Jeffrey′s Prior for Unknown Sinusoidal Noise Model via Cramer-Rao Lower Bound

Samuel A. Phillips, Emmanuel A. Ayanlowo, Rasaki O. Olanrewaju, Olayode Fatoki

Abstract—This paper employs the Jeffrey′s prior technique in the process of estimating the periodograms and frequency of sinusoidal model for unknown noisy time variants or oscillating events (data) in a Bayesian setting. The non-informative Jeffrey′s prior was adopted for the posterior trigonometric function of the sinusoidal model such that Cramer-Rao Lower Bound (CRLB) inference was used in carving-out the minimum variance needed to curb the invariance structure effect for unknown noisy time observational and repeated circular patterns. An average monthly oscillating temperature series measured in degree Celsius (^{0}C) from 1901 to 2014 was subjected to the posterior solution of the unknown noisy events of the sinusoidal model via Markov Chain Monte Carlo (MCMC). It was not only deduced that two minutes period is required before completing a cycle of changing temperature from one particular degree Celsius to another but also that the sinusoidal model via the CRLB-Jeffrey′s prior for unknown noisy events produced a miniature posterior Maximum A Posteriori (MAP) compare to a known noisy events.

Keywords—Cramer-Rao Lower Bound (CRLB), Jeffrey′s prior, Sinusoidal, Maximum A Posteriori (MAP), Markov Chain Monte Carlo (MCMC), Periodograms.

I. Introduction

FREQUENCY domain type of time series expresses the fusion of trigonometric functions (e.g. sine or cosine or both) of undulation that possessed different periods of completing circular patterns with maximum and minimum value of some quantities that varies (amplitude) [1]. It is otherwise known as spectral density model – meant for putrefying changes (variations) in form, position, and in state of processes into examining the complete periodical components along with its different frequencies [2], [3]. It is a noisy type of time series model for examining and discovering repeated cycles, revolutions, and regular intervals (periodic) of signals in turbulent or boisterous (noisy) distributed observations in time series events. The assessment of repeated cycles or patterns is by whether the pinnacle of periodograms derived from the sinusoidal model that is affected by random (stochastic) component of a data noisy spectrum or spectral density [4]-[6]. However, periodogram (amplitude) parameters embedded in sinusoidal model are different form of modified spectral models, hyperbolic time series models, or structural time series models to establish, carve-out the prevailing periods of the observational time events. It is a tool meant for the establishment and prevailing of cyclical traits encounter in timely observational series especially in seasonal events (uniform interval of monthly and quarterly data). Similarly, another indispensable subset parameter of the sinusoidal model is the frequency–a fraction of the complete cycle that is completed in a single period [7], [8].

Substantial applications and simulation studies have been applied to Fast Fourier Transformation (FFT) model, modified spectral density models, sinusoidal model, and its variants. Among the few applications of sinusoidal model and its variants are the periodical behaviour of index reected series of cyclic warming (negative Southern Oscillation Index (OSI)) and cooling (positive OSI) of the eastern and central Pacific, which affects the sea level pressure at two locations that were subjected to structural spectral time series model [2]. It was inferred that the spectral peak of the OSIs arose at a zero frequency and clarification was made for a clear mode of the corresponding frequency not achievable at the period of four years. Reference [9] showed that the observe peak in the periodogram was highly unlikely under the spectrum assumptions than the underlying power spectrum continuum. They divided the used timely observational series into two parts and used the largest periodogram residuals of the two different series to derive a redefined Fourier frequency for the detection of Quasi-Periodic Oscillation (QPO) via Bayesian method of periodogram estimation by Markov Chain Monte Carlo (MCMC). In addition, the choice of prior distribution was affirmed to be uniform distribution when the choice parameterization is needed with the use of Bayes factor [10]. He applied the derived solution of the posterior to 180 terrestrial impact craters repeated oscillating time series, and concluded the absence of periodic variation in the cratering rate; that is, no strong evidence of periodicity was superimposed on the constant rate of the craters with the believe of the presence of intricate signals. Furthermore, [11] described the Bayesian approach as an alternative technique in estimating time axes and periodiogram associated to sinusoidal with or without known noise effect. It was noted that the Bayesian method of parameter estimation technique for sinusoidal (periodogram) or Fourier frequency model not only adopted non-informative prior of either the uniform distribution or the Gaussian distribution but also failed to diagnose invariance structure effect that might have associated to the time repeated cycles (data) when incorporating into sinusoidal model. In order to curb the problem of invariance structure effect for unknown noisy time

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events, the Cramer-Rao Lower Bound (CRLB) for deriving the minimum variance is needed by the distributional time events. The improper (non-informative) prior of the Jeffrey′s prior will be adopted in estimating the unknown noisy (sigma) for the sinusoidal or periodogram model via the CRLB inference.

II. Notation and Definition of Related Terms

To address the problem of this choice of invariance structure associated with Bayesian sinusoidal (periodogram) or Fourier frequency model, the Jeffrey′s improper (non-informative) prior approached will be adopted in this paper. The concept of Cramer-Rao Lower Bound (CRLB) as defined by [12], [13], the inverse of the Fisher information will be introduced to solve the choice of invariance structure characterized by the Jeffrey′s prior in Bayesian periodogram with unknown noise model. The variance of any parameter under consideration for any estimation problem with likelihood function $f_x(x/\theta)$ is the inverse of Fisher information as defined [14], [15] as

$$\mathrm{I}(\overset{\wedge}{\theta}) = E_y\left[\left(\frac{\partial \log f_y(y/\theta)}{\partial\theta}\right)\right]^2 = - E_y\left[\left(\frac{\partial^2 \log f_y(y/\theta)}{\partial\theta^2}\right)\right] \tag{1}$$

Given a sinusoidal trigonometric like model to be

$$f(t_i) = C\cos(wt_i) + D\sin(wt_i) \tag{2}$$

For parameters $\theta = \{C,\, D,\, \omega\}$ for a single variable time, series set of events with wave of a single quantity over time $''y''$ with time $''t''$ for a general model data

$$y_i = f(t_i) + \xi(t_i) \tag{3}$$

Then the conditional distribution of the invariance structure is needed in computing the Fisher information. $\xi(t_i), y_i \sim i.i.d(0,\sigma^2)\,\forall,\ i = 1,\cdots,t$

III. Parameter Estimation

A. The Fisher Information from the Jeffrey′s Prior in Obtaining Bayesian Periodogram with Unknown Noise Model

The resulting Jeffrey′s prior for the unknown variance $''\sigma^2''$ of a set of zero mean Independent and Identical Distribution (I.I.D) of Gaussian is

$$p(\sigma^2) \propto \frac{1}{\sigma^2} \tag{4}$$

$$P(\sigma^2, /H, M) = \prod_{i=1}^{t} \frac{1}{\sqrt{\sigma^2 2\pi}}\, e^{-\left(\frac{y_i - f(t_i)}{2\sigma}\right)^2} \tag{5}$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{t}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{t}{2}} e^{-\sum_{i=1}^{t}\left(\frac{y_i - f(t_i)}{2\sigma}\right)^2} \tag{6}$$

The resulting posterior density is,

$$p(\sigma^2/\, y_i) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{t}{2}+1} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{t}(y_i - f(t_i))^2} \tag{7}$$

$$\log p(\sigma^2/\, y_i) \propto \left(\frac{t}{2}+1\right)\log\left(\frac{1}{\sigma^2}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{t}(y_i - f(t_i))^2 \tag{8}$$

$$let,\ \sigma^2 = \lambda$$

$$p(\lambda/\, y_i) \propto -\left(\frac{t}{2}+1\right)\log(\lambda) - \frac{1}{2\lambda}\sum_{i=1}^{t}(y_i - f(t_i))^2 \tag{9}$$

$$\frac{\partial \log p\left(\lambda/_{y_i}\right)}{\partial\lambda} = -\frac{\left(\frac{t}{2}+1\right)}{\lambda} + \frac{1}{2\lambda^2}\sum_{i=1}^{n}(y_i - f(t_i))^2 \tag{10}$$

$$\frac{\partial^2 \log p\left(\lambda/_{y_i}\right)}{\partial\lambda^2} = \frac{\left(\frac{t}{2}+1\right)}{\lambda^2} - \frac{1}{\lambda^3}\sum_{i=1}^{n}(y_i - f(t_i))^2 \tag{11}$$

$$E_x\left[\frac{\partial^2 \log p\left(\lambda/_{y_i}\right)}{\partial\lambda^2}\right] = \frac{\left(\frac{t}{2}+1\right)}{\lambda^2} - \frac{1}{\lambda^3}\,E_x\left[\sum_{i=1}^{n}(y_i - f(t_i))^2\right] \tag{12}$$

$$= \frac{\left(\frac{t}{2}+1\right)}{\lambda^2} - \frac{1}{\lambda^3}\,E_x\left[\sum_{i=1}^{n}(y_i - f(t_i))^2\right] \tag{13}$$

recall,

$$\sum_{i=1}^{t}(y_t - f(t_i))^2 = \sum_{i=1}^{t} y_i^2 - 2\sum_{i=1}^{t} y_i f(t_i) + \sum_{i=1}^{t} f(t_i)^2 \tag{14}$$

$$\sum_{i=1}^{t} y_i^2 - 2\sum_{i=1}^{t} y_i\,(C\cos(wt_i) + D\sin(wt_i)) + \sum_{i=1}^{t}((C\cos(wt_i) + D\sin(wt_i)))^2$$

but,

$$\sum_{i=1}^{t}((C\cos(wt_i) + D\sin(wt_i)))^2 =$$

$$C^2\sum_{i=1}^{t}\cos^2(\omega t_i) + D^2\sum_{i=1}^{t} sin^2(\omega t_i) + 2CD\sum_{i=1}^{t}\cos(\omega t_i)sin(\omega t_i)$$

for $'i'$ that is greater than one, and from approximating solutions of trigonometry.

$$\sum_{i=1}^{t}\sin^2(\omega t_i) = \frac{t}{2} - \frac{1}{2}\sum_{i=1}^{t}\cos(2\omega t_i) \approx \frac{t}{2}$$

$$\sum_{i=1}^{t}\cos^2(\omega t_i) = \frac{t}{2} + \frac{1}{2}\sum_{i=1}^{t}\cos(2\omega t_i) \approx \frac{t}{2}$$

$$\sum_{i=1}^{t}\cos(\omega t_i)\,\sin(\omega t_i) = \frac{1}{2}\sum_{i=1}^{t}\sin(2\omega t_i) \approx< \frac{t}{2}$$

for a large sample size, $(i \gg 1)$ such that its variation frequency is zero or no more, that is, the data has been de-trend [16]. It implies,

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$$\sum_{i=1}^{t}\left((C\cos(wt)\,+\,D\sin(wt))\right)^2 \approx \frac{t}{2}\left(C^2+\,D^2\right) \tag{15}$$

let,

$$h(\omega)\,=\,\sum_{i=1}^{t} y_i\cos(wt);\quad n(\omega)\,=\,\sum_{i=1}^{t} y_i\sin(wt)$$

(14) becomes,

$$\sum_{i=1}^{t}\left(y_t-f(t)\right)^2=\sum_{i=1}^{t} y_i^2+\frac{t}{2}\left(C^2+\,D^2\right)-2\left[Ch(\omega)+Dn(\omega)\right] \tag{16}$$

$$E_x\left[\sum_{i=1}^{t}\left(y_t-f(t)\right)^2\right] = E_x\left[\sum_{i=1}^{t} y_i^2\right] + E_x\left[\frac{t}{2}\left(C^2+\,D^2\right)\right]-2E_x\left[Ch(\omega)+Dn(\omega)\right]$$

Mean and variance of $\xi(t_i), y_i$ are zero and σ^2 respectively. But,

$$E_x\left[\sum_{i=1}^{t}\left(y_t-f(t)\right)^2\right]=$$

$$E_x\left[\sum_{i=1}^{t} y_i^2\right]+E_x\left[\frac{t}{2}\left(C^2+\,D^2\right)\right]=t\sigma^2+\left[\frac{t}{2}\left(C^2+\,D^2\right)\right] \tag{17}$$

$$E_x(\varepsilon_t^2)=\,E_x(y_i^2)=\sigma^2$$

(13) becomes,

$$E_x\left[\frac{\partial^2\log p\left(\lambda/_{y_i}\right)}{\partial\lambda^2}\right]\;=\;\frac{\left(\frac{t}{2}\,+1\right)}{\lambda^2}\;-\;\frac{t\sigma^2+\left[\frac{t}{2}\left(C^2+\,D^2\right)\right]}{\lambda^3} \tag{18}$$

since, $\sigma^2\,=\,\lambda$

$$E_x\left[\frac{\partial^2\log p\left(\lambda/_{y_i}\right)}{\partial\lambda^2}\right]=\frac{\sigma^2\left(\frac{t}{2}+1\right)-t\left(\sigma^2+\left[\frac{1}{2}\left(C^2+D^2\right)\right]\right)}{\sigma^6} \tag{19}$$

$$I(\overset{\wedge}{\sigma^2})=-E_x\left[\frac{\partial^2\log p\left(\sigma^2/_{y_i}\right)}{\partial\lambda^2}\right]=\frac{t\left(1+\left[\frac{1}{2}\left(C^2+D^2\right)\right]\right)-\left(\frac{t}{2}+1\right)}{\sigma^4} \tag{20}$$

The Fisher Information of the Bayesian periodogram for Jeffrey′s prior for unknown noise model is

$$\frac{t\left(1+\left[\frac{1}{2}\left(C^2+\,D^2\right)\right]\right)-\left(\frac{t}{2}\,+1\right)}{\sigma^4}$$

The CRLB (Minimum Variance of the Unbiased Estimator) for $''\sigma^{2''}$, which is the inverse of Fisher information is

$$Var(\overset{\wedge}{\sigma^2})\;=\;\frac{1}{I(\sigma^2)}\;\leq\;\frac{\sigma^4}{t\left(1+\left[\frac{1}{2}\left(C^2+\,D^2\right)\right]\right)-\left(\frac{t}{2}\,+1\right)}\;\forall t\;>2 \tag{21}$$

In other words, the unknown variance structure for Jeffrey′s prior when dealing with the unknown noise model for Bayesian periodogram can now be approximately switched to Bayesian periodogram for known noise model for estimated C and D.

B. Estimation of Frequency Parameters of the Bayesian Periodograms

Assuming,

$$f(ti)=\,C\cos(wti)\,+\,D\sin(wti)\quad for\;\;i=1,\ldots,t$$

$$X(\omega)\,=\,\begin{bmatrix}\cos(i\omega) & \sin(i\omega)\\ \vdots & \vdots\\ \cos(n\omega) & \sin(n\omega)\end{bmatrix}$$

$$such\;\;that\;\;\beta=\,(C,D)^T,\quad Y=(\,y_1,\,\cdots\,,\,y_n)$$

$$L_1(C,D,\omega)=\,\left[Y-X(\omega)\beta\right]^T\;\left[Y-X(\omega)\beta\right] \tag{22}$$

Minimizing C, D, ω

$$\overset{\wedge}{\beta}(\omega)=\,\left[X^T(\omega)X(\omega)\right]^{-1}X^T(\omega)Y \tag{23}$$

Substituting $\beta\;by\;\overset{\wedge}{\beta}(\omega)$ in (22)

$$Z_1(\omega)=\;\;L_1\left(\hat{C}(\omega),\,\hat{D}(\omega),\,\omega\right)\;=\;Y^T\left(1-P_x(\omega)\right)Y \tag{24}$$

$$where\;\;\;P_x(\omega)=\;\;X(\omega)\left[X^T(\omega)X(\omega)\right]^{-1}X^T(\omega)$$

If $\overset{\wedge}{\omega}$ minimizes (24) we denoted $\left(\hat{C}(\omega),\,\hat{D}(\omega),\,\omega\right)$ considering

$$Y^{(1)}\,=\,Y-X(\overset{\wedge}{\omega})\,\overset{\wedge}{\beta}_1$$

where, $\overset{\wedge}{\beta}_1\;=\;(\hat{C},\,\hat{D})^T,\;Y\;will\;now\;be\;replaced\;by\;Y^{(1)}$ such that

$$L_2(C,D,\,\omega)\;=\;\left[Y^{(1)}-X(\omega)\beta\right]^T\;\left[Y^{(1)}-X(\omega)\beta\right] \tag{25}$$

$L_2(C,D,\,\omega)$ would be minimized with respect to $C,\;D,\;\&\;\omega$ and repeated continuous (iteratively) the process until convergence is achieved.

IV. Application

The periodical noise data of temperature recorded in degree Celsius (^{0}C) by the Lagos State ministry of environment were a good example of oscillating pattern events. The variability in the dataset was cyclical. The dataset is a monthly temperature recorded from 1901 to 2014 for meteorological forecast and comparison for seasonal effects. It consists of 114 data monthly points. It is regarded as a single noisy quantity of time variant series of events. It stationary process was confirmed and was in line with possessed traits of noisy data for spectral or frequency domain analysis, that is, the oscillating patterns of the temperature series.

Fig. 1 indicates a blue coated-cluster colour between -1.0 and 0.5, suggesting that the recorded temperature for the 114 years was ranged between 22^0C and 30^0C with

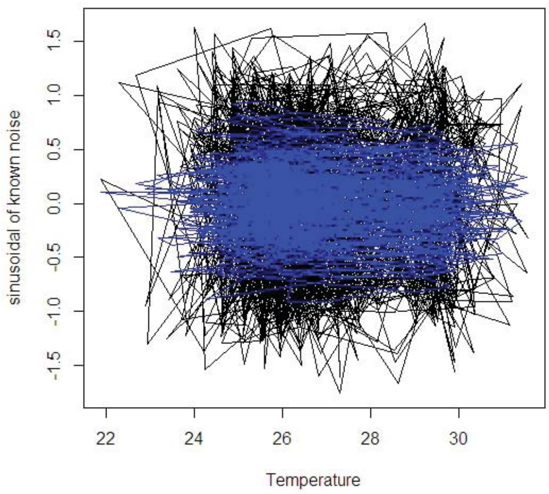


Fig. 1 The Noisy measured series of the Temperature in ^{0}C

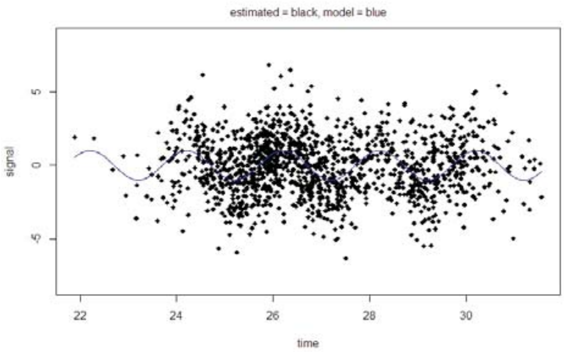


Fig. 2 The Timely Periodical Signal of the Temperature in ^{0}C

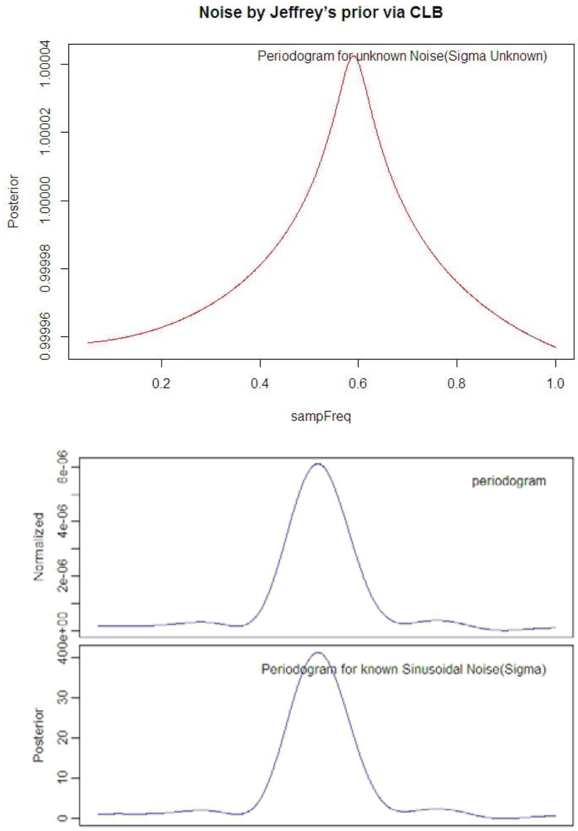
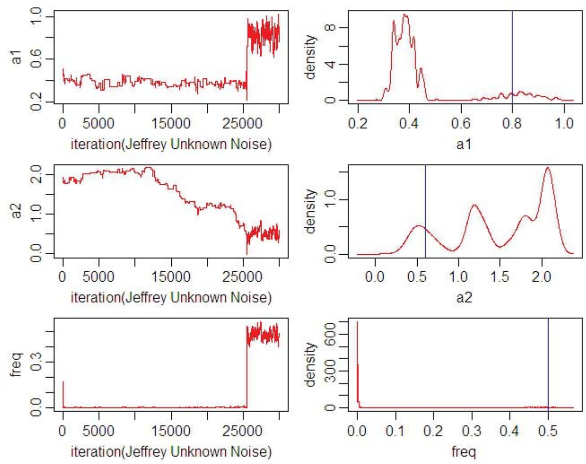


Fig. 3 The Periodiogram Curve for Unknown, Fourier, and Known Noise

clustered-mean around 24 ^{0}C and 29 ^0C repeated patterns. In addition, the periodiograms parameters are between -1.5 and 1.5, however molded around -1.0 and 1.0 amplitude. Both Figs. 1 and 2 reveal the temperature noisy series incorporated into the periodogram model in colour blue.

Figs. 1-3 confirm the dominant apex area around sample frequency of 0.5. This suggested that the apex periodiogram parameters occurred at apex value for the sample frequency and the Fourier transformation adopt frequency values of zero to one. This resulted into $\omega = 0.5$, this connotes a periodical effect of $\frac{1}{0.5} = 2$ periods. This implies that two minutes period is required before completing a cycle of temperature changing from one particular ^{0}C to another. A more robust and bold peak was associated to the Fourier and known noisy sinusoidal periodograms (that is, a proper bell-shape for a normal curve) compare to a taper peak of the periodogram curve experienced via unknown noise but estimated by CRLB. The extraneous periodogram effect of frequency in the latter might be due to shrinking a consistency larger set to a normally distributed variate with the tendency of large variability and unknown variance; and when the timing of the frequency or amplitude changes periodically from longer to longest or vice versa (cyclical changes of patterns).



The posterior periodograms and frequencies of the two innovations of unknown and know noise where initiated by the

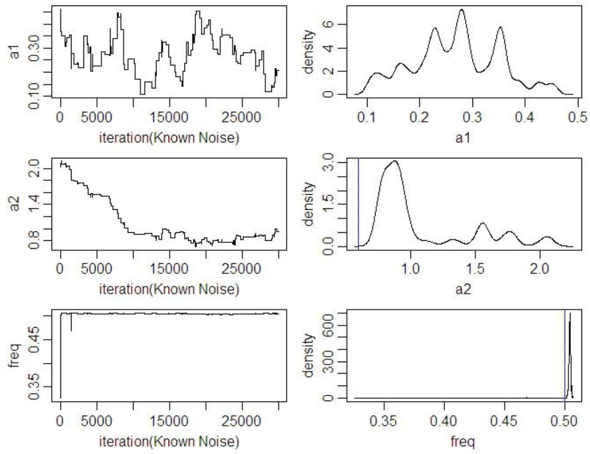


Fig. 4 (a) & (b) Posterior Periodogram Estimates of the Sinusoidal Model Unknown and Known Noise

TABLE I
MAXIMUM A POSTERIORI (MAP) OF PERIODOGRAMS AND PERFORMANCES OF THE BAYESIAN NOISY INNOVATIONS

	Known Noise			Unknown Noise		
	T. M	Per.MAP	S.E	T. M	Per.MAP	S.E
C	0.2688	0.4250	0.0025	0.8644	0.8292	0.0025
D	1.0834	0.7680	0.0025	0.4664	0.4321	0.0025
ω	0.5042	0.5029	0.01	0.4801	0.4647	0.01

Known Noise	Unknown Noise
BIC= 999.766	BIC= 960.1774
posMAP=19169	posMAP=12953
C-Prior=-0.5380	C-Prior= -0.5380
D-Prior= -0.4772	D-Prior=-0.4772
ω-Prior= -0.3152	ω-Prior= -0.3152

Index: S.E=Standard Error; BIC= Bayesian Information Criterion; Per.MAP=Periodiograms Maximum A Posteriori (MAP); T.M=True Means

$$f(t_i) = 0.8\cos(0.5t_i) + 0.6\sin(0.5t_i)$$

MCMC approach. The Bayesian posterior amplitudes of the known noise used the presume information (that is, the prior information) from the Fourier or Schuster periodograms as its conjugate prior. In other words, the noise (variance) of the data or for the Fourier transformation were assumed known. Unlike the Bayesian posterior amplitudes of the unknown noise, the noise (variance) of the data or for the Fourier transformation was assumed unknown. The Jeffrey′s non-conjugate prior via CRLB was adopted in circumvent an estimate for the noise. The MCMC converged to close form solution after the 25,000th iteration for the periodogram parameters. The blue dropdown line of the left hand side of the rectangular Figs. 4(a) and (b) to the index base are the estimates. Both converged to; $\{C, D, \omega\} = \{0.8, 0.6, 0.5\}$

From Table I, the means of periodogram parameters (C, D, ω), for the unknown noise generated by CRLB via Jeffrey′s prior (0.8644, 0.4664, 0.4801) are deeply centered on the range of values of the estimated posterior values compare to a slightly deviation from the periodogram parameter of D=1.0834 generated for the known noise. However, the standard errors of both the two innovations (unknown and known noise) are approximately the same. The overall posterior Maximum A Posteriori (posMAP), an index similar to the Maximum Likelihood (ML) of the classical approach is 19169 for known noise compare to a relatively small index of 12953 associated to the CRLB-Jeffrey′s prior for the unknown repeated pattern noisy series. The lower the posMAP of estimator(s), the lower the associated error of the model; the stability in the performance of the subjected model or incorporated data. This led to a miniature model performance of BIC of 960.1774 for the CRLB-Jeffrey′s prior for the unknown noise compared to a higher of 999.766 by the known noisy sinusoidal model, as shown in Fig. 5.

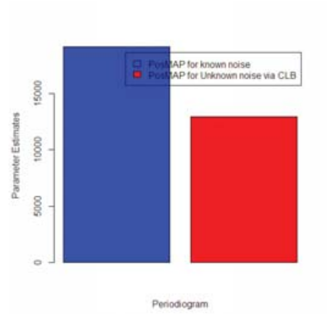


Fig. 5 Posterior Maximum A Posteriori for the Known and Unknown Noisy Sinusoidal model

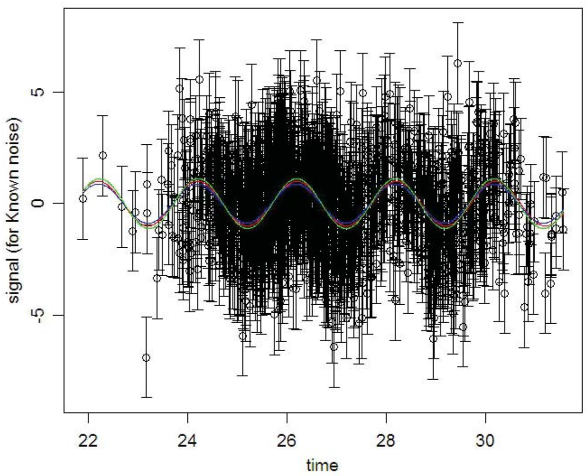


Fig. 6 Predicted ^{0}C for periodogram Model Unknown and Known Noise

Fig. 6 predicts the data in coated blue, green and red are for the known and unknown noise for the sinusoidal; and the temperature series itself respectively. However, the two sinusoidal lines give a near perfect prediction of the temperature series as if it was raw. This indicated a good signal capture of the periodogram model.

V. CONCLUSION

To resolve the limitation of choices of non-informative prior invariance associated to estimating periodograms (amplitudes) and frequency with Bayesian technique of sinusoidal model, known variance (noise) by Jeffrey′s prior of the repeated patterns has been used and adopted via analytical and marginal density approach. In case of an improper prior or unknown variance (noise), CRLB marginalized density approach of extracting the noise via Jeffrey′s prior is an ideal and alternative approach for large dataset and invariance structure for Bayesian sinusoidal or periodogram model.

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