_t + \frac{\mu - r_t}{\sigma} t$ (applying Girsanov Theorem)

Using B_t as the numeraire, the discounted stock price $\tilde{X}_t = \frac{X_t}{B_t}$ and $\tilde{X}_t$ will be a martingale. Applying Ito's Lemma to $\tilde{X}_t$ which follows the SDE, we have

$$d\tilde{X}_t = \frac{\partial \tilde{X}}{\partial B} dB_t + \frac{\partial \tilde{X}}{\partial X} dX_t \quad (13)$$

All terms involving the second order derivatives are zero. Expanding (13), we have

$$d\tilde{X}_t = -\frac{X_t}{B_t^2} dB_t + \frac{1}{B_t} dX_t = -\frac{X_t}{B_t^2}(r_t B_t dt) + \frac{1}{B_t}(r_t X_t dt + \sigma X_t^{\alpha} dW_t^{\mathbb{B}}) = \underbrace{-\frac{X_t r_t dt}{B_t} + \frac{1}{B_t} r_t X_t dt}_{=0} + \frac{\sigma X_t^{\alpha} dW_t^{\mathbb{B}}}{B_t} \Rightarrow \ d\tilde{X}_t = \sigma \tilde{X}_t^{\alpha} dW_t^{\mathbb{B}}$$

The solution to the SDE is,

$$\tilde{X}_t^{\alpha} = \tilde{X}_0^{\alpha} exp\left[-\frac{\sigma^2}{2}t + \sigma W_t^{\mathbb{B}}\right].$$

To show that $\tilde{X}_t^{\alpha}$ is a martingale under $\mathbb{B}$, we consider the expectation under $\mathbb{B}$ for $s < t$, hence we have,

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t\right). E^{\mathbb{B}}\left[\exp(\sigma W_t^{\mathbb{B}})|\mathcal{F}_s\right] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{B}}\right). E^{\mathbb{B}}\left[\exp(\sigma(W_t^{\mathbb{B}} - W_s^{\mathbb{B}}))|\mathcal{F}_s\right]$$

at time s we have that $W_t^{\mathbb{B}} - W_s^{\mathbb{B}}$ is distributed as $N(0,t)$ which is identical in distribution to $W_{t-s}^{\mathbb{B}}$ at time zero. Hence, we can write

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{B}}\right). E^{\mathbb{B}}\left[\exp(\sigma(W_{t-s}^{\mathbb{B}})|\mathcal{F}_0\right]$$

Now, the moment generating function (mgf) of a random variable X with normal distribution $N(\mu, \sigma^2)$ is given as

$$E\left[e^{\phi x}\right] = \exp\left(\mu\phi + \frac{1}{2}\phi^2\sigma^2\right)$$

Under $\mathbb{B}$, we have that $W_{t-s}^{\mathbb{B}}$ is $\mathbb{B}$- Brownian motion and distributed as $N(0, t-s)$. Therefore, the mgf of $W_{t-s}^{\mathbb{B}}$ is

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{B}}\right). \exp\left(\frac{1}{2}\sigma^2(t-s)\right)$$

where $\sigma = \phi$ and we can then write

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{B}}\right) exp\left(\frac{1}{2}\sigma^2(t-s)\right)$$

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_0^{\alpha} \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{B}}\right)$$

$$\therefore \ E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_s^{\alpha}$$

We thus have that

$$E^{\mathbb{B}}[\tilde{X}_t^{\alpha}|\mathcal{F}_s] = \tilde{X}_s^{\alpha}$$

which shows that $\tilde{X}_t^{\alpha}$ is a $\mathbb{B}$ martingale. Hence, we have that

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \xi_s d\tilde{X}_s = \tilde{V}_0(0) + \int_0^t \xi_s \, \sigma \tilde{X}_t^{\alpha} d\widetilde{W}_t$$

and $\tilde{V}_t(\phi)$ is a stochastic integral with respect to a Brownian motion under $\mathbb{B}$. Hence, under the integrability condition as stated in Theorem 2, we have

$$E^{\mathbb{B}}\left[\int_0^T \left|\xi_t \sigma \tilde{X}_t^{\alpha}\right|^2 dt\right] < \infty$$

Hence, we have shown that $\tilde{V}_t(\phi)$ is a martingale under $\mathbb{B}$. Now since,

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \xi_s d\tilde{X}_s \quad t \in \mathbb{R}_+$$

is a martingale under $\mathbb{B}$, it follows from the martingale properties of $\tilde{V}_t(\phi)$ under $\mathbb{B}$ that,

$$\tilde{V}_t(\phi) = E^{\mathbb{B}}[\tilde{V}_T|\mathcal{F}_t] = e^{-rT}E^{\mathbb{B}}[V_T|\mathcal{F}_t] = e^{-rT}E^{\mathbb{B}}[C|\mathcal{F}_t] \quad (14)$$

where C is a contingent claim, $u(T, X_T)$. Note that $\phi = (\eta_t, \xi_t)_{t \in [0,T]}$ hedges the claim C, i.e. we have $V_T = C \implies V_T = u(T, X_T)$. Hence (14) becomes

$$V_t = e^{-rt}\tilde{V}_t \quad = e^{-r(T-t)}E^{\mathbb{B}}[u(T, X_T)|\mathcal{F}_t]$$

Since the process $(X_t)_{t \in \mathbb{R}_+}$ has the Markov property, the value

$$V_t = e^{-r(T-t)}E^{\mathbb{B}}[\phi(X_T)|\mathcal{F}_t] = u(t, X_t) \quad (15)$$

here $\phi(X_T) = u(T, X_T)$ of the portfolio at $t \in [0,T]$ can be written from (15) as a function $u(t, X_t)$ of t and X_t. Given the payoff function,

$$u(T, X_T) = (K - X_T)^+$$

Hence, (15) becomes

$$u(t, X_t) = e^{-r(T-t)}E^{\mathbb{B}}[(K - X_T)^+|\mathcal{F}_t]$$

$$u(t, X_t) = E^{\mathbb{B}}\left[e^{-r(T-t)}(K - X_T)^+\right]. \quad (16)$$

2. Black-Karasinski Term Structure Model

Let us consider another example which is the pricing of a zero coupon bond in the term structure model of [7]. They describe the short rate r by the SDE

$$d(logr_u) = \varphi(u)(log\mu(u) - logr_u)du + \sigma(u)dW_u, r_0 > 0 \quad (17)$$

with deterministic functions φ, μ, σ. The price at time t of a zero coupon bond with maturity T is then

$$E\left[exp\left(-\int_t^T r_s\, ds\right)|\mathcal{F}_t\right] = u(t, r_t)$$

by the Markov property of r and we want to obtain a Martingale and PDE option price valuation formula for the function u.

To derive the Martingale option price valuation formula for the above SDE, we let $\phi = (\eta_t, \xi_t)_{t\,\epsilon[0,T]}$ be portfolio strategy with price

$$V_t(\phi) = \eta_t B_t + \xi_t X_t$$

and that it satisfies the self-financing condition

$$dV_t(\phi) = \eta_t dB_t + \xi_t dX_t = re^{rt}\eta_t dt + \xi_t dX_t$$

or equivalently

$$V_t(\phi) = V_0(\phi) + \int_0^t \eta_t dB_s + \int_0^t \xi_t dX_s$$

Now we are given that,

$$d(logr_u) = \varphi(u)(log\mu(u) - logr_u)du + \sigma(u)dW_u$$

Let $logr_u = X_t$ and $log\mu(u) - logr_u = M_t$ we have

$$dX_t = \varphi(t)M_t dt + \sigma(t)d\widetilde{W}_t$$

If we now compute $d\tilde{X}_t$, we get

$$d\tilde{X}_t = d(e^{-rt}X_t) = -re^{-rt}X_t dt + e^{-rt}dX_t = -re^{-rt}X_t dt + e^{-rt}\left[\varphi(t)M_t dt + \sigma(t)d\widetilde{W}_t\right] = -r\tilde{X}_t dt + e^{-rt}\varphi(t)M_t dt + e^{-rt}\sigma(t)d\widetilde{W}_t = -r\tilde{X}_t dt + \varphi(t)\widetilde{M}_t dt + \tilde{\sigma}(t)d\widetilde{W}_t$$

$$d\tilde{X}_t = \left[\varphi(t)(\widetilde{M}_t - r\tilde{X}_t)\right]dt + \tilde{\sigma}(t)d\widetilde{W}_t$$
$$\text{let } \varphi(t)(\widetilde{M}_t - r\tilde{X}_t) = \tilde{Z}_t$$

Hence, we have

$$d\tilde{X}_t = \tilde{Z}_t dt + \tilde{\sigma}(t)d\widetilde{W}_t \quad (18)$$

or in explicit form, let

$$\ln\left(\frac{\tilde{X}_t}{\tilde{X}_0}\right)dt + \tilde{\sigma}d\widetilde{W}_t$$

Next, we now seek the solution by applying the Ito's formula. Setting

$$Y(t) = \ln\tilde{X}_t,\ f_t = \frac{\partial(\ln\tilde{X}_t)}{\partial t} = 0,\ f_x = \frac{\partial(\ln\tilde{X}_t)}{\partial\tilde{X}_t} = \frac{1}{\tilde{X}_t},\ f_{xx} = -\frac{1}{\tilde{X}_t^2}$$

Noting that $u(t) = \mu X_t = \tilde{Z}_t\tilde{X}_t$ and $v(t) = \tilde{\sigma}\tilde{X}_t$. So by the Ito's formula, we have

$$d(\ln X\bar{\ }\_t\,) = \left[0 + \frac{\tilde{Z}_t\tilde{X}_t}{\tilde{X}_t} + \frac{1}{2}\tilde{\sigma}^2\tilde{X}_t^2\left(-\frac{1}{\tilde{X}_t^2}\right)dt\right] + \tilde{\sigma}d\widetilde{W}_t = \left[\tilde{Z}_t\tilde{X}_t.\tilde{X}_t^{-1} + \frac{1}{2}\tilde{\sigma}^2\tilde{X}_t^2.\left(-\tilde{X}_t^{-2}\right)\right]dt + \tilde{\sigma}d\widetilde{W}_t$$

$$d\left(\ln\tilde{X}_t\right) = \left(\tilde{Z}_t - \frac{\tilde{\sigma}^2}{2}\right)dt + \tilde{\sigma}d\widetilde{W}_t$$

integrating both sides from $t\varepsilon[0, t]$ we have

$$\ln\tilde{X}_t - ln\tilde{X}_0 = \int_0^t\left(\tilde{Z}_t - \frac{\tilde{\sigma}^2}{2}\right)dt + \int_0^t\tilde{\sigma}d\widetilde{W}_t$$

$$\ln\tilde{X}_t = \ln\tilde{X}_0 + \left(\tilde{Z}_t - \frac{\tilde{\sigma}^2}{2}\right)t + \tilde{\sigma}\widetilde{W}_t$$

Since $W_0 = 0$, taking exponential of both sides, we have that

$$e^{\ln\tilde{X}_t} = e^{\left[\ln\tilde{X}_0 + \left(\tilde{Z}_t - \frac{\tilde{\sigma}^2}{2}\right)t + \tilde{\sigma}\widetilde{W}_t\right]}$$

$$\tilde{X}_t = e^{\ln\tilde{X}_0}.e^{\left[\left(\tilde{Z}_t - \frac{\tilde{\sigma}^2}{2}\right)t + \tilde{\sigma}\widetilde{W}_t\right]}$$

$$\therefore \tilde{X}_t = \tilde{X}_0 exp\left(\tilde{Z}_t t + \tilde{\sigma}\widetilde{W}_t - \frac{\tilde{\sigma}^2}{2}t\right) \quad (19)$$

The next step is to check if the above solution in (19) is a martingale. We want to find a measure $\mathbb{B}$ such that under $\mathbb{B}$, the discounted stock price that uses B_t as a numeraire is a martingale. We write

$$dX_t = \varphi M_t dt + \sigma dW_t^{\mathbb{B}}$$

where, $W_t^{\mathbb{B}} = W_t + \frac{\mu - r_t}{\sigma}t$ (applying Girsanov Theorem) using B_t as the numeraire, the discounted stock price $\tilde{X}_t = \frac{X_t}{B_t}$ and $\tilde{X}_t$ will be a martingale. Applying Ito's Lemma to $\tilde{X}_t$ which follows the SDE, we have

$$d\tilde{X}_t = \frac{\partial\tilde{X}}{\partial B}dB_t + \frac{\partial\tilde{X}}{\partial X}dX_t \quad (20)$$

all terms involving the second order derivatives are zero. Expanding (20) we have

$$d\tilde{X}_t = -\frac{X_t}{B_t^2}dB_t + \frac{1}{B_t}dX_t = -\frac{X_t}{B_t^2}(r_t B_t dt) + \frac{1}{B_t}(\varphi M_t dt + \sigma dW_t^{\mathbb{B}}) = -\frac{X_t r_t dt}{B_t} + \frac{1}{B_t}\varphi M_t dt + \frac{\sigma dW_t^{\mathbb{B}}}{B_t}$$

$$d\tilde{X}_t = -\tilde{X}_t r_t dt + \varphi\widetilde{M}_t dt + \tilde{\sigma}dW_t^{\mathbb{B}} \quad (21)$$

$$d\tilde{X}_t = \tilde{Z}_t dt + \tilde{\sigma}dW_t^{\mathbb{B}}$$

where $\tilde{Z}_t = \varphi\widetilde{M}_t - \tilde{X}_t r_t$.

The solution to the SDE in (21) is given as

$$\tilde{X}_t = \tilde{X}_0 exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}\widetilde{W}_t\right).$$

To show that $\tilde{X}_t$ is a martingale under $\mathbb{B}$, we consider the expectation under $\mathbb{B}$ for $s < t$, hence we have,

$$E^{\mathbb{B}}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_0 \exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t\right). E^{\mathbb{B}}\left[\exp(\tilde{\sigma}W_t^{\mathbb{B}})|\mathcal{F}_s\right] =$$
$$\tilde{X}_0 \exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}W_s^{\mathbb{B}}\right). E^{\mathbb{B}}\left[\exp(\tilde{\sigma}\left(W_t^{\mathbb{B}} - W_s^{\mathbb{B}}\right))|\mathcal{F}_s\right]$$

at time s we have that $W_t^{\mathbb{B}} - W_s^{\mathbb{B}}$ is distributed as $N(0,t)$ which is identical in distribution to $W_{t-s}^{\mathbb{B}}$ at time zero. Hence, we can write

$$E^{\mathbb{B}}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_0 \exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}W_s^{\mathbb{B}}\right). E^{\mathbb{B}}\left[\exp(\tilde{\sigma}\left(W_{t-s}^{\mathbb{B}}\right)|\mathcal{F}_0\right]$$

Now, the moment generating function (mgf) of a random variable X with normal distribution $N(\mu, \sigma^2)$ is given as

$$E\left[e^{\phi x}\right] = \exp\left(\mu\phi + \frac{1}{2}\phi^2\sigma^2\right).$$

Under $\mathbb{B}$, we have that $W_{t-s}^{\mathbb{B}}$ is $\mathbb{B}$- Brownian motion and distributed as $N(0, t-s)$. Therefore, the mgf of $W_{t-s}^{\mathbb{B}}$ is

$$E^{\mathbb{B}}[\exp(\tilde{\sigma}W_{t-s}^{\mathbb{B}})] = \exp\left(\frac{1}{2}\tilde{\sigma}^2(t-s)\right)$$

where $\tilde{\sigma} = \phi$ and we can then write

$$E^{\mathbb{B}}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_0 \exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}W_s^{\mathbb{B}}\right). \exp\left(\frac{1}{2}\tilde{\sigma}^2(t-s)\right)$$
$$E^{\mathbb{B}}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_0 \exp\left(\tilde{Z}_t t - \frac{\tilde{\sigma}^2}{2}t + \tilde{\sigma}W_s^{\mathbb{B}}\right) = \tilde{X}_s.$$

We thus have that

$$E^{\mathbb{B}}[\tilde{X}_t|\mathcal{F}_s] = \tilde{X}_s$$

which shows that $\tilde{X}_t$ is a $\mathbb{B}$ martingale. Hence, we have that

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \tilde{Z}_s\, ds + \int_0^t \tilde{\sigma}(s)\, d\tilde{W}_t$$

and $\tilde{V}_t(\phi)$ is a stochastic integral with respect to a Brownian motion under $\mathbb{B}$. Hence, under the integrability condition as stated in Theorem 2, we have

$$E^{\mathbb{B}}\left[\int_0^T \left|\tilde{Z}_t + \tilde{\sigma}(t)\right|^2 dt\right] < \infty$$

Hence, we have shown that $\tilde{V}_t(\phi)$ is a Martingale under $\mathbb{B}$. Now since,

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \tilde{Z}_s\, ds + \int_0^t \tilde{\sigma}(s)\, d\widetilde{W}_t, \qquad t\varepsilon\mathbb{R}$$

is a Martingale under $\mathbb{B}$ it follows from the Martingale properties of $\tilde{V}_t(\phi)$ under $\mathbb{B}$ that,

$$\tilde{V}_t(\phi) = E^{\mathbb{B}}\left[\tilde{V}_T\middle|\mathcal{F}_t\right] = e^{-rT}E^{\mathbb{B}}[V_T|\mathcal{F}_t] = e^{-rT}E^{\mathbb{B}}[C|\mathcal{F}_t] \quad (22)$$

where C is a contingent claim, $u(T, X_T)$.

Note that $(\phi) = (\eta_t, \xi_t)_{t\varepsilon[0,\pi]}$ hedges the claim C, i.e. we have

$$V_T = C \Rightarrow V_T = u(T, X_T)$$

Hence, we have (22) that

$$V_t = e^{-r(T-t)}E^{\mathbb{B}}[u(T, X_T)|\mathcal{F}_t]$$

Since the process $(X_t)_{t\varepsilon\mathbb{R}^+}$ has Markov property, the value

$$V_t = e^{-r(T-t)}E^{\mathbb{B}}[\phi(X_T)|\mathcal{F}_t] = u(t, X_t) \quad (23)$$

of the portfolio at $t\varepsilon[0,T]$ can be written from (8) as a function $u(t, X_t)$ of t and X_t. Given the payoff function $u(T, X_T) = (K - X_T) = 1$, for zero coupon bonds. Hence (23) becomes

$$u(t, X_t) = e^{-r(T-t)}E^{\mathbb{B}}[(K - X_T)|\mathcal{F}_t] \Rightarrow u(t, X_t)$$
$$= e^{-r(T-t)}E^{\mathbb{B}}[(1)|\mathcal{F}_t]$$

$$u(t, X_t) = E\left[e^{-\int_t^T X_s ds}\right]$$

But $X_s = logr_s$, hence,

$$u(t, X_t) = E\left[e^{-\int_t^T r_s ds}\middle| \mathcal{F}_t\right].$$

3. Heston Stochastic Volatility Model

We consider the Heston stochastic volatility model with two assets B and S. The bank account B is given by $B_t = e^{rt}$ where r is the instantaneous riskless interest rate. The stock S satisfies the SDE

$$dS_t = \mu S_t dt + \sqrt{v_t}S_t dW_t^{(1)}, \quad S_0 > 0$$

where the volatility v is itself stochastic and given as in [8] by

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^{(2)}, \quad v_0 > 0$$

for non-negative constants κ, θ, σ. The process S and v are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{B})$ and $W_t^{(1)}$ *and* $W_t^{(2)}$ are B –Brownian motions with instantaneous correlation $dW_t^{(1)}dW_t^{(2)} = \rho dt$.

We want to obtain the Martingale and PDE option price valuation formula for the function u.

To derive the Martingale options price valuation formula for the above SDE, we let $\phi = (\eta_t, \xi_t)_{t\,\epsilon[0,T]}$ be portfolio strategy with price

$$V_t(\phi) = \eta_t B_t + \xi_t X_t$$

and that it satisfies the self-financing condition

$$dV_t(\phi) = \eta_t dB_t + \xi_t dX_t = re^{rt}\eta_t dt + \xi_t dX_t$$

or equivalently,

$$V_t(\phi) = V_0(\phi) + \int_0^t \eta_t dB_s + \int_0^t \xi_t dX_s$$

We are also given that the riskless investment (bank account) is given by

$$B_t = e^{rt}$$

Now under the Black-Scholes model, the Martingale options price of a financial derivative is given by

$$V_t = N_t E^{\mathbb{N}}\left[\frac{V_T}{N_T}\middle| \mathcal{F}_t\right] \quad (24)$$

With numeraire N_t as the bond with constant interest rates,

$$B_t = e^{rt}$$

Hence, the above becomes with $N_t = B_t$ and

$$V_t = u_t = u(S_t, t) = e^{-r(T-t)} E^{\mathbb{B}}\left[u(S_{T,}T)\middle|\mathcal{F}_t\right] \quad (25)$$

where $\mathbb{B}$ is the measure under which the discounted stock price

$\frac{S_t}{B_t} = e^{-rt} S_t$ is a Martingale.

But, we know that the value of the option is the expected value of the payoff at expiry, discounted by the numeraire. The European Put option has payoff,

$$u_T = u(T, S_T) = (K - S_T)^+,$$

So, in accordance with (25), the time $-\,t$ price of the Put option for the Heston's model is given as

$$u_t = u(t, S_t, v_t) = B_t E^{\mathbb{B}}\left[\frac{(K-S_T)^+}{B_T}\middle| \mathcal{F}_t\right]$$

$$u_t = e^{-r(T-t)} E^{\mathbb{B}}[(K - S_T)^+ | \mathcal{F}_t]$$

Since the process $(X_t)_{t\epsilon\mathbb{R}^+}$ has Markov property, the value of the Martingale option price valuation formula is given by

$$u_t = u(t, S_t, v_t) = e^{-r(T-t)} E^{\mathbb{B}}[(K - S_T)^+ | \mathcal{F}_t] \quad (26)$$

IV. Numerical Solutions

A. Derivation of Monte Carlo Scheme for Martingales Option Price Valuation Formulas

Let us consider the SDE given by the CEV model,

$$dX_t = rX_t dt + \sigma X_t^\alpha dW_t, x > 0 \quad (27)$$

where r is the interest rate which is assumed to be constant, σ is the volatility rate, α is the elasticity, and W_t is the standard Brownian motion process. The value parameter of European Put option at $t\epsilon\,[0, T]$ is then given by

$$u_t = \max\ E\left[e^{-r(T-t)}(K - X_T)^+\right] \quad (28)$$

The payoff function of the above equation is given as

$$P(X_T, t) = (K - X_T)^+$$

We denote $X_i = X(t = t_i)$; $i\epsilon(0,1,2, \dots, M)$ as the state of asset price process at the ith exercise opportunity. The Euler-Maruyama scheme discussed by Wu, in [8] can be used to approximate the asset price process in (27) over the time interval $[0, T]$ given by

$$X_{n+1}^{(k)} = X_n{}^{(k)} + rX_n{}^{(k)}\Delta t + \sigma X_n^{\alpha(k)} \Delta W_t \quad (29)$$

$$X_0{}^{(k)} = x, \quad n = 0,1,2, \dots, N-1$$

where $\Delta t = \frac{T}{N}$ and $\Delta W_t = W_t - W_{t-1}$ is the independent Brownian increment which follows a normal distribution $N(0, \sqrt{\Delta t})$. The discretized process X_n given in this way is essentially a Markov chain.

We know that for a European Put option, the payoff function is given as

$$Y = P(X) = \max[(K - X), 0] \quad (30)$$

where $X = X(T)$ is the price of the underlying stock at the time T when the option expires and (30) produces one possible option value at expiration of the option. The overall aim is to determine the correct and fair price of the option at the time the holder and the writer enter into the contract [9].

To estimate price of the Put option of the CEV model in (27) using the Monte Carlo method, we consider a collection of M stock prices at expiration generated by using (27). That is

$$\{X_N{}^{(k)} = X^{(k)}(T), \quad k = 1, \dots, M\}$$

Option pricing theory requires that the average value of the payoffs,

$$\left\{P\left(X_n^{(k)}\right), \qquad k = 1, \dots, M\right\}$$

be equal to the compounded total return obtained by investing the option premium $\hat{P}(x)$, at rate r over the life of the option. Hence we have,

$$P\left(x_n^{(k)}\right) = \max\left(K - x_n^{(k)}, 0\right), k = 1, \dots, M \Rightarrow \frac{1}{M}\sum_{k=1}^{M} P\left(x_n^{(k)}\right) = (1 + r\Delta t)^N \hat{P}(x) \quad (31)$$

Solving (31) for $\hat{P}(x)$ yields the Monte Carlo estimate for the option price given as

$$\therefore \hat{P}(x) = (1 + r\Delta t)^{-N}\left[\frac{1}{M}\sum_{k=1}^{M} P\left(x_n^{(k)}\right)\right] \quad (32)$$

So the Monte Carlo estimate $\hat{P}(x)$ is the present value of the average of the payoffs computed using rules of compound interest. Equation (32) is the general Monte Carlo estimate formula for computing the estimate $\hat{P}(x)$ [9]. Similarly, for the Black- Karasinski model given below as

$$d(logr_u) = \varphi(u)(log\mu(u) - logr_u)du + \sigma(u)dW_u \quad (33)$$

$$d(logr_t) = \varphi(t)log\mu(t)dt - \varphi(t)logr_u dt + \sigma(u)dW_t$$

Let $logr_t = X_t$ and $\varphi(t)log\mu(t) = M_t$. Therefore, we have

$$X_t = M_t dt - \varphi(t)X_t dt + \sigma(t)dW_t \quad (34)$$

The discretized form of (34) using the Euler-Maruyama scheme over time interval $[0, T]$is given by

$$X_{n+1}^{(k)} = X_n{}^{(k)} + M_n^{(k)}\Delta t - \varphi X_n{}^{(k)}\Delta t + \sigma\Delta W_t \quad (35)$$

$$X_0{}^{(k)} = x, \quad n = 0,1,2, \dots, N-1$$

We can then repeat the steps for the computation of Monte Carlo estimator $\hat{P}(x)$ as enumerated above for that of the CEV model.

Finally, we consider the Heston's model given as

$$dS_t = rS_t dt + \sqrt{v_t}S_t dW_t^i \quad (36)$$

$$dv_t = (\kappa(\theta - v_t) - \lambda v_t)dt + \sigma\sqrt{v_t}dW_t^2 \quad (37)$$

$$dW_t^1 dW_t^2 = \rho dt$$

where θ the long-term running is mean of the variance process, κ is the speed of mean-reversion of the variance process and ρ is the instantaneous correlation between the state process and the volatility process.

To perform a standard Monte Carlo simulation in the above model, we will split the time to maturity T into N steps with step size δt (i.e. $T = N\delta t$). Then, we have a time-stepping scheme using the Euler-Maruyama time-stepping with the initial value S_0, v_o of

$$S_{n+1} = S_n\left[1 + r\delta t + \sqrt{v_n}\left(\rho N_{0,1}^1 + \sqrt{1-\rho^2}N_{0,1}^2\right)\sqrt{\delta t}\right] \quad (38)$$

$$v_{n+1} = v_n + (\kappa(\theta - v_n) - \lambda v_n)\delta t + \sqrt{v_n}\sigma N_{0,1}^2\sqrt{\delta t} \quad (39)$$

where $N_{0,1}^{i's}$ are realization of two independent $N(0,1)$ variables. Then M realizations of the stock priceA paths $\{S_n^m\}_{n=0,m=1}^{N,M}$ and the variance paths $\{v_n^m\}_{n=0,m=1}^{N,M}$ are simulated following the necessary steps required for the computation of Monte Carlo estimator $\hat{P}(x)$ as stipulated above.

B. Numerical Examples

Empirical data obtained from the NSE will be used to plot some graphs to investigate the effect of increase in the underlying asset (i.e. positive correlation) on the option value (price) for the three financial models examined in this paper. The parameter values are shown in Tables I-III Computer programs coded in MATLAB were used for solving the systems of the derived Monte Carlo scheme in (29), (35) and (38) … (39). The graphs for the various parameter values for the CEV model in Table I, the Black-Karasinski model in Table II, and the Heston model in Table IIII are presented in Figs. 1-6, respectively.

TABLE I
PARAMETER VALUES FOR THE CEV MODEL [10]

Parameters \ Cases	1	2	3
K	200	200	200
dX	1	1	1
X	20	40	60
T	1	1	1
α	0.2	0.2	0.2
dT	0.1	0.1	0.1
r	0.1	0.1	0.1
σ	0.5	0.5	0.5

TABLE II
PARAMETER VALUES FOR THE BLACK-KARASINSKI MODEL [10]

Parameters \ Cases	1	2	3
K	200	200	200
dX	1	1	1
X	20	40	60
φ	0.2	0.2	0.2
T	1	1	1
α	0.2	0.2	0.2
dT	0.1	0.1	0.1
r	0.1	0.1	0.1
σ	0.5	0.5	0.5

TABLE III
PARAMETER VALUES FOR HESTON MODEL [10]

Parameters \ Cases	1	2	3
$\boldsymbol{K}$	200	200	200
$\boldsymbol{S}$	20	40	60
$\boldsymbol{V}$	1	1	1
$\boldsymbol{\lambda}$	1	1	1
$\boldsymbol{\kappa}$	2	2	2
$\boldsymbol{T}$	1	1	1
$\boldsymbol{\theta}$	0.2	0.2	0.2
$\boldsymbol{\eta}$	0.2	0.2	0.2
$\boldsymbol{r}$	0.1	0.1	0.1
$\boldsymbol{\sigma}$	0.5	0.5	0.5
$\boldsymbol{\rho}$	0.8	0.8	0.8

V. RESULTS AND DISCUSSION

In this section, we will discuss the result of our numerical experiments carried out by increasing the value of the underlying assets for the CEV, Black-Karasinski, and Heston models respectively at various parametric values. The parameter values for the experiments are shown in Tables I-III. The graphs, plotted using these values, are shown in Figs. 1-6. In the curves, blue represents the value of the Option at expiry, green represents half a year before expiration, and red represents one year before expiration that is when the contract is signed.

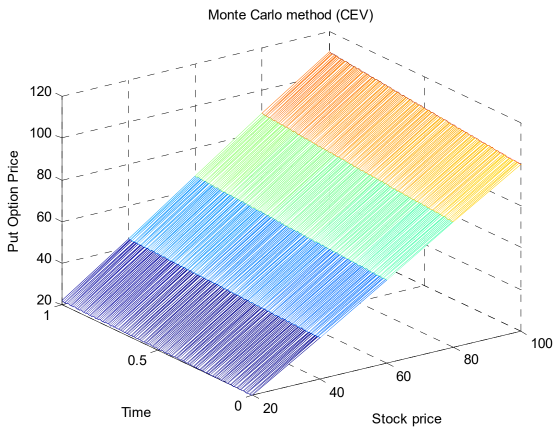

Fig. 1 Put Option Price, Expiration Time and Stock Price at the following Parameter values: $X = 20{:}100$; $K = 200$; $T = 1$; $\boldsymbol{\alpha} = 0.2$; $t = 0$; $r = 0.1$; and $\sigma = 0.5$ for CEV model

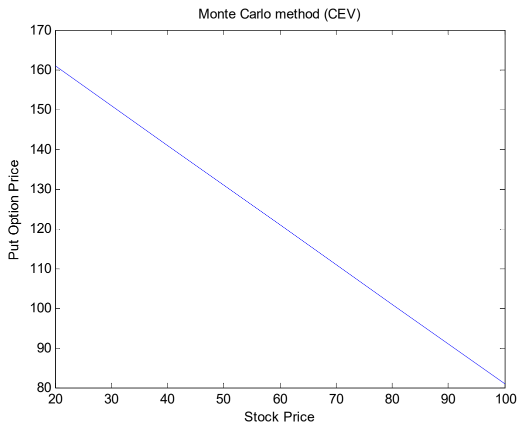

Fig. 2 Put Option Price against the Stock Price (X) at the following Parameter values : $X = 20{:}100$; $K = 200$; $T = 1$; $t = 0$; ; $r = 0.1$; $\sigma = 0.5$ and $\alpha = 0.2$ for the CEV model

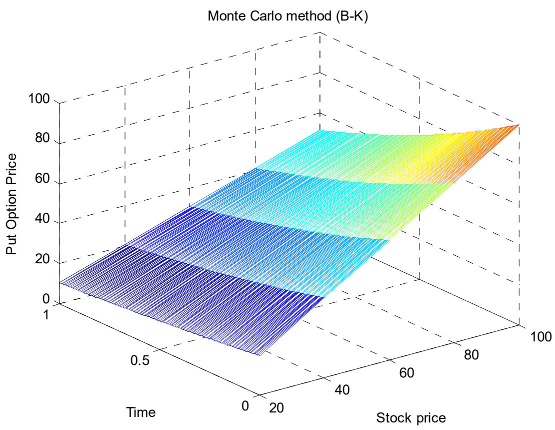

Fig. 3 Put Option Price, Expiration Time and Stock Price at the following Parameter values: $X = 20{:}100$; $K = 200$; $T = 1$; $\varphi = 0.2$; $t = 0$; $r = 0.1$; and $\sigma = 0.5$ for Black-Karasinski model

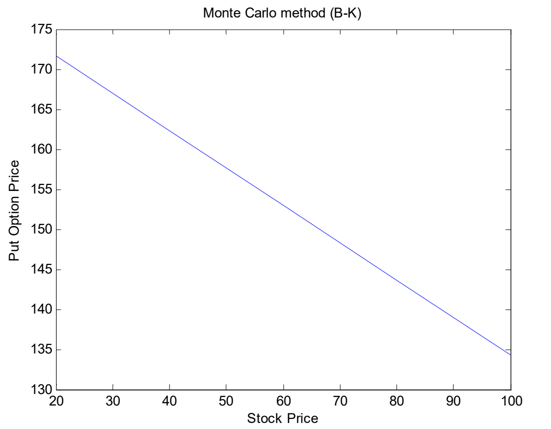

Fig. 4 Put Option Price against the Stock Price (X) at the following Parameter values: $X = 20{:}100$; $K = 200$; $T = 1$; $\varphi = 0.2$; $t = 0$; $r = 0.1$; and $\sigma = 0.5$ for Black-Karasinski model.

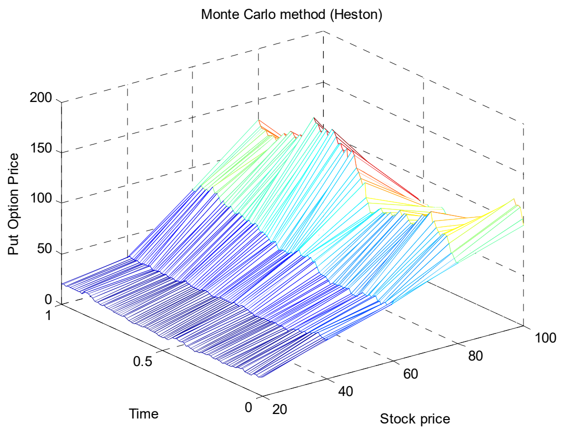

Fig. 5 Put Option Price, Expiration Time and Stock Price at the following Parameter values: $S = 20{:}100$; $K = 200$; $V = 0.2$; $\lambda = 1$; $\kappa = 2$; $T = 1$; ; $\theta = 0.2$; $r = 0.1$; $\sigma = 0.5$ *and* $\rho = 0.8$ for the Heston model

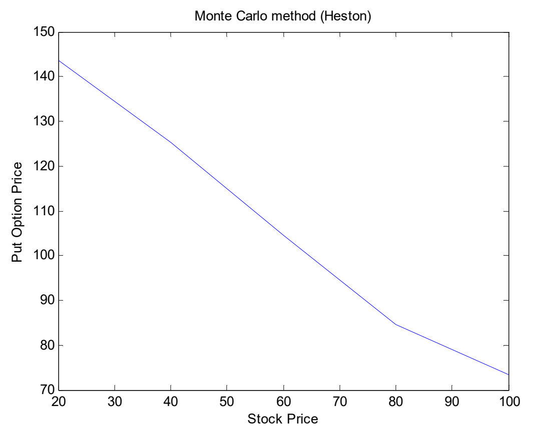

Fig. 6 Put Option Price against the Stock Price (S) at the following Parameter values: $S = 20{:}100$; $K = 200$; $V = 0.2$; $\lambda = 1$; $\kappa = 2$; $T = 1$; $\theta = 0.2$; $r = 0.1$; $\sigma = 0.5$ and $\rho = 0.8$ for the Heston model

A. Experiment One

In this case, we looked at the situation where the underlying asset (stock price) value, X or $S = 20$ for the CEV, Black-Karasinski, and Heston models, respectively. The result displayed in Figs. 1-6 showed that the value of the European Put option price using the Monte Carlo method has values of 160 for the CEV model, 172 for the Black-Karasinski model and 142 for the Heston model respectively.

B. Experiment Two

Also, we reviewed the situation where the underlying asset (stock price) value, X or $S = 40$ for the CEV, Black-Karasinski, and Heston models, respectively. The result displayed in Figs. 1-6 showed that the value of the European Put option price using the Monte Carlo method has values of 140 for the CEV model, 164 for the Black-Karasinski model and 120 for the Heston model respectively.

C. Experiment Three

Finally, we investigated the scenario where the underlying asset (stock price) value, X or $S = 60$ for the CEV, Black-Karasinski and Heston models respectively. The result displayed in Figs. 1-6 showed that the value of the European Put option price using the Monte Carlo method has values of 120 for the CEV model, 150 for the Black-Karasinski model and 110 for the Heston model respectively.

VI. Conclusion

In this paper, we have derived the Martingale European Put Options valuation formulas for three SDE models in finance which are the CEV model, the Black-Karasinski term structure model and the Heston model. The Monte Carlo method of numerical solution for the derived Martingales option price valuation formulas for the three distinct SDE models was used in the implementation of numerical experiments. From this study, we observed that the Martingales approach presents option price valuation formulas in form of conditional expectations of the payoff function discounted by the numeraires. Furthermore, numerical experiments, using published data from the NSE show that as the price of the underlying asset (stock price) increases, the value of the European Put option decreases. Hence, the right to sell at a fixed price (Puts) will become less valuable and the buyer decides not to exercise his right on the options by allowing the option to expire.

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