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Research of Amplitude-Frequency Characteristics of Nonlinear Oscillations of the Interface of Two-Layered Liquid

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***Abstract*—**The problem of nonlinear oscillations of a two-layer liquid completely filling a limited volume is considered. Using two basic asymmetric harmonics excited in two mutually perpendicular planes, ordinary differential equations of nonlinear oscillations of the interface of a two-layer liquid are investigated. In this paper, hydrodynamic coefficients of linear and nonlinear problems in integral relations were determined. As a result, the instability regions of forced oscillations of a two-layered liquid in a cylindrical tank occurring in the plane of action of the disturbing force are constructed, as well as the dynamic instability regions of the parametric resonance for different ratios of densities of the upper and lower liquids depending on the amplitudes of liquids from the excitations frequencies. Steady-state regimes of fluid motion were found in the regions of dynamic instability of the initial oscillation form. The Bubnov-Galerkin method is used to construct instability regions for approximate solution of nonlinear differential equations.

***Keywords*—**Hydrodynamic coefficients, instability region, nonlinear oscillations, resonance frequency, two-layered liquid.

I. Introduction

THE nonlinear theory of motion of a limited volume of liquid with a free surface, as well as with the interface of two liquids, which is a special branch of mechanics, is used in solving a number of practical problems. Ensuring the stable flight of modern and advanced aircraft of rocket and space technology and achieving precision control is impossible without a thorough description of the dynamic processes occurring in a complex mechanical system. A large number of papers are devoted to the study of layered liquids [4], [5], [7], [12], [13], to the theory of wave motions [3], [8], [9], [16], [17] and to the nonlinear problems, for example [1], [2], [6], [14], [15].

The purpose of this article is to study the stability of nonlinear oscillations of the interface of a two-layer liquid in a movable cylindrical tank.

II. Problem Statement

We present a coordinate system $Oxyz$ starting at a point O on the undisturbed interface (Fig. 1). Liquids of density ρ_1 and ρ_2 are assumed to be ideal and incompressible. h_1 and h_2 denote the depth of each layer of the liquid in undisturbed state. The coordinate system $Oxyz$ is arranged so that in the undisturbed position of the mechanical body-liquid system, the axis Ox is perpendicular to the undisturbed liquid interface Γ_0.

The moistened surfaces of the cavity are denoted by $S^{(i)}(i=1,2)$, and the perturbed interface of liquids is denoted by Γ (see Fig. 1). The equation of the perturbed interface can be represented as resolved with respect to the coordinate x

$$\zeta = x - f(y,z,t) = 0 . \quad (1)$$

Assuming the absence of vortex motion in each fluid, we formulate a problem of considerable oscillations of the fluid interface, consisting of the Laplace equation, non-flow conditions on wetted surfaces, as well as kinematic and dynamic conditions on the perturbed interface

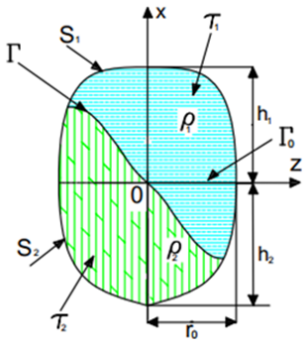

Fig. 1 Coordinate system and basic designations for a body with a two-layer liquid

$$\nabla^2\Phi^{(1)} = 0 \text{ , at } \tau_1 \text{ , } \nabla^2\Phi^{(2)} = 0 \text{ , at } \tau_2 \text{ , } \quad (2)$$

$$\frac{d\Phi^{(1)}}{d\nu} = 0 \text{ , on } S_1 \text{ , } \frac{d\Phi^{(2)}}{d\nu} = 0 \text{ ,on } S_2 \text{ , } \frac{d\Phi^{(1)}}{d\nu} = \frac{d\Phi^{(2)}}{d\nu} \text{ on } \Gamma \quad (3)$$

$$\left(\rho_2 \frac{\partial\Phi^{(2)}}{\partial t} - \rho_1 \frac{\partial\Phi^{(1)}}{\partial t}\right) + \frac{1}{2}\left[\rho_2(\nabla\Phi^{(2)})^2 - \rho_1(\nabla\Phi^{(1)})^2\right] = (\rho_1 - \rho_2)\vec{g}\times\vec{r}. \quad \text{on } \Gamma \quad (4)$$

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$\vec{r}\big|_{\Gamma} = \vec{r}\big|_{\Gamma_0} + \vec{i}_x f$, ν -external normal to the corresponding boundary of the area occupied by the liquid.

Imagine the velocity potentials of each fluid as the following sum:

$$\Phi^{(k)}(x,y,z,t) = \sum_{i=1}^{\infty} \dot{\alpha}_i(t) B_i^{(k)}(x,y,z)\,,\ (k=1,2) \qquad (5)$$

$\Phi^{(k)}$ - velocity potentials of upper and lower liquids, $B_i^{(k)}$ - function of the coordinates of the upper and lower fluids, α_i - generalized coordinates of wave motions of liquids on the surfaces of the *i*-th harmonic section. Here, *i*-th is made on a natural series of numbers from one to infinity. The upper indices of the parameters (1) and (2) refer to the upper and lower liquids, respectively.

Let us further assume that we know a system of functions $f_i\ (y,z)$ orthogonal to the domain Γ_0, which together with the constant constitute a complete system of functions. The deviation of the interface of liquids is decomposed by the system of functions f_i :

$$f = \sum_{i=1}^{\infty} \alpha_i(t) f_i\ (y,z)\,, \qquad (6)$$

Let's represent functions $B_i^{(k)}$ as expansion on parameters α_i up to the second order inclusive

$$B_i^{(k)} = B_{i0}^{(k)} + \sum_j \alpha_j B_{ij}^{(k)} + \sum_j \sum_k \alpha_j \alpha_k B_{ijk}^{(k)} + \ldots; \qquad (7)$$

where the functions $B_{i0}^{(k)}$, $B_{ij}^{(k)}$, $B_{ijk}^{(k)}$ depend only on spatial coordinates and do not depend on time.

Expressing all the functions included in the kinematic and dynamic conditions using Taylor's formula through their values on the undisturbed interface, and multiplying (4) by f_i , integrate Γ_0, finally we obtained infinite system of ordinary differential equations for generalized coordinates α_i at index *i*:

$$\begin{aligned}
&(\mu_i^{(2)} - \mu_i^{(1)})\frac{d^2\alpha_i}{dt^2} + j(N_i^{(2)2} - N_i^{(1)2})\alpha_i + \\
&+\sum_j \sum_k (\mu_{jik}^{(2)} - \mu_{jik}^{(1)})\alpha_k \frac{d^2\alpha_j}{dt^2} + \\
&+\sum_j \sum_k (N_{jik}^{(2)} - N_{jik}^{(1)})\frac{d\alpha_j}{dt}\frac{d\alpha_k}{dt} + \\
&+\sum_j \sum_k \sum_l (\mu_{jikl}^{(2)} - \mu_{jikl}^{(1)})\alpha_k \alpha_l \frac{d^2\alpha_j}{dt^2} + \\
&+\sum_j \sum_k \sum_l (N_{jikl}^{(2)} - N_{jikl}^{(1)})\alpha_l \frac{d\alpha_j}{dt}\frac{d\alpha_k}{dt} = 0\ \ (i=1,2,3,\ldots).
\end{aligned} \qquad (8)$$

where

$$\mu_i^{(k)} = \rho_k \int_{\Gamma_0} B_i^{(k)} f_i^{(k)} d\Gamma_0\,,\ N_i^{(k)2} = \rho_k \int_{\Gamma_0} (f_i^{(k)})^2 d\Gamma_0\,, \qquad (9)$$

$$\mu_{jik}^{(k)} = \rho_k \int_{\Gamma_0} (B_{ik}^{(k)} + f_k^{(2)} \frac{\partial B_{i0}^{(k)}}{\partial x_k}) f_j^{(k)} d\Gamma_0\,, \qquad (10)$$

$$\begin{aligned}
&\mu_{jikl}^{(k)} = \rho_k \\
&\cdot \int_{\Gamma_0} [B_{j0}^{(k)} \frac{\partial B_{ikl}^{(k)}}{\partial x_k} + (\frac{\partial B_{ik}^{(k)}}{\partial x_k} + \frac{1}{2} f_k^{(2)} \frac{\partial^2 B_{i0}^{(k)}}{\partial x_k^{\,2}}) f_i^{(k)} f_j^{(k)}] d\Gamma_0\,,
\end{aligned} \qquad (11)$$

$$N_{jik}^{(k)} = \rho_k \int_{\Gamma_0} (B_{jk}^{(k)} + \frac{1}{2} \nabla B_{j0}^{(k)} \cdot \nabla B_{k0}^{(k)}) f_i^{(k)} d\Gamma_0\,, \qquad (12)$$

$$N_{jikl}^{(k)} = \rho_k \left\{ \int_{\Gamma_0} \begin{array}{l} [2B_{ikl}^{(k)} + f_l^{(k)} \frac{\partial B_{jk}^{(k)}}{\partial x_2} + \nabla B_{jk}^{(k)} \cdot \nabla B_{kl}^{(k)} + \\ +\nabla B_{k0}^{(k)} \cdot \nabla B_{jl}^{(k)} + \\ +\frac{1}{2} f_l^{(k)} \frac{\partial}{\partial x_2} (\nabla B_{j0}^{(k)} \cdot \nabla B_{k0}^{(k)}) f_i^{(k)}] d\Gamma_0 \end{array} \right\}. \qquad (13)$$

III. Determination of Hydrodynamic Coefficients of the Equations of Motion for the Case of a Cylindrical Cavity

Let us consider a solid body with a cylindrical cavity completely filled with liquids. We introduce the coordinate system *Oxyz*, associated with the center of mass point *O* the body-liquid system in the undisturbed state. The axis of the cylinder coincides with the longitudinal axis *Ox*. The flat bottom of the tank S_2 is determined by the coordinate at the interface Γ_0 is $(x=-h_2)$ at a distance $(x=0)$ of the radius of the cylindrical side surface is equal to (see Fig. 2).

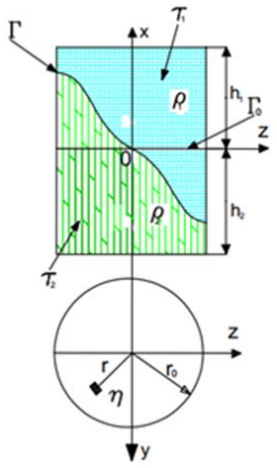


Fig. 2 Designations and coordinate systems for the case of a straight circular fixed cylinder with a two-layer liquid

We introduce a cylindrical coordinate system x, r, η associated with Cartesian x, y, z by the following formulas: $x = x$, $y = r\cos\eta$, $z = r\sin\eta$.

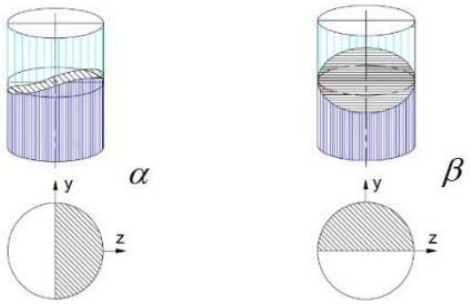


Fig. 3 The shapes of surfaces of section in the excitation of the main and auxiliary harmonics α, β.

We distinguish only two basic asymmetric harmonics excited in two mutually perpendicular planes and defined by generalized coordinates and forms $\alpha_i = \alpha_i\ (i = 1,2) : \alpha_1 = \alpha$, $\alpha_2 = \beta$, $f_1 = f_\alpha$, $f_2 = f_\beta$.

In the cylindrical coordinate system, the shape of the ground tone of the vibrations of the liquid interface is represented in the form similar to the representation of the free surface of the liquid at nonlinear oscillations [15] (see Fig. 3).

$$f_\alpha^{(k)}(y,z) = \varphi(r)^{(k)}\sin\eta,\ \ f_\beta^{(k)}(y,z) = \varphi(r)^{(k)}\cos\eta. \qquad (14)$$

Let's present the required functions in the form

$$B_{\alpha 0}^{(k)} = \psi^{(k)}(x,r)\sin\eta;\ \ B_{\beta 0}^{(k)} = \psi^{(k)}(x,r)\cos\eta; \qquad (15)$$

$$B_{\alpha\alpha}^{(k)} = \Psi_0^{(k)} - \Psi_2^{(k)}\cos 2\eta;\ \ B_{\alpha\beta}^{(k)} = B_{\beta\alpha}^{(k)} = \Psi_2^{(k)}\sin 2\eta;$$
$$B_{\beta\beta}^{(k)} = \Psi_0^{(k)} + \Psi_2^{(k)}\cos 2\eta. \qquad (16)$$

The functions $\psi^{(k)}(x,r)$ are solutions of boundary value linear problems and are given in [10], [11].

Having solutions of boundary value problems, it is not difficult to calculate the hydrodynamic coefficients of linear equations of motion of a body with liquids (9) corresponding to the case of small displacements of particles of liquids:

$$N_{11}^{(1)2} = \rho_1\delta_{11},\ N_{11}^{(2)2} = \rho_2\delta_{11},\ \mu_0^{(k)} = \rho_k\frac{\delta_{11}}{\ell_{11}^{(k)}}, \qquad (17)$$

$$\delta_{11} = \pi\frac{r_0^2(\xi_{11}^2 - 1)}{2\xi_{11}^2},\ \ell_{11}^{(k)} = -k_{11}th(k_{11}h_k). \qquad (18)$$

Solutions of boundary value nonlinear problems (16) will be sought in the form of corresponding Fourier series expansions on Bessel functions of zero and second orders, so

$$\Psi_m^{(k)}(x,r) = \sum_n c_{nm}^{(k)} X_{nm}^{(k)}(x) Y_{nm}(r);\ (m = 0,2). \qquad (19)$$

The coefficients of these expansions are determined by the following integrals:

$$c_{n0}^{(k)} = \frac{1}{2\ell_{11}^{(k)} N_{n0}^{(k)2}}\int_0^{r_0}[Y_{11}'^2 + (\frac{1}{r^2} - k_{11}^2)Y_{11}^2]Y_{n0}rdr; \qquad (20)$$

$$c_{n2}^{(k)} = \frac{1}{2\ell_{11}^{(k)} N_{n0}^{(k)2}}\int_0^{r_0}[Y_{11}'^2 - (\frac{1}{r^2} + k_{11}^2)Y_{11}^2]Y_{n2}rdr;$$
$$N_{n2}^{(k)2} = \frac{r_0^2(\xi_{n2}^2 - 4)}{2\xi_{n2}^2},\ N_{n0}^{(k)2} = \frac{r_0^2}{2}. \qquad (21)$$

Collecting values of (19)-(21), we calculate values of parameters $\mu_1^{(k)}$ and $\mu_2^{(k)}$, determining the nonlinearity of wave motions of the liquid, accounting for which allows us to describe the phenomenon of rotation of the liquid interface

$$\mu_1^{(k)} = \rho_k\frac{\pi}{4\ell_{11}^{(k)}}\int_0^{r_0}\{[6Y_{11}'^2 + (\frac{2}{r^2} - 3k_{11}^2)Y_{11}^2]Y_{11} +$$
$$\sum_n\frac{4c_{n0}^{(k)}}{\ell_{n0}^{(k)}}(Y_{n0}'Y_{11}' - k_{n0}^2Y_{n0}Y_{11}) +$$
$$\sum_n\frac{2c_{n2}^{(k)}}{\ell_{n2}^{(k)}}[Y_{n2}'Y_{11}' + (\frac{2}{r^2} - k_{n2}^2)Y_{n2}Y_{11}]\}Y_{11}rdr, \qquad (22)$$

$$\mu_2^{(k)} = \rho_k\frac{\pi}{4\ell_{11}^{(k)}}\int_0^{r_0}\{[2Y_{11}'^2 - (\frac{2}{r^2} + k_{11}^2)Y_{11}^2]Y_{11} +$$
$$\sum_n\frac{2c_{n2}^{(k)}}{\ell_{n2}^{(k)}}[Y_{n2}'Y_{11}' + (\frac{2}{r^2} - k_{n2}^2)Y_{n2}Y_{11}]\}Y_{11}rdr \qquad (23)$$

We rewrite the equations of motion of liquids on the interface in the form

$$\ddot{\alpha} + \sigma^2\alpha + d_1(\alpha^2\ddot{\alpha} + \dot{\alpha}^2\alpha + \alpha\beta\ddot{\beta} + \alpha\dot{\beta}^2) +$$
$$d_2(\beta^2\ddot{\alpha} + 2\beta\dot{\alpha}\dot{\beta} - \alpha\beta\ddot{\alpha} - 2\alpha\dot{\beta}^2) = 0; \qquad (24)$$

$$\ddot{\beta} + \sigma^2\beta + d_1(\beta^2\ddot{\beta} + \dot{\alpha}^2\beta + \alpha\beta\ddot{\alpha} + \beta\dot{\beta}^2) +$$
$$d_2(\beta^2\ddot{\beta} + 2\alpha\dot{\alpha}\dot{\beta} - \alpha\beta\ddot{\alpha} - 2\beta\dot{\alpha}^2) = 0; \qquad (25)$$

here $\sigma^2 = \frac{g(N_{11}^{(2)2} - N_{11}^{(1)2})}{\mu_0^*}$, $d_1 = \frac{(\mu_1^{(2)} - \mu_1^{(1)})}{\mu_0^*}$, $d_2 = \frac{(\mu_2^{(2)} - \mu_2^{(1)})}{\mu_0^*}$, $\mu_0^* = (\mu_0^{(2)} - \mu_0^{(1)})$.

IV. Investigation of Stability of Forced and Parametric Oscillations of Liquids in the Simplest Case

Let the cavity make the given movements in the direction of

the axis Oz according to the law: $U(t) = S\cos\omega t$, here $U(t)$ - the displacement of a given movement of the vessel, see Fig. 4. S and ω - amplitude and frequency of perturbation. The corresponding system of nonlinear equations takes the form

$$L_1(\alpha,\beta) = \ddot{\alpha} + \sigma^2\alpha + d_1(\alpha^2\ddot{\alpha} + \dot{\alpha}^2\alpha + \alpha\beta\ddot{\beta} + \alpha\dot{\beta}^2) + d_2(\beta^2\ddot{\alpha} + 2\beta\dot{\alpha}\dot{\beta} - \alpha\beta\ddot{\alpha} - 2\alpha\dot{\beta}^2) - \omega^2 P\cos\omega t = 0; \quad (26)$$

$$L_2(\alpha,\beta) = \ddot{\beta} + \sigma^2\beta + d_1(\beta^2\ddot{\beta} + \dot{\alpha}^2\beta + \alpha\beta\ddot{\alpha} + \beta\dot{\beta}^2) + d_2(\beta^2\ddot{\beta} + 2\alpha\dot{\alpha}\dot{\beta} - \alpha\beta\ddot{\alpha} - 2\beta\dot{\alpha}^2) = 0; \quad (27)$$

Here $P = \frac{S\lambda}{\mu_0^*}$, $\lambda = \lambda^{(2)} - \lambda^{(1)}$, $\lambda^{(k)} = \rho_k\pi\int_0^{r_0} r^2 Y_{11}dr = \frac{\rho_k\pi r_0^3}{\xi_{11}^2}$, $(k = 1,2)$ [15].

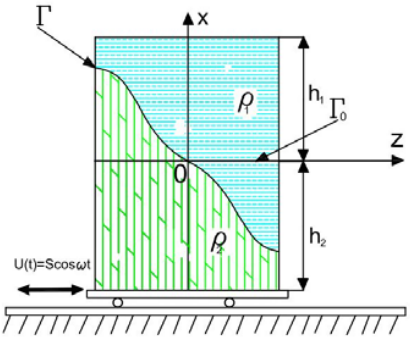


Fig. 4 Cylindrical vessel performing translational motion

This system roughly describes the forced and parametric excited oscillations of the liquid interface. In the case where parametric oscillations do not occur in the system $(\beta \equiv 0)$, the forced oscillations are described by a nonlinear differential equation

$$L_1(\alpha) = \ddot{\alpha} + \sigma^2\alpha + d_1(\alpha^2\ddot{\alpha} + \dot{\alpha}^2\alpha) = \omega^2 P\cos\omega t; \quad (28)$$

The approximate solution of this equation is found by the Bubnov-Galerkin method, presenting the solutions as (29)

$$\alpha(t) = \alpha_0 + \sum_{k=1}^{n}(\alpha_k\cos k\omega t + \bar{\alpha}_k\sin k\omega t), \quad (29)$$

here α_k и $\bar{\alpha}_k$ - the unknown constants. When the main harmonics only keep in (29)

$$\alpha(t) = A\cos\omega t + \bar{A}\sin\omega t, \quad (30)$$

We get the following equation

$$(\bar{\sigma}^2 - 1)A - m_1A^3 = P,\ \bar{A} \equiv 0, \quad (31)$$

here $m_1 = d_1/2$, $\bar{\sigma}^2 = \sigma^2/\omega^2$. Equation (31) is used to determine the amplitudes of forced oscillations of a two-layer liquid depending on the parameter P and ω. Putting in (31) $P = 0$, to determine the dependence of the amplitudes of free oscillations of liquids on the frequency, (32):

$$(\bar{\sigma}^2 - 1) - m_1A^2 = 0. \quad (32)$$

In the future, when determining the steady-state modes of motion of liquids and the boundaries of the regions of their stability, the main terms in the expansions of the type (29) are used.

Let us consider the question of the stability of the periodic solution (30). The answer to it is to find those values of the parameters S and ω at which the steady state

$$\alpha(t) = A\cos\omega t,\ \beta \equiv 0, \quad (33)$$

described by a system of nonlinear equations (26)-(27), physically realizable. To this end, along with the movement (33), which was taken for undisturbed, we consider also the movements close to it

$$\alpha(t) = \bar{\alpha}(t) + \xi(t),\ \beta(t) = \bar{\beta}(t) + \eta(t). \quad (34)$$

The initial conditions for $\xi(t)$ and $\eta(t)$ differ little from the initial conditions for $\bar{\alpha}(t)$ and $\bar{\beta}(t)$ (33).

In accordance with the general theory of stability, we make equations in variations corresponding to a given system of nonlinear equations (26), (27). Substituting (33) into the system of equations (26), (27) and taking that $\bar{\alpha}(t)$ it is a particular solution (28), we obtain the equations of perturbed motion in the form

$$(d_1\bar{\alpha}^2 + 1)\ddot{\xi} + 2d_1\bar{\alpha}\dot{\bar{\alpha}}\dot{\xi} + (\sigma^2 + 2d_1\bar{\alpha}\ddot{\bar{\alpha}} + d_1\dot{\bar{\alpha}}^2)\xi + F_1(\xi,\beta,\dot{\xi},\dot{\beta},\ddot{\xi},\ddot{\beta}) = 0; \quad (35)$$

$$(1 + d_2)\ddot{\eta} + 2d_2\bar{\alpha}\dot{\bar{\alpha}}\dot{\eta} + (\sigma^2 + c\bar{\alpha}\ddot{\bar{\alpha}} + k_3\dot{\bar{\alpha}}^2)\eta + F_2(\xi,\beta,\dot{\xi},\dot{\beta},\ddot{\xi},\ddot{\beta}) = 0; \quad (36)$$

here F_1 and F_2 - functions containing perturbations and their derivatives in degrees above the first. Leaving in (35) and (36) only linear terms, taking into account (33) we come to equations in variations

$$L_3(\xi) = (p + q\cos\theta t)\ddot{\xi} - \varepsilon_1\dot{\xi}\sin\theta t + (\gamma - \delta\cos\theta t)\xi = 0 \quad (37)$$

$$L_4(\eta) = (\bar{p} + \bar{q}\cos\theta t)\ddot{\eta} - \varepsilon\dot{\eta}\sin\theta t + (\bar{\gamma} - \bar{\delta}\cos\theta t)\eta = 0 \quad (38)$$

called equations of the first approximation. The following designations are accepted here:

$$q=(d_1A^2)/2,\ p=1+q,\ \varepsilon_1=d_1\omega A^2,\ \gamma=(\bar{\sigma}^2-(d_1A^2)/2)\omega^2,$$
$$\delta=(3d_1\omega^2A^2)/2,\ \bar{p}=1+\bar{q},\ \bar{q}=(d_2A^2)/2,$$
$$\bar{\gamma}=(\bar{\sigma}^2-(d_2A^2)/2)\omega^2,\ \bar{\delta}=\omega^2A^2(2d_1-3d_2),\ \varepsilon=d_2\omega A^2,$$
$$\theta=2\omega. \quad (39)$$

Let us proceed to the construction of instability regions of solutions of (37) and (38). Let us first consider (37) with respect to the functions $\xi(t)$, characterizing the perturbation of the periodic solution $\bar{\alpha}(t)=A\cos\omega t$.

The amplitude *A* of this solution is determined by (31). It is necessary to determine which pair of values *A* and ω satisfying (31) leads to stable solutions and which pair to unstable ones. Bearing in mind the construction of the main instability domain of the solution of (37), we present it as

$$\xi(t)=a_1\cos\frac{\theta t}{2}+b_1\sin\frac{\theta t}{2}. \quad (40)$$

From the Bubnov-Galerkin equation

$$\int_0^{\frac{4\pi}{\theta}} L_3(\alpha)\cos\frac{\theta t}{2}dt=0,\ \int_0^{\frac{4\pi}{\theta}} L_3(\alpha)\sin\frac{\theta t}{2}dt=0, \quad (41)$$

to determine the boundaries of the instability region we obtain:

$$(\bar{\sigma}^2-1)-3m_1A^2=0; \quad (42)$$

$$(\bar{\sigma}^2-1)-m_1A^2=0. \quad (43)$$

Equation (43) coincides with the skeletal line equation (32). For Fig. 5, this line corresponds to the *ABC* curve. Equation (42) corresponds to the curve of the *AMO*. Comparing (42) with the equation of amplitude-frequency characteristics (31), it is easy to notice that the stable branch of the resonance curve *RNM* is separated from the unstable point *M*. in which the amplitude curves have a vertical tangent. On the stable part of the branch of the amplitude-frequency response, located to the left of the glass line, the derivative must be necessarily positive. At *M* it tends to infinity, and on the unstable left branch it is negative. In the region *I* bounded by the *AMO* and *ABC* curves, the solution $\bar{\alpha}(t)=A\cos\omega t$ unstably. From the physical point of view, the stability condition of a periodic solution with a period of disturbing force means that the amplitude of the forced oscillations increases with the increase of the external force *P*.

Consider next (38) in variations with respect to the perturbation $\eta(t)$ of the trivial solution $\beta(t)\equiv 0$. The study of the solutions of (38) should answer the question of the stability of this trivial solution. In accordance with the above, the instability region of (38) corresponds to the regions of parametrically excited oscillations $(\beta\neq 0)$, so the regions of dynamic instability of the motion mode (33). To construct the main domain of instability we assume

$$\eta(t)=a_0\cos\frac{\theta t}{2}+b_0\sin\frac{\theta t}{2}. \quad (44)$$

Putting (44) into the equations of the Bubnov-Galerkin

$$\int_0^{\frac{4\pi}{\theta}} L_4(\eta)\sin\frac{\theta t}{2}dt=0,\ \int_0^{\frac{4\pi}{\theta}} L_4(\eta)\cos\frac{\theta t}{2}dt=0, \quad (45)$$

to determine the boundaries of the dynamic stability domains we obtain:

$$\bar{\sigma}^2=m_2A^2+1; \quad (46)$$

$$\bar{\sigma}^2=m_1A^2+1, \quad (47)$$

here $m_2=(4d_2-d_1)$. On Fig. 5, (47) is the *ADE* curve and (47) is the *ABC* curve. Consequently, the instability regions of the solutions of (37) and (38) continuously move into one another.

In region *II* bounded by *ABC* and *ADE* curves, the solution $\beta(t)\equiv 0$ is unstable. The second stable branch of the amplitude-frequency characteristic *KLD* adjoins the region *II* on the right and is separated from it by point *D*. in the region of dynamic unstable *II*, the steady-state mode, if it exists, is described by a nonlinear system of equations (26)-(27).

Let us now consider the construction of steady-state regimes of fluid motion occurring in the main region of dynamic instability *II*. It was mentioned above that on the boundaries of odd instability regions the solutions of linear equations (37) and (38) have the form. It is in this form that steady-state movements in areas of dynamic instability are usually sought.

Suppose that in the region of the main resonance the approximate solution of the system of nonlinear equations (26)-(27) can be represented as

$$\alpha(t)=A\cos\omega t+\bar{A}\sin\omega t,\ \beta(t)=\bar{B}\cos\omega t+B\sin\omega t. \quad (48)$$

Taking to the solution of this system the Bubnov-Galerkin method, for constant A, $\bar{A}$, B and $\bar{B}$ we obtain the following algebraic relations:

$$(\bar{\sigma}^2-1)\mathrm{A}-m_1A^3-m_2AB^2=P; \quad (49)$$

$$(\bar{\sigma}^2-1)-m_1B^2-m_2A^2=0;\ (\bar{A}=0,\ \bar{B}=0). \quad (50)$$

Excluding *B* from (50) and substituting the result in (49), we find an equation for determining the amplitude-frequency characteristics in region *II*

$$(\bar{\sigma}^2-1)\mathrm{A}-m_4A^3=m_5P;\ \text{where}\ m_4=2d_2,\ m_5=m_1/(m_1-m_2) \quad (51)$$

Solution (51) corresponds to the regime of rotation of the liquid interface observed in the experiment. The corresponding resonance curves are shown in Figs. 5 and 6 (a), (b), line *FGQ*.

In Fig. 5, the amplitude-frequency characteristics (AFC) and instability regions of forced oscillations of liquids are constructed from the obtained all relations at the value of the upper liquid density (air) $\overline{\rho}_1$=0.0012, which completely coincided with the result of the problem for one liquid [14].

In Figs. 6 (a), (b) are presented the AFC and instability regions of forced oscillations of two liquids at different density ratios for the pair of petrol fuel $\overline{\rho}_1$=0.75 and water $\overline{\rho}_1$=1 (a) and for the pair of sunflower oil $\overline{\rho}_1$=0.93 and water $\overline{\rho}_1$=1 (b).

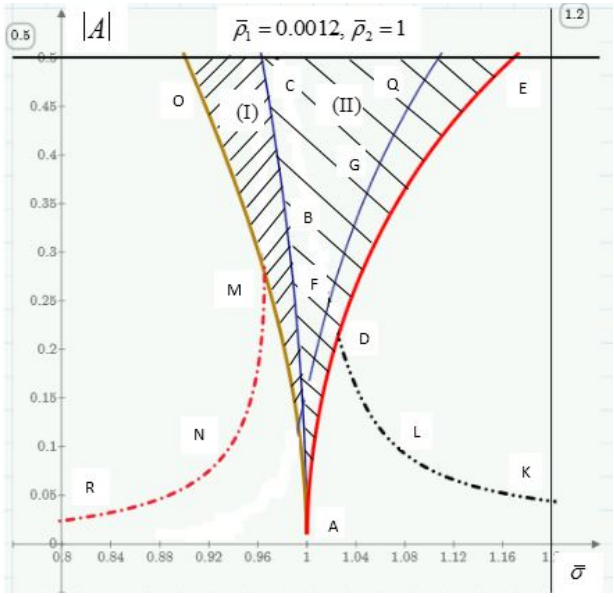
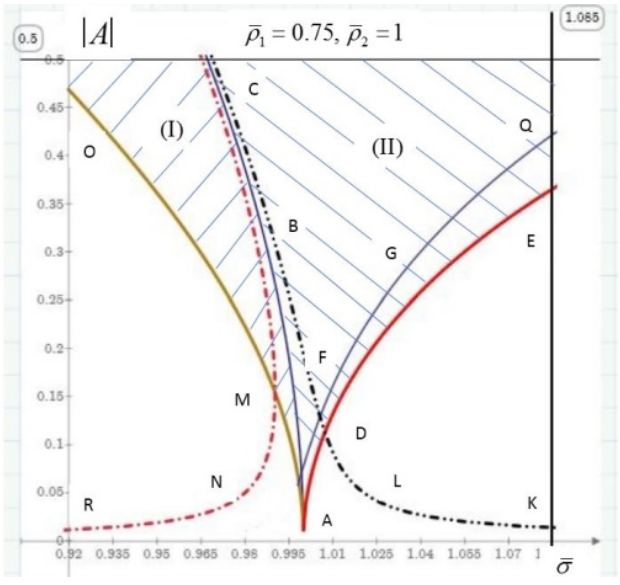
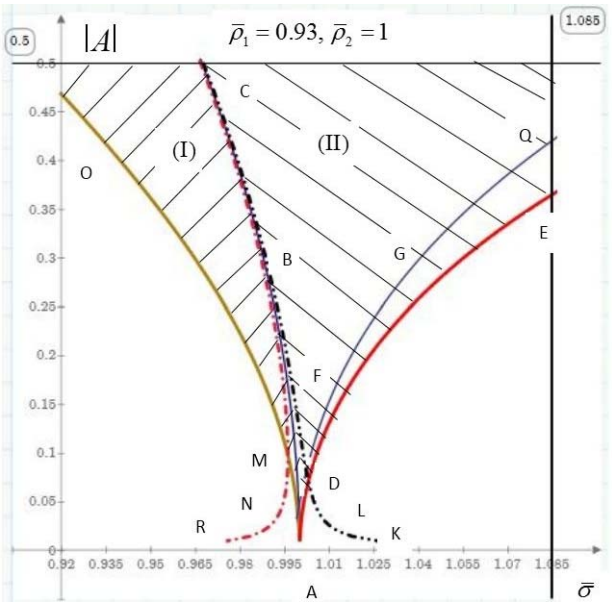


Fig. 5 AFC and instability regions of forced oscillations of liquids in a cylindrical tank under excitation of basic harmonics α for the case $\overline{\rho}_1 = 0.0012$ with $\overline{S} = 0.005,\ \overline{h}_1 = \overline{h}_2 = 2$



(a)



(b)

Fig. 6 AFC and instability regions of forced oscillations of liquids in a cylindrical tank under excitation of basic harmonics α for the case $\overline{\rho}_1 = 0.75$ (*a*), for the case $\overline{\rho}_1 = 0.93$ (*b*) with $\overline{S} = 0.005,\ \overline{h}_1 = \overline{h}_2 = 2$

Regions *I* and *II* are regions of instability of forced oscillations of liquids occurring in the plane of action of the disturbing force. In region *II*, this instability is due to the instability of the trivial solution $\beta(t) \equiv 0$, so the parametric excitation of the generalized coordinate β is possible. The *FGQ* line corresponds to the rotational motion of the liquid interface observed in the experiment.

V. Conclusion

In this paper, the nonlinear effects resulting from the interaction of liquids with a rigid vessel that performs harmonic oscillations are theoretically investigated. The most interesting from the practical side is the case of vibrations of liquids in the vicinity of the lowest frequency of natural oscillations of the interface. Here, a number of characteristics essentially nonlinear features of fluid motion are observed, among which can indicate the dependence of the oscillation frequency on the amplitude, the limited oscillation amplitudes in the resonance mode, the mobility of the nodal lines of the interface, the appearance of a peculiar rotation of the interface in a certain frequency range of the disturbing force.

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