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A Study of Two Disease Models: With and Without Incubation Period

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***Abstract*—**The incubation period is defined as the time from infection with a microorganism to development of symptoms. In this research, two disease models: one with incubation period and another without incubation period were studied. The study involves the use of a $S-I-S$ mathematical model with a single incubation period. The test for the existence and stability of the disease free and the endemic equilibrium states for both models were carried out. The fourth order Runge-Kutta method was used to solve both models numerically. Finally, a computer program in MATLAB was developed to run the numerical experiments. From the results, we are able to show that the endemic equilibrium state of the model with incubation period is locally asymptotically stable whereas the endemic equilibrium state of the model without incubation period is unstable under certain conditions on the given model parameters. It was also established that the disease free equilibrium states of the model with and without incubation period are locally asymptotically stable. Furthermore, results from numerical experiments using empirical data obtained from Nigeria Centre for Disease Control (NCDC) showed that the overall population of the infected people for the model with incubation period is higher than that without incubation period. We also established from the results obtained that as the transmission rate from susceptible to infected population increases, the peak values of the infected population for the model with incubation period decrease and are always less than those for the model without incubation period.

***Keywords*—**Asymptotic stability, incubation period, Routh-Hurwitz criterion, Runge Kutta method.

I. Introduction

THE current trend in the spread and causes of disease out breaks has made the mathematical study of epidemics imperative. Studies have come up with astonishing and quality number of models with explanations for epidemic outbreaks [1]. It is indeed a known fact that the high rate of deaths caused by diseases is presently on the alarming rate in the world. In recent years several studies have come up, which have not only explained various diseases due to socioeconomic aspects but gained triumphs for developing medicine [2].

The emergence of new diseases and reoccurence of old ones makes the idea for interdisciplinary involvement more pressing. Modeling disease infections is gaining great interest in the study of epidemiology. The main objective of modeling is to ascertain the role of infectious diseases in regulating natural population [3]. In the study of disease models, the entire population could be grouped into two categories which are namely; the susceptible and infected class. The infected population can transmit the infection to the susceptible one. In the $S-I-S$ models, the total population size at any instant is $N=S+\mathrm{I}$, where S is the susceptible population and I is the infected population at that point.

As the simple $S-I-S$ model suggests, the population from the susceptible class joins or transfers to the infected class continuously. But in reality this process is not always the case, in fact, in the case of any viral disease and many other diseases, the susceptible individual stays for some definite period after leaving the susceptible class and joining the infected class. This intermediate period may be termed as the incubation period. The incubation period is defined as the time from exposure to onset of disease and when limited to infectious disease, corresponds to the time from infection with a micro-organism to symptoms development [4].

A careful study of the role of incubation period in a disease model by assuming an intermediate class called the incubated population would be useful in understanding the treatment strategies required in tackling and eradicating disease epidemics [5]. Moreover, during an outbreak of a newly emerged directly transmitted disease, the incubation period distribution permits determination of the length of quarantine required for a potentially exposed individual (i.e., by restricting movement of an exposed individual for a duration sufficiently longer than the incubation period). Further, if the time lag between acquiring infectiousness and symptom onset appears long (i.e., if the incubation period is relatively long compared to the latent period), it implies that isolation measures (e.g. restriction of movement until the infectious individual loses infectiousness) are likely to be ineffective, complicating disease control [6].

In view of the above, the role of incubation period in a disease model is studied by assuming an intermediates class in the $S-I-S$ model which is the incubated class, I. The paper is presented as follows: Section II describes a "susceptible → infected → susceptible" and a "susceptible → incubation → infected → susceptible" mathematical model. In Section III, we studied the existence of the equilibrium system for both the model with and without incubation period. In Section IV, we studied the dynamical behavior and stability analysis of both models. In Section V, the result of the numerical analysis of both models was studied. Finally, in Sections VI and VII, we

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discussed the results of the numerical experiments and the came up with a conclusion on the study.

II. The Mathematical Model

The model by [2] considered the density at any time t of the susceptible and infected (or disease) populations to be $S(t)$ and $D(t)$, respectively. Let, $b =$ Disease contact rate, $\delta =$ Rate of removal from disease class due to natural death (natural death rate), $\gamma =$ Fraction of the infected population recovering from disease that will rejoin the susceptible class, $K =$ Carrying Capacity of the population, $r =$ Intrinsic growth rate of the population, $\mu =$ Death rate of the infected class due to infection.

Then the model without incubation period by [2] is explicitly given as follows:

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - bSD + \gamma D \tag{1}$$

$$\frac{dD}{dt} = bSD - \delta D - \gamma D - \mu D \tag{2}$$

with initial populations $S(0) > 0$, $D(0) > 0$ and $N(t) = S(t) + D(t)$.

Therefore, using the following transformations for the systems (1) and (2).

$$x = \frac{S}{K}; y = \frac{D}{K};\ \tau = rt$$

we get the following re-scaled system for the model without incubation,

$$\frac{dx}{d\tau} = x(1-x) - axy + yc \tag{3}$$

$$\frac{dy}{d\tau} = axz - ye \tag{4}$$

where,

$$a = \frac{bK}{r}; c = \frac{\gamma}{r}; e = \frac{\delta_1}{r}$$

and $x(0) > 0, y(0) > 0$.

The model by [2] also examined the case in which the susceptible class instead of going straight to the infected class but rather goes through a middle class called the incubated class.

Let, $\beta =$ Fraction of the incubated population that will go to the disease class.

Then the model with incubation period by [2] is explicitly given as follows:

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - bSD + \gamma D \tag{5}$$

$$\frac{dI}{dt} = bSD - \beta I - \delta I \tag{6}$$

$$\frac{dD}{dt} = \beta I - \delta D - \gamma D - \mu D \tag{7}$$

where,

$S(0) > 0, I(0) > 0, D(0) > 0$ and $N(t) = S(t) + I(t) + D(t)$

Also, using the below transformations for the systems (5)-(7)

$$x = \frac{S}{K}; y = \frac{I}{K}; z = \frac{D}{K};\ \tau = rt$$

we get the following re-scaled system for the model with incubation period:

$$\frac{dx}{d\tau} = x(1-x) - axz \tag{8}$$

$$\frac{dy}{d\tau} = axz - dy \tag{9}$$

$$\frac{dz}{d\tau} = d_1 y - ez \tag{10}$$

where,

$$a = \frac{bK}{r}; c = \frac{\gamma}{r}; d = \frac{\beta_1}{r};\ d_1 = \frac{\beta}{r}; e = \frac{\delta_1}{r}$$

and $x(0) > 0, y(0) > 0\ and\ z(0) > 0$.

In the next section, we will study the existence of all possible steady and endemic equilibrium states of both systems.

III. Existence of Equilibrium States

There are three feasible equilibrium states for systems (1) and (2), and (5)-(7), namely: Model without incubation, $E_0 = (0,0)$ is the trivial state; $E_1 = (K, 0)$ is the disease free state and $E_2 = (\bar{S}, \bar{D})$ is the endemic equilibrium state, where,

$$\bar{S} = \frac{\delta_1}{b} \quad \text{and} \quad \bar{D} = \frac{r\delta_1(\delta_1 - bK)}{b^2K(\gamma - \delta_1)} \tag{11}$$

It is clear from equation that $E_2 \in R_+^3$, if $\delta_1 - bK$ and $\gamma - \delta_1$ have the same sign and are non-zero.

Also, for model with incubation period we have, $E_0 = (0,0,0)$, $E_1 = (K, 0,0)$ and $E_2 = (\bar{S}, \bar{I}, \bar{D})$, where,

$$\bar{S} = \frac{MD}{b};\ \bar{I} = \frac{\delta_1 bK(\gamma b - rM)}{\beta(Kb^2C + rM^2)} \text{ and } \bar{D} = \frac{(\gamma b + rM)bK}{Kb^2M + rM^2} \tag{12}$$

and

$$M = \delta_1 + \frac{\delta\delta_1}{\beta}$$

Also, It is clear from (12) that $E_2 \in R_+^3, if\ b > \frac{rC}{\gamma}$

Theorem 1: For the model without incubation period, if $K, \delta_1, b, \gamma, r > 0$, then, there exist equilibrium state

$$E_0 = (0,0);\ E_1 = (K, 0) \text{ and } \quad E_2 = \left(\frac{\delta_1}{b}, \frac{r\delta_1(\delta_1 - bK)}{b^2K(\gamma - \delta_1)}\right)$$

provided that $\delta_1 > bK$ and $\gamma > \delta_1$.

Theorem 2: For the model with incubation period, if, $K, \delta, b, \beta, \beta_1, r, \gamma_1, \mu > 0$, then there exists equilibrium states

$$E_0 = (0,0,0)\ E_1 = (K,0,0)\ and\ E_2 = \left(\frac{MD}{b}, \frac{\delta_1 bK(\gamma b - rM)}{\beta(Kb^2M + rM^2)}, \frac{(\gamma b + rM)bK}{Kb^2M + rM^2}\right)$$

provided that $b > \frac{rM}{\gamma}$.

IV. Dynamical Behavior of the System

We have already established the fact that the system (1) and (2) has three equilibrium states as stated above in the previous section. The general Jacobian Matrix corresponding for systems (1) and (2) is given by

$$J = \begin{pmatrix} r - \frac{2r\bar{S}}{K} - b\bar{D} & -b\bar{S} + \gamma \\ b\bar{D} & b\bar{S} - \delta_1 \end{pmatrix} \quad (13)$$

Corresponding to the trivial steady state $E_0 = (0,0)$ the Jacobian matrix J in (13) has the following eigenvalues $\lambda_1 = r > 0$ or $\lambda_2 = -\delta_1 < 0$, thus the equilibrium state, $E_0 = (0,0)$ is unstable.

Again, evaluating the Jacobian matrix in (13) for the disease free equilibrium state, $E_1 = (K,0,)$, the following eigenvalues are obtained, $\lambda_1 = -r < 0$ or $\lambda_2 = bK - \delta_1 < 0$ provided that $bK < \delta_1$. Thus, we conclude that the equilibrium state $(K,0)$ is locally asymptotically stable since both eigenvalues are negative.

Finally, corresponding to the endemic equilibrium state

$$E_2\left(\frac{\delta_1}{b}, \frac{r\delta_1(\delta_1 - bK)}{b^2K(\gamma - \delta_1)}\right),$$

we have,

$$trace(J_{E_2}) = \frac{r(\gamma(bK - 2\delta_1) + {\delta_1}^2)}{bK(\gamma - \delta_1)} > 0,$$
$$bK < \delta_1 \text{ and } \gamma < \delta_1,$$

by Theorem 1.

$$det(J_{E_2}) = -\frac{r\delta_1(\delta_1 - bK) \times (\gamma - \delta_1)}{bK(\gamma - \delta_1)} = \frac{-r\delta_1(\delta_1 - bK)}{bK} = \frac{r\delta_1(bK - \delta_1)}{bK} < 0, \text{since}$$

$\delta_1 > bK$ by Theorem 1.

Therefore, we conclude, by Hartman-Grobman stability criterion that the equilibrium state

$$E_2 = \left(\frac{\delta_1}{b}, \frac{r\delta_1(\delta_1 - bK)}{b^2K(\gamma - \delta_1)}\right)$$

is an unstable equilibrium state.

We have also established that the system (5)-(7) has three equilibrium states as stated above in the previous section. The general Jacobian Matrix corresponding for systems (5)-(7) is given by

$$J = \begin{pmatrix} r - \frac{2r\bar{S}}{K} - b\bar{D} & 0 & \gamma - b\bar{S} \\ b\bar{D} & -\beta_1 & b\bar{S} \\ 0 & \beta & -\delta_1 \end{pmatrix} \quad (14)$$

Corresponding to the trivial steady state $E_0 = (0,0,0)$, the Jacobian matrix J in (14) has the following eigenvalues $\lambda_1 = r > 0, \lambda_2 = -\beta_1 < 0$ and $\lambda_3 = -\delta_1 < 0$, thus, we conclude that the equilibrium state, $E_0 = (0,0,0)$ is an unstable equilibrium state.

Again, corresponding to the disease free equilibrium state $E_0 = (K,0,0)$, the Jacobian matrix J in (14) has the following eigenvalues $\lambda_1 = -r < 0$ and $\lambda_{2,3}$ are the roots of the quadratic equation $\lambda^2 + \lambda(\beta + \delta_1) + \beta\delta_1 - \beta bK = 0$.

$$\lambda_2 = \frac{-(\beta + \delta_1) + \sqrt{(\beta + \delta_1)^2 - 4(\beta\delta_1 - \beta bK)}}{2}$$

If $bk < \delta_1$, then

$$\lambda_2 < \frac{-(\beta + \delta_1) + \sqrt{(\beta + \delta_1)^2}}{2} = 0$$

and

$$\lambda_3 < \frac{-(\beta + \delta_1) - \sqrt{(\beta + \delta_1)^2}}{2} < 0$$
$$\therefore \lambda_2, \quad \lambda_3 < 0.$$

Hence, we conclude, that the equilibrium state $E_1 = (K,0,0)$ is locally asymptotically stable since all the roots of the quadratic equation are negative.

Finally, corresponding to the endemic equilibrium state,

$$E_2 = \left(\frac{MD}{b}, \frac{\delta_1 bK(\gamma b - rM)}{\beta(Kb^2M + rM^2)}, \frac{(\gamma b + rM)bK}{Kb^2M + rM^2}\right)$$

The Jacobian matrix J in (14) has the following characteristics equation.

$$\lambda^3 - \lambda^2\left(r - \frac{2r\bar{S}}{K} - b\bar{D} - \beta_1 - \delta_1\right) - \lambda\left(\beta_1 r + \delta_1 r - \frac{2r\beta_1\bar{S}}{K} - \frac{2r\delta_1\bar{S}}{K} - \beta_1 b\bar{D} - b\delta_1\bar{D} - \beta_1\delta_1 + \beta b\bar{S}\right) - r\beta_1\delta_1 + \beta br\bar{S} + \frac{2r\beta_1\delta_1\bar{S}}{K} - \frac{2r\beta b\bar{S}^2}{K} + b\beta_1\delta_1\bar{D} - \beta b^2\bar{D}\bar{S} - b\gamma\beta_1\bar{D} + \beta b^2\bar{D}\bar{S} = 0$$

Clearly, the above equation is of the form

$$\lambda^3 + \lambda^2 A + \lambda B + C = 0 \quad (15)$$

where,

$$A = r - \frac{2r\bar{S}}{K} - b\bar{D} - \beta_1 - \delta_1$$
$$B = \beta_1 r + \delta_1 r - \frac{2r\beta_1\bar{S}}{K} - \frac{2r\delta_1\bar{S}}{K} - \beta_1 b\bar{D} - b\delta_1\bar{D} - \beta_1\delta_1 + \beta b\bar{S}$$

$$C = \beta b^2 \bar{D}\bar{S} + \beta br\bar{S} + \frac{2r\beta_1\delta_1\bar{S}}{K} - \frac{2r\beta b\bar{S}^2}{K} + b\beta_1\delta_1\bar{D} - r\beta_1\delta_1 - \beta b^2\bar{D}\bar{S} - b\gamma\beta_1\bar{D}$$

Now from Routh-Hurwitz criterion [7], a necessary and sufficient condition for all roots of (15) to have negative real part are:

i. $A, B, C > 0$ and

ii. $AB - C > 0$

Clearly, under stable choice of parameter values, we were able to establish the above stated two conditions, thus E_2 is locally asymptotically stable.

Theorem 3: Given that $K, \delta, b, \beta, \beta_1, r, \gamma_1, \mu > 0$, then the equilibrium states $E_0 = (0,0,0)$, $E_1 = (K, 0,0)$ and $E_2 = (\bar{S}, \bar{I}, \bar{D})$ of the model with incubation period is locally asymptotically stable under suitable choice of parameter values.

V. Numerical Experiments

The classical **Runge-Kutta** method (**RK4**) was used to solve for the numerical solution for the models with and without incubation period as represented in (1) and (2), and (5)-(7) respectively. Computer programs using in-built MATLAB function ode45 [8] was used for solving the systems of non-linear ODE. Let, b =Transmission rate from susceptible to infected population, $\gamma =$ Fraction of the disease population recovering from the disease that will again join the susceptible class $c = 0.01; d = 0.11;\ d_1 = 0.1$ and $e = 0.08$.

TABLE I
Parameter Values for the Numerical Experiments for Model without Incubation Period [9]

Parameter \ Case	1	2	3	4
b	0.001	0.002	0.003	0.004
K	500	500	500	500
r	0.1	0.1	0.1	0.1
γ	0.001	0.001	0.001	0.001
δ	0.006	0.006	0.006	0.006
μ	0.001	0.001	0.001	0.001
$S(0)$	200	200	200	200
$D(0)$	120	120	120	120

TABLE II
Parameter Values for the Numerical Experiments for Model with Incubation Period [9]

Parameter \ Case	1	2	3	4
b	0.001	0.002	0.003	0.004
K	500	500	500	500
r	0.1	0.1	0.1	0.1
β	0.011	0.011	0.011	0.011
δ	0.006	0.006	0.006	0.006
γ	0.001	0.001	0.001	0.001
μ	0.001	0.001	0.001	0.001
$S(0)$	200	200	200	200
$I(0)$	120	120	120	120
$D(0)$	20	20	20	20

From the data obtained [2] and based on some hypothetical values, some graphs will be plotted to investigate the effects of incubation period on the disease models as the transmission rate from susceptible to infected population increases.

In order to carry out the above task, the parameters $b, \mu, \delta, \beta, r, K, \gamma$ as defined in Section II are assigned some data values as well as the initial values for $S(0)$, $I(0)$ and $D(0)$ as shown in Tables I and II.

The numerical values generated from the various parameter values in Tables I and II and the figures obtained from the corresponding tables are given in Figs.1-4.

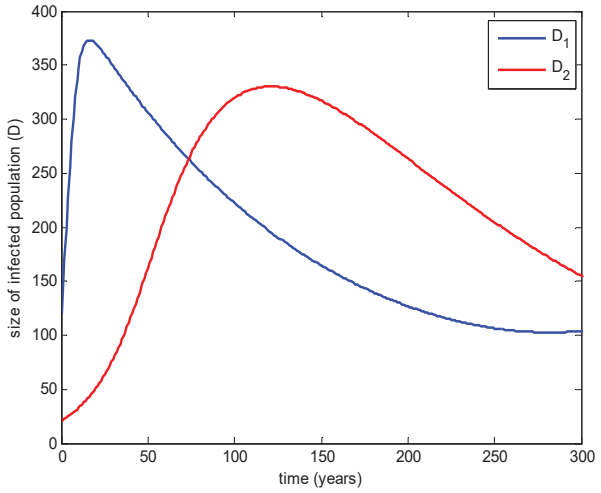

Fig. 1 The graph of the infected population for model without incubation (D_1) and with incubation (D_2) versus time at $b = 0.001$

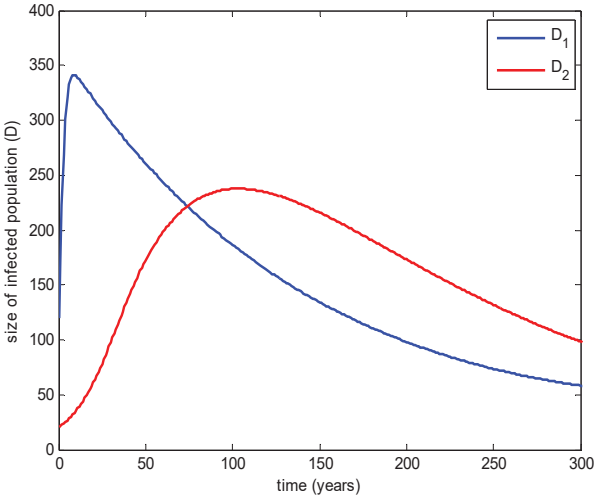

Fig. 2 The graph of the infected population for model without incubation (D_1) and with incubation (D_2) versus time at $b = 0.002$

VI. Results and Discussion

In this section, we will discuss the result of our numerical experiments carried out by increasing the transmission rate from susceptible to infected population at various parametric values. The parameter values for the experiments are shown in Tables I and II. The graphs, plotted using these results, are shown in Figs. 1-4.

Experiment one: Here, we investigate the situation where the transmission rate from susceptible to infected population, b

is 0.001. The result displayed in Fig. 1 showed that the population of the infected people for both models increases and peaked at different values before decreasing. At the initial stage, the population of the infected people for the model without incubation period is higher than that with incubation period but with time, the final results shows that the overall population of the infected people for the model with incubation period becomes higher than that without incubation period. Furthermore, the result also shows that the peak values of the population of the infected people for the model with incubation period (D_2) is 340 in Fig. 1 as compared to a peak value of 380 also in Fig. 1 for the model without incubation period (D_1).

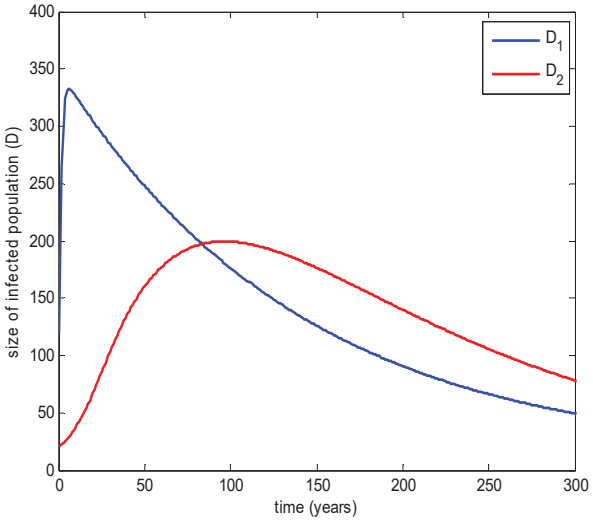

Fig. 3 The graph of the infected population for model without incubation (D_1) and with incubation (D_2) versus time at $b = 0.003$

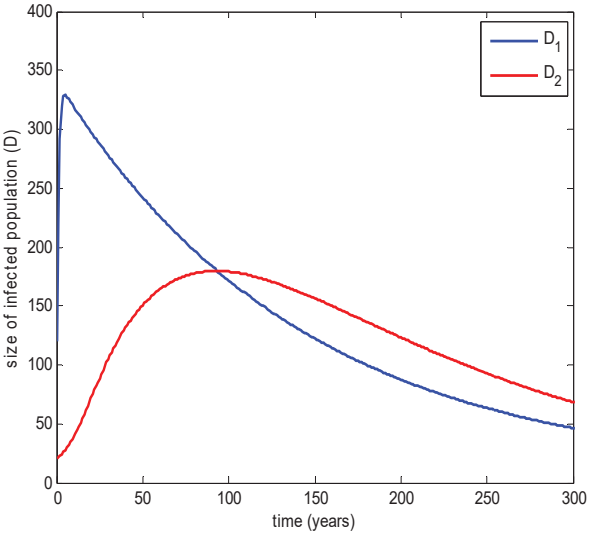

Fig. 4 The graph of the infected population for model without incubation (D_1) and with incubation (D_2) versus time at $b = 0.004$

Experiment two: In this case, we investigate the situation where the transmission rate from susceptible to infected population, b is 0.002. The result displayed in Fig. 2 showed that the population of the infected people for both models increases and peaked at different values before decreasing. At the initial stage, the population of the infected people for the model without incubation period is higher than that with incubation period but with time, the final results shows that the overall population of the infected people for the model with incubation period becomes higher than that without incubation period. Furthermore, the result also shows that the peak values of the population of the infected people for the model with incubation period (D_2) is 240 in Fig. 2 as compared to a peak value of 340 also in Fig. 2 for the model without incubation period (D_1).

Experiment three: Here, we investigate the situation where the transmission rate from susceptible to infected population, b is 0.003. The result displayed in Fig. 3 showed that the population of the infected people for both models increases and peaked at different values before decreasing. At the initial stage, the population of the infected people for the model without incubation period is higher than that with incubation period but with time, the final results shows that the overall population of the infected people for the model with incubation period becomes higher than that without incubation period. Furthermore, the result also shows that the peak values of the population of the infected people for the model with incubation period (D_2) is 200 in Fig. 3 as compared to a peak value of 340 also in Fig. 3 for the model without incubation period (D_1).

Experiment four: Here, we investigate the situation where the transmission rate from susceptible to infected population, b is 0.004. The result displayed in Fig. 4 showed that the population of the infected people for both models increases and peaked at different values before decreasing. At the initial stage, the population of the infected people for the model without incubation period is higher than that with incubation period but with time, the final results shows that the overall population of the infected people for the model with incubation period becomes higher than that without incubation period. Furthermore, the result also shows that the peak values of the population of the infected people for the model with incubation period (D_2) in Fig. 4 is 180 as compared to a peak value of 330 also in Fig. 4 for the model without incubation period (D_1).

VII. Conclusion

In this research work, we have reviewed and studied two disease models: the model without incubation period and the model with incubation period. The model parameters are given in Section III. Runge-Kutta method for numerical analysis was used in the numerical experiments.

From this study the following conclusions may be drawn:

1) The stability of the disease free state and the endemic equilibrium states for both models were studied and we are able to prove that the endemic equilibrium state of the model with incubation period is locally asymptotically stable (LAS) under certain conditions on the given model parameters whereas the endemic equilibrium state for the model without incubation period is unstable.

2) The Local Asymptotic Stability (LAS) of the disease free equilibrium state of the model with and without incubation period were ascertained.
3) Numerical experiments, using published data from the Nigeria Centre for Disease Control also shows that as the transmission rate from susceptible to infected population increases, the overall population of the infected people for the model with incubation period is higher than that without incubation period.
4) Numerical results also show that as the transmission rate from susceptible to infected population "*b*" increases, the peak values of the infected population for the model with incubation period decreases sharply and always less than the peak values of the infected population for the model without incubation period. This is as a result of the presence of incubation period which acts as a delay constant to the development of the disease symptoms.

REFERENCES

[1] Li, Z., Shuai, Z. S and Wang, K (2004). Persistence and Extinction of Single Population in a Populated Environment. Electronic Journal of Differential Equations, pp. 1-5.
[2] Dhar, J and Sharma, A. K. (2009). The Role of the Incubation Period in a Disease Model. Applied Mathematics E-Notes, Vol. 9, pp. 146-153.
[3] Watts, S. (1998). An Ancient Scourge: The End of Dracunculiasis in Egypt. Social Science and Medicine, pp. 811-819.
[4] Armenian, H. K and Lilienfeld, A. M. (1983). Incubation Period of Diseases. Epidemiology Review, Vol. 5, pp. 1–15.
[5] Hethcote, H., Zhien, M. and Shengbing, L. (2002). Effects of Quarantine in Six Endemic Models for Infectious Diseases. Elsevier Science Incorporated, Mathematical Biosciences, pp. 141-160.
[6] Nishiura, H. (2007). Early efforts in modeling the incubation period of infectious diseases with an acute cause of illness. Emerging Themes in Epidemiology, Vol. 4, No. 2.
[7] Allen, L. J. S. (2006). An Introduction to Mathematical Biology. Pearson Press, Chap. 4, pp. 150 - 151.
[8] Yang, W. Y., Cao, W., Chung, T. and Morris, J. (2005). Applied Numerical Methods Using MATLAB. A John Wiley & Sons, Inc. pp. 267-268.
[9] https://www.ncdc.gov.ng/reports/7/2016-september-week-37.