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Single Valued Neutrosophic Hesitant Fuzzy Rough Set and Its Application

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Abstract**—**In this paper, we proposed the notion of single valued neutrosophic hesitant fuzzy rough set, by combining single valued neutrosophic hesitant fuzzy set and rough set. The combination of single valued neutrosophic hesitant fuzzy set and rough set is a powerful tool for dealing with uncertainty, granularity and incompleteness of knowledge in information systems. We presented both definition and some basic properties of the proposed model. Finally, we gave a general approach which is applied to a decision making problem in disease diagnoses, and demonstrated the effectiveness of the approach by a numerical example.

Keywords**—**Single valued neutrosophic hesitant set, single valued neutrosophic hesitant relation, single valued neutrosophic hesitant fuzzy rough set, decision making method.

I. Introduction

THE notion of rough set theory has been proposed by Pawlak in 1982 [5] and the theory of fuzzy set proposed by Zadeh in 1965 [13], they are generalizations of the classical set theory. Rough set theory is a mathematical approach concerning uncertainty that comes from noisy, inexact or incomplete informations. In Zadeh's fuzzy set theory and the membership function play the important role, whereas in Pawlak's rough set theory and equivalence classes of a set are the significant part for the upper and lower approximations of the set.

As a generalization of fuzzy sets, intuitionistic fuzzy set [1] and the concept of neutrosophic set (NS) were introduced by Smarandache [6] in 1999. The concept of neutrosophic set handles indeterminate data whereas fuzzy set theory and intuitionistic fuzzy set theory failed when the relation is indeterminate. Neutrosophic set is described by three functions: true-membership function, indeterminacy-membership function and falsity-membership function that are connected independently. The neutrosophic set theory has been very successful in several areas, such as medical diagnosis, database, topology and decision making problem [3], [12]. While the neutrosophic set is an important tool for handing the indeterminate and inconsistent data, the theory of rough set is a powerful mathematics tool to deal with incompleteness.

Without a particular description, it is hard to use the NS in real scientific and different domain. Therefore, researchers presented the interval neutrosophic set (INS) [8], multi-valued neutrosophic set (MVNS) [4] and rough neutrosophic set (RNS) [2].

Wang et al. [9] proposed single valued neutrosophic set (SVNSs) by simplifying NSs. SVNSs can also be considered as an extension of intuitionistic fuzzy sets, in which the three membership functions are unrelated and their function values belong to the unit closed interval.

As another generalization of fuzzy sets, the hesitant fuzzy set (HF) was defined by Torra [7], which allows its membership function to have a set of possible values. Hesitant fuzzy set is also important concept used to deal with imperfect information [10]. By combining the advantages of the SVNS and HFS, Ye [12] introduced the notion of single valued neutrosophic hesitant fuzzy set (SVNHFS) which allows its membership function to have sets of possible values, which are called truth, indeterminacy, and falsity membership hesitant functions and discussed some properties of SVNHFS to solve multiple attribute decision making problems. In addition, many researchers have studied hesitant fuzzy decision making problems by utilizing plenty of classical decision making tools. Among them, since the rough set approach owns advantages in attribute selection, we aim to deal with the situation by virtue the rough set theory.

In this paper, we apply rough set model to decision making involving single valued neutrosophic hesitant fuzzy sets. Moreover, we also propose an illustrative example to interpret the basic principal and method of the application of the rough set model in single valued neutrosophic hesitant fuzzy decision making.

Section II recalls some basic concepts of rough sets, single valued neutrosophic hesitant fuzzy sets. In Section III, we present rough set model based on SVNHF relation over two universes and examine some properties of this model. In Section IV, we establish a general approach to decision making based on SVNHF rough set over two universes. Section IV illustrates the principal steps of the proposed decision method by a numerical example. Finally, in Section VI we conclude the paper with a summary and outlook for further research.

II. Preliminaries

In this section we recall some basic notions and properties which will be used in this paper.

A. Pawlak Rough Sets

Let U be a non-empty finite universe, R be an equivalence relation on U. We use U/R to denote the family of all equivalence classes of R (or classifications of U), and $[x]_R$

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to denote an equivalence class of R containing the element $x \in U$. The pair (U, R) is called an approximation space. For any $X \subseteq U$, we can define the lower and upper approximations of X [5] as:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}.$$

The pair $(\underline{R}(X), \overline{R}(X))$ is referred to as the rough set of X. The rough set $(\underline{R}(X), \overline{R}(X))$ gives rise to a description of X under the present knowledge, i.e., the classification of U. Furthermore, the positive region, negative region, and boundary region of X about the approximation space (U, R) are defined as follows, respectively,

$$pos(X) = \underline{R}(X) \quad neg(X) = \sim \overline{R}(X), \quad bn(X) = \overline{R}(X) - \underline{R}(X),$$

where $\sim X$ stands for complementation of the set X.

B. Single Valued Neutrosophic Hesitant Fuzzy Sets

Wang et al. [7] proposed the concept of single valued neutrosophic sets defined as follows:

Definition 1 [9]. Let U be a space of points (objects), with a generic element in U denoted by x. A SVNS A in U is characterized by three membership functions, truth membership function T_A, an indeterminacy membership function I_A and falsity membership function F_A where $\forall x \in U$ $, T_A(x), I_A(x), F_A(x) \in [0, 1]$.

Ye [10] defined the concept of single valued neutrosophic hesitant fuzzy sets (SVNHFS), which is an extension of hesitant fuzzy set.

Definition 2 [12]. Let X be a non-empty fixed set, a SVNHFS on X is expressed by:

$$N = \{(x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x) | x \in X)\},$$

where $\tilde{t}(x) = \{\gamma | \gamma \in \tilde{t}(x)\}$, $\tilde{i}(x) = \{\delta | \delta \in \tilde{i}(x)\}$,$\tilde{f}(x) = \{\eta | \eta \in \tilde{f}(x)\}$ are three sets with value in $[0, 1]$, representing truth, indeterminacy and falsity membership hesitant degrees of the element $x \in X$, which satisfy limits: $\gamma \in [0, 1]$, $\delta \in [0, 1]$,$\eta \in [0, 1]$ and $0 \leq$ sup γ + sup δ + sup $\eta \geq 3$.

For any $y \in U$, several special SVNHF sets $1_y, 1_U - y$ and 1_M are defined, respectively, as follows: For $x \in U, M \subseteq U$,

$$\tilde{t}_{1_y}(x) = \begin{cases} 1 & \text{for} \quad x = y \\ 0 & \text{for} \quad x \neq y \end{cases} \quad \tilde{i}_{1_y}(x) = \begin{cases} 0 & \text{for} \quad x = y \\ 1 & \text{for} \quad x \neq y \end{cases}$$

$$\tilde{f}_{1_y}(x) = \begin{cases} 0 & \text{for} \quad x = y \\ 1 & \text{for} \quad x \neq y \end{cases}$$

$$\tilde{t}_{1_{U-y}}(x) = \begin{cases} 0 & \text{for} \quad x = y \\ 1 & \text{for} \quad x \neq y \end{cases} \quad \tilde{i}_{1_{U-y}}(x) = \begin{cases} 1 & \text{for} \quad x = y \\ 0 & \text{for} \quad x \neq y \end{cases}$$

$$\tilde{f}_{1_{U-y}}(x) = \begin{cases} 1 & \text{for} \quad x = y \\ 0 & \text{for} \quad x \neq y \end{cases}$$

$$\tilde{t}_{1_M}(x) = \begin{cases} 1 & \text{for} \quad x \in M \\ 0 & \text{for} \quad otherwise \end{cases} \quad \tilde{i}_{1_M}(x) = \begin{cases} 0 & \text{for} \quad x \in M \\ 1 & \text{for} \quad otherwise \end{cases}$$

$$\tilde{f}_{1_M}(x) = \begin{cases} 0 & \text{for} \quad x \in M \\ 1 & \text{for} \quad otherwise \end{cases}$$

Let U be a discrete universe of discourse, A and B be two SVNHF sets on U denoted as $A = \{(x, \tilde{t_A}(x), \tilde{i_A}(x), \tilde{f_A}(x) | x \in U)\}$and $B = \{(x, \tilde{t_A}(x), \tilde{i_A}(x), \tilde{f_A}(x) | x \in U)\}$, respectively. It should be noted that the number of values in different SVNHF elements may be different and the values of SVNHF element are usually given in a disorder. Suppose that $\ell(\tilde{t_A}(x))$, $\ell(\tilde{i_A}(x))$ and $\ell(\tilde{f_A}(x))$ stands for the number of values in $\tilde{t_A}(x)$, $\tilde{t_A}(x)$ and $\tilde{t_A}(x)$, respectively.

Some basic operations of SVNHFEs are defined by Ye [10], as

Definition 3 [12]. Let $\tilde{n_1} = (\tilde{t_1}, \tilde{i_1}, \tilde{f_1})$ and $\tilde{n_2} = (\tilde{t_2}, \tilde{i_2}, \tilde{f_2})$ be two SVNHFEs, then:

1) $\tilde{n_1} \cup \tilde{n_2} = \{\tilde{t_1} \cup \tilde{t_2}, \tilde{t_1} \cap \tilde{t_2}, \tilde{t_1} \cap \tilde{t_2}\}$;
2) $\tilde{n_1} \cap \tilde{n_2} = \{\tilde{t_1} \cap \tilde{t_2}, \tilde{t_1} \cup \tilde{t_2}, \tilde{t_1} \cup \tilde{t_2}\}$;
3) $\tilde{n_1} \oplus \tilde{n_2} = \bigcup_{\gamma_1 \in \tilde{t_1}, \delta_1 \in \tilde{i_1}, \eta_1 \in \tilde{f_1} \gamma_2 \in \tilde{t_2}, \delta_2 \in \tilde{i_2}, \eta_2 \in \tilde{f_2}} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2, \delta_1\delta_2, \eta_1\eta_2\}$
4) $\tilde{n_1} \otimes \tilde{n_2} = \bigcup_{\gamma_1 \in \tilde{t_1}, \delta_1 \in \tilde{i_1}, \eta_1 \in \tilde{f_1} \gamma_2 \in \tilde{t_2}, \delta_2 \in \tilde{i_2}, \eta_2 \in \tilde{f_2}} \{\gamma_1\gamma_2, \delta_1 + \delta_2 - \delta_1\delta_2, \eta_1 + \eta_2 - \eta_1\eta_2\}$
5) $K\tilde{n_1} = \bigcup_{\gamma_1 \in \tilde{t_1}, \delta_1 \in \tilde{i_1}, \eta_1 \in \tilde{f_1}} \{1 - (1 - \gamma_1)^k, \delta_1^k, \eta_1^k\}$
6) $\tilde{n_1}^k = \bigcup_{\gamma_1 \in \tilde{t_1}, \delta_1 \in \tilde{i_1}, \eta_1 \in \tilde{f_1}} \{\gamma_1^k, 1 - (1 - \delta_1)^k, 1 - (1 - \eta_1)^k\}$

III. Single Valued Neutrosophic Hesitant Fuzzy Rough Sets

Yang et al. [9] proposed a hesitant fuzzy relation as:

Definition 4 [11]. Let U be a nonempty and finite universe. A hesitant fuzzy relation R over U is a hesitant fuzzy subset such that $R \in HF(U \times U)$ where $R = \{(x, y), h_R(x, y) | (x, y) \in U \times U\}$. For all $(x, y) \in U \times U$, $h_R(x, y)$ is a set of the values in $[0, 1]$, which is used to denote the possible membership degrees of the relationships between x and y.

Inspired by the concept of the hesitant fuzzy relation, we will further extend the hesitant fuzzy relation into SVNHF environment, and introduce the concept of SVNHF relation over two universes which is used to construct SVNHF rough approximation operators. Firstly, we present the concept of a SVNHF relation as:

Definition 5. Let U, V be two nonempty and finite universes. A SVNHF subset R of the universe $U \times V$ is called a SVNHF relation from U to V, namely, R is given by $R = \{(x, y), \tilde{t}_R(x, y), \tilde{i}_R(x, y), \tilde{f}_R(x, y) | (x, y) \in U \times V\}$ where $\tilde{t}_R, \tilde{i}_R, \tilde{f}_R : U \times V \longrightarrow [0, 1]$ are triple sets of some values in $[0, 1]$, denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the relationships between x and y, respectively, with the conditions: $0 \leq \gamma, \delta, \eta \geq 1$ and $0 \leq \gamma^+ + \delta^+ + \eta^+ \geq 3$, where for all $(x, y) \in U \times V, \gamma \in \tilde{t}_R(x, y)$, $\delta \in \tilde{i}_R(x, y)$, $\eta \in \tilde{f}_R(x, y)$, $\gamma^+ \in \tilde{t}_R^+(x, y) = \cup_{\gamma \in \tilde{t}_R(x,y)} max\{\gamma\}$, $\delta^+ \in \tilde{i}_R^+(x, y) = \cup_{\delta \in \tilde{i}_R(x,y)} max\{\delta\}$, $\eta^+ \in \tilde{f}_R^+(x, y) = \cup_{\eta \in \tilde{f}_R(x,y)} max\{\eta\}$.

Definition 6. The SVNHF relation R from U to V is said to be serial if for each $x \in U$, there exists a $y \in V$ such that $\tilde{t}_R(x, y) = \{1\} and \tilde{i}_R(x, y) = \tilde{f}_R(x, y) = 0$, R is said to be reflexive on U if $\tilde{t}_R(x, x) = \{1\} and \tilde{i}_R(x, x) = \tilde{f}_R(x, x) = 0$ for all $x \in U, R$ is referred to as a symmetric SVNHF relation on U if $\tilde{t}_R(x, y) = \tilde{t}_R(y, x), \tilde{i}_R(x, y) =$

$\tilde{i}_R(y,x) and \tilde{f}_R(x,y) = \tilde{f}_R(y,x) forall 4 x, y \in U$, R is said to be transitive on U if $\bigvee_{y\in U}\{\tilde{t}_R(x,y) \wedge \tilde{t}_R(y,z)\} \leq \tilde{t}_R(x,z)$, $\bigwedge_{y\in U}\{\tilde{i}_R(x,y) \vee \tilde{i}_R(y,z)\} \leq \tilde{i}_R(x,z)$ and $\bigwedge_{y\in U}\{\tilde{f}_R(x,y) \vee \tilde{f}_R(y,z)\} \leq \tilde{f}_R(x,z)$ for all $x, z \in U$.

Alternatively, R is transitive if the following conditions are satisfied:
$\bigvee_{y\in U}\{\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_R^{\sigma(s)}(y,z)\} \leq \tilde{t}_R^{\sigma(s)}(x,z)$, $1 \leq s \leq k$;
$\bigwedge_{y\in U}\{\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_R^{\sigma(t)}(y,z)\} \leq \tilde{i}_R^{\sigma(t)}(x,z)$, $1 \leq t \leq m$;
$\bigwedge_{y\in U}\{\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_R^{\sigma(v)}(y,z)\} \leq \tilde{f}_R^{\sigma(v)}(x,z)$, $1 \leq v \leq n$.
where $\tilde{t}^{\sigma(s)}$ denote the sth largest value in $\tilde{t}$, $\tilde{i}^{\sigma(t)}$ denote the tth largest value in $\tilde{i}$ and $\tilde{f}^{\sigma(v)}$ denote the vth largest value in $\tilde{f}$, $k = max\{\ell(\tilde{t}_R(x,y)), \ell(\tilde{t}_R(y,z)), \ell(\tilde{t}_R(x,z))\}$, $m = max\{\ell(\tilde{i}_R(x,y)), \ell(\tilde{i}_R(y,z)), \ell(\tilde{i}_R(x,z))\}$ and $n = max\{\ell(\tilde{f}_R(x,y)), \ell(\tilde{f}_R(y,z)), \ell(\tilde{f}_R(x,z))\}$ Based on the above SVNHF relation, lower and upper SVNHF approximation operators are defined as:

Definition 7. Let U and V be two nonempty and finite universes and R be SVNHF relation from U to V. The triple (U,V,R) is called a SVNHF approximations space. For any $A \in SVNHF(V)$ the lower and upper approximations of A with respect to (U,V,R), denote by $\underline{R}(A)$ and $\overline{R}(A)$ are two SVNHF sets of U and are, respectively, defined as:
$\underline{R}(A) = \{x, < \tilde{t}_{\underline{R}}(x), \tilde{i}_{\underline{R}}(x), \tilde{f}_{\underline{R}}(x) > | x \in U\}$
$\overline{R}(A) = \{x, < \tilde{t}_{\overline{R}}(x), \tilde{i}_{\overline{R}}(x), \tilde{f}_{\overline{R}}(x) > | x \in U\}$
where
$\tilde{t}_{\underline{R}(A)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{t}_A(y)\}$,
$\tilde{i}_{\underline{R}(A)}(x) = \bigvee_{y\in V}\{\tilde{i^c}_R(x,y) \wedge \tilde{i}_A(y)\} = \bigvee_{y\in V}\{1 - \tilde{i}_R(x,y) \wedge \tilde{i}_A(y)\}$,
$\tilde{f}_{\underline{R}(A)}(x) = \bigvee_{y\in V}\{\tilde{t}_R(x,y) \wedge \tilde{f}_A(y)\}$,
$\tilde{t}_{\overline{R}(A)}(x) = \bigvee_{y\in V}\{\tilde{t}_R(x,y) \wedge \tilde{t}_A(y)\}$,
$\tilde{i}_{\overline{R}(A)}(x) = \bigwedge_{y\in V}\{\tilde{i}_R(x,y) \vee \tilde{i}_A(y)\}$,
$\tilde{f}_{\overline{R}(A)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{f}_A(y)\}$,

The pair $(\underline{R}(A), \overline{R}(A))$ is called the SVNHF rough set of A with respect to (U,V,R), and $\underline{R}(A), \overline{R}(A) : SVNHF(V) \longrightarrow SVNHF(U)$ are referred to as lower and upper SVNHF rough approximation operators,respectively.

Clearly, the above definition implies equivalences of the following form:

$\tilde{t}_{\underline{R}(A)}(x) = \bigwedge_{y\in V} \tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y) | 1 \leq s \leq max\{\ell(\tilde{f}_R(x,y)), \ell(\tilde{t}_A(y))\}$,
$\tilde{i}_{\underline{R}(A)}(x) = \bigvee_{y\in V}\{1 - \tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y) | 1 \leq t \leq max\{\ell(1 - \tilde{i}_R(x,y)), \ell(\tilde{i}_A(y))\}\}$,
$\tilde{f}_{\underline{R}(A)}(x) = \bigvee_{y\in V}\{\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y) | 1 \leq v \leq max\{\ell(\tilde{t}_R(x,y)), \ell(\tilde{f}_A(y))\}\}$,
$\tilde{t}_{\overline{R}(A)}(x) = \bigvee_{y\in V}\{\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_A^{\sigma(s)}(y) | 1 \leq s \leq max\{\ell(\tilde{t}_R(x,y)), \ell(\tilde{t}_A(y)\}$,
$\tilde{i}_{\overline{R}(A)}(x) = \bigwedge_{y\in V}\{\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_A^{\sigma(t)}(y) | 1 \leq t \leq max\{\ell(\tilde{i}_R(x,y)), \ell(\tilde{i}_A(y)\}$,
$\tilde{f}_{\overline{R}(A)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_A^{\sigma(v)}(y) | 1 \leq v \leq max\{\ell(\tilde{f}_R(x,y)), \ell(\tilde{f}_A(y)\}$.
where the $\ell(.)$ stands for the number of values in hesitant fuzzy elements.

Definition 8. Let U be a nonempty and finite universe of discourse. Denote $k = max\{\ell(\tilde{t}_A(x), \ell(\tilde{t}_B(x)\}$, $m = max\{\ell(\tilde{i}_A(x)), \ell(\tilde{i}_B(x))\}$ and $n = max\{\ell(\tilde{f}_A(x), \ell(\tilde{f}_B(x)\}$ $\forall A, B \in SVNHF(V)$, A is said to be a SVNHF subset of B, if $\tilde{t}_A(y) \leq \tilde{t}_B(y)$, $\tilde{i}_A(y) \geq \tilde{i}_B(y)$ and $\tilde{f}_A(y) \geq \tilde{f}_B(y)$ hold for any $x \in U$; such that
$\tilde{t}_A(y) \leq \tilde{t}_B(y)$, $\tilde{i}_A(y) \geq \tilde{i}_B(y)$ and $\tilde{f}_A(y) \geq \tilde{f}_B(y) \Longleftrightarrow$
$\tilde{t}_A^{\sigma(s)}(y) \leq \tilde{t}_B^{\sigma(s)}(y)$, $\tilde{i}_A^{\sigma(t)}(y) \geq \tilde{i}_B^{\sigma(t)}(y)$ and $\tilde{f}_A^{\sigma(v)}(y) \geq \tilde{f}_B^{\sigma(v)}(y)$ with $1 \leq s \leq k$, $1 \leq t \leq m$ and $1 \leq v \leq n$ we denote it by $A \subseteq B$.

Theorem 1. Let (U, V, R) be a single valued neutrosophic hesitant fuzzy approximation space over two universes. Then the lower and upper SVNHF rough approximation operators induced from (U, V, R) satisfy the following properties for all $A, B \in SVNHF(V)$

1) $(SVNHFL_1)\underline{R}(A^c) = (\overline{R}(A))^c$
2) $(SVNHFU_1)\overline{R}(A^c) = (\underline{R}(A^c))^c$
3) $(SVNHFL_2)A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$
4) $(SVNHFU_2)A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$
5) $(SVNHFL_3)\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$
6) $(SVNHFU_3)\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$
7) $(SVNHFL_4)\underline{R}(V) = U$
8) $(SVNHFU_4)\overline{R}(\phi) = \phi$

Proof.

1) $(SVNHFL_1)$ For all $x \in U$, we have,
$\tilde{t}_{\underline{R}(A^c)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{t}_{A^c}(y)\}$
$= \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{f}_A(y)\}$
$= \bigwedge_{y\in V}\{\tilde{f^{\sigma(v)}}_R(x,y) \vee \tilde{f^{\sigma(v)}}_A(y)\}$
$= \tilde{f}_{\overline{R}(A)}(x) = \tilde{t}_{(\overline{R}(A))^c}(x)$

$\tilde{i}_{\underline{R}(A^c)}(x) = \bigvee_{y\in V}\{1 - \tilde{i}_R(x,y) \wedge \tilde{i}_{A^c}(y)\}$=$\bigvee_{y\in V}\{1 - \tilde{i^{\sigma(t)}}_R(x,y) \wedge \tilde{i^{\sigma(t)}}_{A^c}(y)\}$
$= \tilde{i}_{(\overline{R}(A))^c}(x)$
$\tilde{f}_{\underline{R}(A^c)}(x) = \bigvee_{y\in V}\{\tilde{t}_R(x,y) \wedge \tilde{f}_{A^c}(y)\}$
$= \bigvee_{y\in V}\{\tilde{t}_R(x,y) \wedge \tilde{t}_A(y)\}$
$= \bigvee_{y\in V}\{\tilde{t^{\sigma(s)}}_R(x,y) \wedge \tilde{t^{\sigma(s)}}_A(y)\}$
$\tilde{t}_{\overline{R}(A)}(x) = \tilde{f}_{(\overline{R}(A))^c}(x)$

From above discussions, it follows that $\underline{R}(A^c) = (\overline{R}(A))^c$

2) $(SVNHFL_2)$ Since $A \subseteq B$ then, by Definition 8, we have
$\tilde{t}_A^{\sigma(s)}(y) \leq \tilde{t}_B^{\sigma(s)}(y)$, $\tilde{i}_A^{\sigma(t)}(y) \geq \tilde{i}_B^{\sigma(t)}(y)$ and $\tilde{f}_A^{\sigma(v)}(y) \geq \tilde{f}_B^{\sigma(v)}(y)$ with $1 \leq s \leq k$, $1 \leq t \leq m$ and $1 \leq v \leq n$ for all $y \in U$. So it follows that
$\bigwedge_{y\in V} \tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y) \leq \bigwedge_{y\in V} \tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_B^{\sigma(s)}(y)$
$\bigvee_{y\in V}\{1 - \tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y) \geq \bigvee_{y\in V}\{1 - \tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_B^{\sigma(t)}(y)$
$\bigvee_{y\in V}\{\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y) \geq \bigvee_{y\in V}\{\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_B^{\sigma(v)}(y)$
hence, for each $x \in U$, we conclude that $\tilde{t}_{\underline{R}(A)}(x) \leq \tilde{t}_{\underline{R}(B)}(x)$, $\tilde{i}_{\underline{R}(A)}(x) \geq \tilde{i}_{\underline{R}(B)}(x)$ and $\tilde{f}_{\underline{R}(A)}(x) \geq \tilde{f}_{\underline{R}(B)}(x)$.
Consequently, $\underline{R}(A) \subseteq \underline{R}(B)$

3) $(SVNHFL_3)$ For all $x \in U$, we have
$\tilde{t}_{\underline{R}(A\cap B)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{t}_{A\cap B}(y)\}$
$= \bigwedge_{y\in V}\{\tilde{f}_R(x,y)\vee(\tilde{t}_A(y)\wedge\tilde{t}_B(y))\}$
$= \bigwedge_{y\in V}\{\tilde{f}_R^{\sigma(s)}(x,y) \vee (\tilde{t}_A^{\sigma(s)}(y)\wedge \tilde{t}_B^{\sigma(s)}(y))\}, s = 1,2,....,k$
$\left\{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(s)}(x,y)\vee\tilde{t}_A^{\sigma(s)}(y))\right\} \wedge \left\{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(s)}(x,y)\vee\tilde{t}_B^{\sigma(s)}(y))\right\}, s=1,2,....,k$
$= \tilde{t}_{\underline{R}(A)}(x)\cap\tilde{t}_{\underline{R}(B)}(x) = \tilde{t}_{\underline{R}(A\cap B)}(x)$
$\tilde{i}_{\underline{R}(A\cap B)}(x) = \bigvee_{y\in V}\{1-\tilde{i}_R(x,y)\wedge\tilde{i}_{A\cap B}(y)\}$
$=\bigvee_{y\in V}\{1-\tilde{i}_R(x,y)\wedge(\tilde{i}_A(y)\vee\tilde{i}_B(y))\}$
$=\bigvee_{y\in V}\{1-\tilde{i}_R^{\sigma(t)}(x,y)\wedge(\tilde{i}_A^{\sigma(t)}(y)\vee\tilde{i}_B^{\sigma(t)}(y))\}, t=1,2,....,m$
$\left\{\bigvee_{y\in V}(1-\tilde{i}_R^{\sigma(t)}(x,y)\wedge\tilde{t}_A^{\sigma(t)}(y))\right\}\vee\left\{\bigvee_{y\in V}(\tilde{t}_{R^c}^{\sigma(t)}(x,y)\vee\tilde{t}_B^{\sigma(t)}(y))\right\}, t=1,2,....,m$
$=\tilde{i}_{\underline{R}(A)}(x)\vee\tilde{i}_{\underline{R}(B)}(x)=\tilde{i}_{\underline{R}(A\cap B)}(x)$

$\tilde{f}_{\underline{R}(A\cap B)}(x) = \bigvee_{y\in V}\{\tilde{t}_R(x,y)\wedge\tilde{f}_{A\cap B}(y)\}$
$=\bigvee_{y\in V}\{\tilde{t}_R(x,y)\wedge(\tilde{f}_A(y)\vee\tilde{f}_B(y))\}$
$=\bigvee_{y\in V}\{\tilde{t}_R^{\sigma(v)}(x,y)\wedge(\tilde{f}_A^{\sigma(v)}(y)\vee t\tilde{f}_B^{\sigma(v)}(y))\}, v=1,2,....,n$
$\left\{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(v)}(x,y)\wedge\tilde{f}_A^{\sigma(v)}(y))\right\}\vee\left\{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(v)}(x,y)\vee\tilde{f}_B^{\sigma(v)}(y))\right\}, v=1,2,....,n$
$=\tilde{f}_{\underline{R}(A)}(x)\vee\tilde{f}_{\underline{R}(B)}(x)=\tilde{f}_{\underline{R}(A\cap B)}(x)$

Where $k = max\{\ell(\tilde{t}_R(x,y)),\ell(\tilde{t}_A(y)),\ell(\tilde{t}_B(x,z))\}$, $m = max\{\ell(1-\tilde{i}_R(x,y)),\ell(\tilde{i}_A(y)),\ell(\tilde{i}_B(y))\}$ and $n = max\{\ell(\tilde{f}_R(x,y)),\ell(\tilde{f}_A(y)),\ell(\tilde{f}_B(y))\}$
Hence, it follows that $(SVNHFL_4)$ holds.

4) $(SVNHFL_4)$ It is easy to prove.

Theorem 2. Let U, V be two nonempty and finite universes. Suppose that R_1 and R_2 are two SVNHF relations from U to V, if $R_1 \subseteq R_2$ then the following holds:

1) $\underline{R_1}(A) \supseteq \underline{R_2}(A), \forall A \in SVNHF(V)$
2) $\overline{R_1}(A) \subseteq \overline{R_2}(A), \forall A \in SVNHF(V)$

Proof.

1) Since $R_1 \subseteq R_2$, then for any $(x,y)\in U\times V$, we have $\tilde{t}_{R_1}^{\sigma(s)}(x,y)\le\tilde{t}_{R_2}^{\sigma(s)}(x,y)$, $\tilde{i}_{R_1}^{\sigma(t)}(x,y)\ge\tilde{i}_{R_2}^{\sigma(t)}(x,y)$ and $\tilde{f}_{R_1}^{\sigma(v)}(x,y)\ge\tilde{f}_{R_2}^{\sigma(v)}(x,y)$ with $1\le s\le k$, $1\le t\le m$ and $1\le v\le n$ for all $y\in U$
$\tilde{t}_{\underline{R_1}(A)}(x)=\bigwedge_{y\in V}\{\tilde{f}_{R_1}(x,y)\vee\tilde{t}_A(y)\}$
$=\bigwedge_{y\in V}\{\tilde{f}_{R_1}^{\sigma(s)}(x,y)\vee\tilde{t}_A^{\sigma(s)}(y)|s=1,2,....,k\}$
$\ge\bigwedge_{y\in V}\{\tilde{f}_{R_2}^{\sigma(s)}(x,y)\vee\tilde{t}_A^{\sigma(s)}(y)|s=1,2,....,k\} = \tilde{t}_{\underline{R_2}(A)}(x)$
$\tilde{i}_{\underline{R_1}(A)}(x) = \bigvee_{y\in V}\{1-\tilde{i}_{R_1}(x,y)\wedge\tilde{i}_A(y)|t=1,2,....,m\} = \bigvee_{y\in V}\{1-\tilde{i}_{R_1}^{\sigma(t)}(x,y)\wedge\tilde{i}_A^{\sigma(t)}(y)|t=1,2,....,m\}\le\bigvee_{y\in V}\{1-\tilde{i}_{R_2}^{\sigma(t)}(x,y)\wedge\tilde{i}_A^{\sigma(t)}(y)|t=1,2,....,m\}=\tilde{i}_{\underline{R_2}(A)}(x)$
$\tilde{f}_{\underline{R_1}(A)}(x)=\bigvee_{y\in V}\{\tilde{t}_{R_1}(x,y)\wedge\tilde{f}_A(y)|v=1,2,....,n\} = \bigvee_{y\in V}\{\tilde{t}_{R_1}^{\sigma(v)}(x,y)\wedge\tilde{f}_A^{\sigma(v)}(y)|v=1,2,....,n\}\le\bigvee_{y\in V}\{\tilde{t}_{R_2}^{\sigma(t)}(x,y)\wedge\tilde{f}_A^{\sigma(t)}(y)|v=1,2,....,n\}=\tilde{f}_{\underline{R_2}(A)}(x)$
Hence, it follows that $\underline{R_1}(A)\supseteq\underline{R_2}(A)$ holds.

2) It follows immediately from the above result (1).

Theorem 3. Let (U,V,R_1) and (U,V,R_2) be two single valued neutrosophic hesitant fuzzy approximation space over two universes and $R = R_1\cup R_2$ then for any $A\in SVNHF(V)$

1) $\overline{R}(A)=\overline{R_1}(A)\cup\overline{R_2}(A)$
2) $\underline{R}(A)=\underline{R_1}(A)\cap\underline{R_2}(A)$

Proof.

1) $\forall x\in U$, we have
$\tilde{t}_{\overline{R}(A)}(x)=\bigvee_{y\in V}\{\tilde{t}_R(x,y)\wedge\tilde{t}_A(y)\}$,
$\tilde{t}_{\overline{R}(A)}(x)=\bigvee_{y\in V}\{\tilde{t}_{R_1\cup R_2}(x,y)\wedge\tilde{t}_A(y)\}$,
$\tilde{t}_{\overline{R}(A)}(x)=\bigvee_{y\in V}\{(\tilde{t}_{R_1}(x,y)\vee\tilde{t}_{R_2}(x,y))\wedge\tilde{t}_A(y)\}$,
$(\bigvee_{y\in V}(\tilde{t}_{R_1}^{\sigma(s)}(x,y)\wedge\tilde{t}_A^{\sigma(s)}(y)))\vee(\bigvee_{y\in V}(\tilde{t}_{R_2}^{\sigma(s)}(x,y)\wedge\tilde{t}_A^{\sigma(s)}(y)))|s=1,2,....,k\}$
$=\tilde{t}_{\overline{R_1}(A)}(x)\vee\tilde{t}_{\overline{R_2}(A)}(x)$
$=\tilde{t}_{\overline{R_1}\cup\overline{R_2}(A)}(x)$

$\tilde{i}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{1-\tilde{i}_R(x,y)\vee\tilde{i}_A(y)\}$,
$\tilde{i}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{\tilde{i}_{R_1^c\cup R_2^c}(x,y)\vee\tilde{i}_A(y)\}$,
$\tilde{i}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{(\tilde{i}_{R_1^c}(x,y)\wedge\tilde{t}_{R_2^c}(x,y))\vee\tilde{i}_A(y)\}$,
$(\bigwedge_{y\in V}(\tilde{i}_{R_1^c}^{\sigma(t)}(x,y)\vee\tilde{i}_A^{\sigma(t)}(y)))\wedge(\bigwedge_{y\in V}(\tilde{i}_{R_2^c}^{\sigma(t)}(x,y)\vee\tilde{i}_A^{\sigma(t)}(y)))|t=1,2,....,k\}$
$=\tilde{i}_{\overline{R_1}(A)}(x)\wedge\tilde{i}_{\overline{R_2}(A)}(x)$
$=\tilde{i}_{\overline{R_1}\cup\overline{R_2}(A)}(x)$

$\tilde{f}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{\tilde{t}_R(x,y)\vee\tilde{f}_A(y)\}$,
$\tilde{f}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{\tilde{t}_{R_1\cup R_2}(x,y)\vee\tilde{f}_A(y)\}$,
$\tilde{f}_{\overline{R}(A)}(x)=\bigwedge_{y\in V}\{(\tilde{t}_{R_1}(x,y)\wedge\tilde{t}_{R_2}(x,y))\vee\tilde{f}_A(y)\}$,
$((\bigwedge_{y\in V}(\tilde{t}_{R_1}^{\sigma(v)}(x,y)\vee\tilde{f}_A^{\sigma(v)}(y)))\wedge(\bigwedge_{y\in V}(\tilde{t}_{R_2}^{\sigma(v)}(x,y)\vee\tilde{f}_A^{\sigma(v)}(y)))|v=1,2,....,k\})$
$=\tilde{f}_{\overline{R_1}(A)}(x)\wedge\tilde{f}_{\overline{R_2}(A)}(x)$
$=\tilde{f}_{\overline{R_1}\cup\overline{R_2}(A)}(x)$

2) It follows immediately from the above conclusion.

Definition 9. Let $G_1 = (U,V,R_1)$ and $G_2(U,W,R_2)$ be two SVNHF approximation spaces over two universes. The composition of SVNHF relations R_1 and R_1 is a SVNHF relation from U to W, denoted by $R = R\circ R$, and is defined as follows: for all $(x,z)\in U\times V$
$R = \{(x,y),\tilde{t}_R(x,z),\tilde{i}_R(x,z),\tilde{f}_R(x,z)|(x,z)\in U\times V\}$, where
$\tilde{t}_R(x,z) = \vee\{\tilde{t}_{R_1}(x,y)\wedge\tilde{t}_{R_2}(y,z)\} = \tilde{t}_R^{\sigma(s)}(x,z) = \vee\{\tilde{t}_{R_1}^{\sigma(s)}(x,y)\wedge\tilde{t}_{R_2}^{\sigma(s)}(y,z)|s=1,2,....k\}$
$\tilde{i}_R(x,z) = \wedge\{\tilde{i}_{R_1}(x,y)\vee\tilde{t}_{R_2}(y,z)\} = \tilde{i}_R^{\sigma(t)}(x,z) = \wedge\{\tilde{i}_{R_1}^{\sigma(i)}(x,y)\vee\tilde{i}_{R_2}^{\sigma(t)}(y,z)|t=1,2,....m\}$
$\tilde{f}_R(x,z) = \wedge\{\tilde{f}_{R_1}(x,y)\vee\tilde{f}_{R_2}(y,z)\} = \tilde{f}_R^{\sigma(v)}(x,z) =$

$\wedge\{\tilde{f}_{R_1}^{\sigma(v)}(x,y) \vee \tilde{f}_{R_2}^{\sigma(v)}(y,z)|v = 1,2,....n\}$, for all $(x,y) \in U \times V$ and $(y,z) \in V \times W$.

The SVNHF approximation space $G = (U,V,R)$ is referred to as the composition of $G_1 = (U,V,R_1)$ and $G_2(U,W,R_2)$, denoted by $G = G_1 \circ G_2$.

Theorem 4. Let $G_1 = (U,V,R_1)$ and $G_2(U,W,R_2)$ be two SVNHF approximation spaces over two universes, and $G = G_1 \circ G_2$ be the composition of G_1 and G_2. Then, for any $A \in SVNHF(W)$,

1) $\overline{R}(A) = (\overline{R_1} \circ \overline{R_2})(A) = \overline{R_1}(\overline{R_2}(A))$,
2) $\underline{R}(A) = (\underline{R_1} \circ \underline{R_2})(A) = \underline{R_1}(\underline{R_2}(A))$.

Proof.

1) $\forall x \in U$, we have
$\tilde{t}_{\overline{R_1}(\overline{R_2}(A))}(x) = \bigvee_{y\in V}\{\tilde{t}_{R_1}(x,y) \wedge \tilde{t}_{\overline{R_2}(A)}(y)\} = \bigvee_{y\in V}\{\tilde{t}_{R_1}(x,y) \wedge (\bigvee_{z\in w}\{\tilde{t}_{R_2}(y,z) \wedge \tilde{t}_A(z)\})\} = \bigvee_{y\in V}\bigvee_{z\in w}(\tilde{t}_{R_1}^{\sigma(s)}(x,y) \wedge \tilde{t}_{R_2}^{\sigma(s)}(y,z) \wedge \tilde{t}_A^{\sigma(s)}(z)|s = 1,2,....k = \bigvee_{z\in w}[\bigvee_{y\in V}(\tilde{t}_{R_1}^{\sigma(s)}(x,y) \wedge \tilde{t}_{R_2}^{\sigma(s)}(y,z))] \wedge \tilde{t}_A^{\sigma(s)}(z)|s = 1,2,....k = \bigvee_{z\in w}(\tilde{t}_R^{\sigma(s)}(x,z) \wedge \tilde{t}_A^{\sigma(s)}(z))|s = 1,2,....k = \tilde{t}_{\overline{R}(A)}(x)$

$\tilde{i}_{\overline{R_1}(\overline{R_2}(A))}(x) = \bigwedge_{y\in V}\{\tilde{i}_{R_1}(x,y) \vee \tilde{i}_{\overline{R_2}(A)}(y)\} = \bigwedge_{y\in V}\{\tilde{i}_{R_1}(x,y) \vee (\bigwedge_{z\in w}\{\tilde{i}_{R_2}(y,z) \vee \tilde{i}_A(z)\})\} = \bigwedge_{y\in V}\bigwedge_{z\in w}(\tilde{i}_{R_1}^{\sigma(t)}(x,y) \vee \tilde{i}_{R_2}^{\sigma(t)}(y,z) \vee \tilde{i}_A^{\sigma(t)}(z)|t = 1,2,....m = \bigwedge_{z\in w}[\bigwedge_{y\in V}(\tilde{i}_{R_1}^{\sigma(t)}(x,y) \vee \tilde{i}_{R_2}^{\sigma(t)}(y,z))] \vee \tilde{i}_A^{\sigma(t)}(z)|t = 1,2,....m = \bigwedge_{z\in w}(\tilde{i}_R^{\sigma(t)}(x,z) \vee \tilde{i}_A^{\sigma(t)}(z))|t = 1,2,....k = \tilde{i}_{\overline{R}(A)}(x)$
$\tilde{f}_{\overline{R_1}(\overline{R_2}(A))}(x) = \bigwedge_{y\in V}\{\tilde{f}_{R_1}(x,y) \vee \tilde{f}_{\overline{R_2}(A)}(y)\} = \bigwedge_{y\in V}\{\tilde{f}_{R_1}(x,y) \vee (\bigwedge_{z\in w}\{\tilde{f}_{R_2}(y,z) \vee \tilde{f}_A(z)\})\} = \bigwedge_{y\in V}\bigwedge_{z\in w}(\tilde{f}_{R_1}^{\sigma(v)}(x,y) \vee \tilde{f}_{R_2}^{\sigma(v)}(y,z) \vee \tilde{f}_A^{\sigma(v)}(z)|v = 1,2,....n = \bigwedge_{z\in w}[\bigwedge_{y\in V}(\tilde{f}_{R_1}^{\sigma(v)}(x,y) \vee \tilde{f}_{R_2}^{\sigma(v)}(y,z))] \vee \tilde{f}_A^{\sigma(v)}(z)|v = 1,2,....n = \bigwedge_{z\in w}(\tilde{f}_R^{\sigma(v)}(x,z) \vee \tilde{f}_A^{\sigma(v)}(z))|v = 1,2,....k = \tilde{f}_{\overline{R}(A)}(x)$

2) It follows immediately from the above result.

Theorem 5. Let R be a SVNHF relation from U to V. Suppose that 1_y,$1_U - y$ and 1_M are three special SVNHF sets; then $\forall x \in U, (x,y) \in U \times V, M \subseteq U$, we have

1) $\tilde{t}_{\underline{R}(1_M)}(x) = \bigwedge_{y\neq M}\tilde{f}_R(x,y)$, $\tilde{i}_{\underline{R}(1_M)}(x) = \bigvee_{y\neq M}\tilde{i^c}_R(x,y)$
and $\tilde{f}_{\underline{R}(1_M)}(x) = \bigvee_{y\neq M}(\tilde{t}_R(x,y)$

2) $\tilde{t}_{\overline{R}(1_M)}(x) = \bigvee_{y\in M}\tilde{t}_R(x,y)$, $\tilde{i}_{\overline{R}(1_M)}(x) = \bigwedge_{y\in M}\tilde{i}_R(x,y)$ and $\tilde{f}_{\overline{R}(1_M)}(x) = \bigwedge_{y\in M}\tilde{f}_R(x,y)$

3) $\tilde{t}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{f}_R(x,y)$, $\tilde{i}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{i^c}_R(x,y)$
and $\tilde{f}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{t}_R(x,y)$

4) $\tilde{t}_{\overline{R}(1_y)}(x) = \tilde{t}_R(x,y)$, $\tilde{i}_{\overline{R}(1_y)}(x) = \tilde{i}_R(x,y)$
and $\tilde{f}_{\overline{R}(1_y)}(x) = \tilde{f}_R(x,y)$

Proof.

1) For all $x \in U$, we have
$\tilde{t}_{\underline{R}(1_M)}(x) = \bigwedge_{y\in V}\{\tilde{f}_R(x,y) \vee \tilde{f}_{1_M}(y)\}$
$= \{1\} \wedge \left(\bigwedge_{y\neq M}\tilde{f}_R(x,y)\right)$
$= \bigwedge_{y\neq M}\tilde{f}_R(x,y)$
$\tilde{i}_{\underline{R}(1_M)}(x) = \bigvee_{y\in V}\{\tilde{i^c}_R(x,y) \wedge \tilde{i}_{1_M}(y)\}$
$= \{0\} \vee \left(\bigvee_{y\neq M}\tilde{i^c}_R(x,y)\right)$
$= \bigvee_{y\neq M}\tilde{i^c}_R(x,y)$
$\tilde{f}_{\underline{R}(1_M)}(x) = \bigvee_{y\in V}\{\tilde{t}_R(x,y) \wedge \tilde{f}_{1_M}(y)\}$
$= \{0\} \vee \left(\bigvee_{y\neq M}\tilde{t}_R(x,y)\right)$
$= \bigvee_{y\neq M}\tilde{t}_R(x,y)$
2) It follows immediately from (1)
3) For all $x \in U$, we have
$\tilde{t}_{\underline{R}(1_{U-\{y\}})}(x) = \bigwedge_{z\in V}\{\tilde{f}_R(x,z) \vee \tilde{f}_{1_{U-\{y\}(z)}}\}$
$= \tilde{f}_R(x,y) \wedge \{1\}$
$= \tilde{f}_R(x,y)$
$\tilde{i}_{\underline{R}(1_{U-\{y\}})}(x) = \bigvee_{z\in V}\{\tilde{i^c}_R(x,z) \wedge \tilde{i}_{1_{U-\{y\}(z)}}\}$
$= \tilde{i^c}_R(x,y) \vee \{0\}$
$= \tilde{i^c}_R(x,y)$
$\tilde{f}_{\underline{R}(1_{U-\{y\}})}(x) = \bigvee_{z\in V}\{\tilde{t}_R(x,z) \wedge \tilde{f}_{1_{U-\{y\}(z)}}\}$
$= \tilde{t}_R(x,y) \vee \{0\}$
$= \tilde{t}_R(x,y)$.

4) It follows immediately from (3)

Theorem 6. Let R be a SVNHF relation from U to V. Suppose that $\underline{R}$ and $\overline{R}$ are the lower and upper SVNHF rough approximation operators given in definition, then R is serial iff one of the following properties hold:

1) $(SVNHFL_1)\underline{R}(\phi) = \phi$
2) $(SVNHFU_1)\overline{R}(V) = U$
3) $(SVNHFLU_1)\underline{R}(A) \subseteq \overline{R}(A), \forall A \in SVNHF(U)$

Proof.

1) It is easy to prove.
2) First we prove that R is serial $\Longleftrightarrow$ $(SVNHFU_1)$ suppose that R is serial. For any $x \in U$, there exists a $z \in V$ such that $\tilde{t}_R(x,z) = 1$ and $\tilde{i}_R(x,z) = \tilde{f}_R(x,z) = 0$,
$\tilde{t}_{\overline{R}(V)}(x) = \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_V^{\sigma(s)}(y))|s = 1,2,....k\}$,

$= \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(s)}(x,y) \wedge 1)|s = 1,2,....k\}$,

$= \{\tilde{t}_R^{\sigma(s)}(x,z) \vee \left(\bigvee_{y\neq z}\tilde{t}_R^{\sigma(s)}(x,y)\right)|s = 1,2,....k\}$,

$= 1 = \tilde{t}_U(x)$
, $\tilde{i}_{\overline{R}(V)}(x) = \{\bigwedge_{y\in V}(\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_V^{\sigma(t)}(y))|t = 1,2,....m\}$,

$= \{\bigwedge_{y\in V}(\tilde{i}_R^{\sigma(t)}(x,y) \vee 0)|t = 1,2,....m\}$,

$= \{\tilde{i}_R^{\sigma(t)}(x,z) \wedge \left(\bigwedge_{y\neq z}\tilde{i}_R^{\sigma(t)}(x,y)\right)|t = 1,2,....m\}$,

$= 0 = \tilde{i}_U(x)$
and
$\tilde{f}_{\overline{R}(V)}(x) = \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_V^{\sigma(v)}(y))|v = 1,2,....n\}$,

$= \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(v)}(x,y) \vee 0)|v = 1,2,....m\}$,

$= \{\tilde{f}_R^{\sigma(v)}(x,z) \wedge \left(\bigwedge_{y\neq z} \tilde{f}_R^{\sigma(v)}(x,y)\right)|v = 1,2,....n\}$,

$= 0 = \tilde{f}_U(x)$

Thus, we conclude that $\overline{R}(V) = U$. Conversely, if $(SVNHFU_1)$ holds, then $\forall x \in U$, $\tilde{t}_{\overline{R}(V)}(x) = 1$ and $\tilde{i}_{\overline{R}(V)}(x) = \tilde{f}_{\overline{R}(V)}(x) = 0$. If R is not serial, then $\forall y \in V$, $\exists x \in U$ such that $\tilde{t}_R(x,y) \neq 1$ and $\tilde{i}_R(x,y) = \tilde{f}_R(x,y) \neq 0$, then we have $\tilde{t}_R(x,y) \cap \tilde{t}_V(y) = \tilde{t}_R(x,y) \neq 1$, $\tilde{i}_R(x,y) \cup \tilde{i}_V(y) = \tilde{i}_R(x,y) \neq 0$ and $\tilde{f}_R(x,y) \cup \tilde{f}_V(y) = \tilde{f}_R(x,y) \neq 0$. That is, $\tilde{t}_{\overline{R}(V)}(x) \neq 1$ and $\tilde{i}_{\overline{R}(V)}(x) = \tilde{f}_{\overline{R}(V)}(x) = 0$ which contradict the assumption.

3) Second we prove that R is serial $\iff (SVNHFLU_1)$. Suppose that R is serial. For any $x \in U$, there exists a $z \in V$ such that $\tilde{t}_R(x,z) = 1$ and $\tilde{i}_R(x,z) = \tilde{f}_R(x,z) = 0$, by definition we have
$\tilde{t}_{\underline{R}(A)}(x) = \{\bigwedge_{y\in V}(\tilde{f}_R(x,y) \vee \tilde{t}_A(y))\}$
$= \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y))|s = 1,2,....k\}$,

$= \{(\tilde{f}_R^{\sigma(s)}(x,z) \vee \tilde{t}_A^{\sigma(s)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y)\right)|s = 1,2,....k\}$,

$= \{\tilde{f}_A^{\sigma(s)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y)\right)|s = 1,2,....k\}$,

$\leq \{\tilde{t}_A^{\sigma(s)}(z)|s = 1,2,....k\} = \tilde{t}_A(z)$
, $\tilde{i}_{\underline{R}(A)}(x) = \{\bigvee_{y\in V}(\tilde{i^c}_R(x,y) \wedge \tilde{i}_A(y))\}$
$= \{\bigvee_{y\in V}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y))|t = 1,2,....m\}$,

$= \{(\tilde{i^c}_R^{\sigma(t)}(x,z) \wedge \tilde{i}_A^{\sigma(t)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y)\right)|t = 1,2,....m\}$,

$= \{\tilde{i}_A^{\sigma(t)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y)\right)|t = 1,2,....m\}$,

$\geq \{\tilde{i}_A^{\sigma(t)}(z)|t = 1,2,....m\} = \tilde{i}_A(z)$
and
$\tilde{f}_{\underline{R}(A)}(x) = \{\bigvee_{y\in V}(\tilde{t}_R(x,y) \wedge \tilde{f}_A(y))\}$
$= \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y))|v = 1,2,....n\}$,

$= \{(\tilde{t}_R^{\sigma(v)}(x,z) \wedge \tilde{f}_A^{\sigma(v)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y)\right)|v = 1,2,....n\}$,

$= \{\tilde{f}_A^{\sigma(t)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y)\right)|v = 1,2,....n\}$,

$\geq \{\tilde{f}_A^{\sigma(v)}(z)|v = 1,2,....m\} = \tilde{f}_A(z)$

On the other hand, we have

$\tilde{t}_{\overline{R}(A)}(x) = \{\bigvee_{y\in V}(\tilde{t}_R(x,y) \wedge \tilde{t}_A(y))\}$
$= \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_A^{\sigma(s)}(y))|s = 1,2,....k\}$,

$= \{(\tilde{t}_R^{\sigma(s)}(x,z) \wedge \tilde{t}_A^{\sigma(s)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_A^{\sigma(s)}(y)\right)|s = 1,2,....k\}$,

$= \{\tilde{t}_A^{\sigma(s)}(z) \vee \left(\bigvee_{y\neq z}(\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_A^{\sigma(s)}(y)\right)|s = 1,2,....k\}$,

$\geq \{\tilde{t}_A^{\sigma(s)}(z)|s = 1,2,....k\} = \tilde{t}_A(z)$,
$\tilde{i}_{\overline{R}(A)}(x) = \{\bigwedge_{y\in V}(\tilde{i}_R(x,y) \vee \tilde{i}_A(y))\}$
$= \{\bigwedge_{y\in V}(\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_A^{\sigma(t)}(y))|t = 1,2,....m\}$,

$= \{(\tilde{i}_R^{\sigma(t)}(x,z) \vee \tilde{i}_A^{\sigma(t)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_A^{\sigma(t)}(y)\right)|t = 1,2,....m\}$,

$= \{\tilde{i}_A^{\sigma(t)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_A^{\sigma(t)}(y)\right)|t = 1,2,....m\}$,

$\leq \{\tilde{i}_A^{\sigma(t)}(z)|t = 1,2,....m\} = \tilde{i}_A(z)$

and
$\tilde{f}_{\overline{R}(A)}(x) = \{\bigwedge_{y\in V}(\tilde{f}_R(x,y) \vee \tilde{f}_A(y))\}$
$= \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_A^{\sigma(v)}(y))|v = 1,2,....n\}$,

$= \{(\tilde{f}_R^{\sigma(v)}(x,z) \vee \tilde{f}_A^{\sigma(v)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_A^{\sigma(v)}(y)\right)|v = 1,2,....n\}$,

$= \{\tilde{f}_A^{\sigma(v)}(z) \wedge \left(\bigwedge_{y\neq z}(\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_A^{\sigma(v)}(y)\right)|v = 1,2,....n\}$,

$\leq \{\tilde{f}_A^{\sigma(v)}(z)|v = 1,2,....n\} = \tilde{f}_A(z)$

From the above discussions, we can conclude that $\tilde{t}_{\underline{R}(A)}(x) \leq \tilde{t}_{\overline{R}(A)}(x)$, $\tilde{i}_{\underline{R}(A)}(x) \geq \tilde{i}_{\overline{R}(A)}(x)$ and $\tilde{f}_{\underline{R}(A)}(x) \geq \tilde{f}_{\overline{R}(A)}(x)$ which means that $\underline{R}(A) \subseteq \overline{R}(A)$. Conversely, if $(SVNHFLU_1)$ holds, then $\forall x \in U$, we have $\tilde{t}^{\sigma(s)}_{\underline{R}(A)}(x) \leq \tilde{t}^{\sigma(s)}_{\overline{R}(A)}(x)$, $\tilde{i}^{\sigma(t)}_{\underline{R}(A)}(x) \geq \tilde{i}^{\sigma(t)}_{\overline{R}(A)}(x)$ and $\tilde{f}^{\sigma(v)}_{\underline{R}(A)}(x) \geq \tilde{f}^{\sigma(v)}_{\overline{R}(A)}(x)$. Thus it follows that $\tilde{t}^{\sigma(s)}_{\underline{R}(\phi)}(x) \leq \tilde{t}^{\sigma(s)}_{\overline{R}(\phi)}(x)$, $\tilde{i}^{\sigma(t)}_{\underline{R}(\phi)}(x) \geq \tilde{i}^{\sigma(t)}_{\overline{R}(\phi)}(x)$ and $\tilde{f}^{\sigma(v)}_{\underline{R}(\phi)}(x) \geq \tilde{f}^{\sigma(v)}_{\overline{R}(\phi)}(x)$. On the other hand, we have
$\tilde{t}_{\underline{R}(\phi)}(x) = \bigwedge_{y\in V}(\tilde{f}_R(x,y) = \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(s)}(x,y)|s = 1,2,...,k\}$ and $\tilde{t}_{\overline{R}(\phi)}(x) = 0$. Meanwhile, $\tilde{i}_{\underline{R}(\phi)}(x) = \bigvee_{y\in V}(\tilde{i^c}_R(x,y) = \{\bigvee_{y\in V}(\tilde{i^c}_R^{\sigma(t)}(x,y)|t = 1,2,...,m\}$ and $\tilde{i}_{\overline{R}(\phi)}(x) = 1$. and $\tilde{f}_{\underline{R}(\phi)}(x) = \bigvee_{y\in V}(\tilde{t}_R(x,y) = \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(v)}(x,y)|v = 1,2,...,n\}$ and $\tilde{f}_{\overline{R}(\phi)}(x) = 1$. Therefore, for any $x \in U$ there exists a $y \in V$ such that $\tilde{t}_R^{\sigma(s)}(x,y) = 1$ and $\tilde{i}_R^{\sigma(t)}(x,y) = \tilde{f}_R^{\sigma(v)}(x,y) = 0$ which implies that $\tilde{t}_R(x,y) = 1$ and $\tilde{i}_R(x,y) = \tilde{f}_R(x,y) = 0$ so R is serial.

Theorem 7. Let R be a SVNHF relation on U. For all $A \in SVNHF(U)$, then

1) R is reflexive $\iff (SVNHFL1)\underline{R}(A) \subseteq A$, $\iff (SVNHFL1)A \subseteq \overline{R}(A)$
2) R is symmetric $\iff (SVNHFL2)\tilde{t}_{\underline{R}(1_{U-x})}(y) =$

$\tilde{t}_{\underline{R}(1_{U-y})}(x)$, $\tilde{i}_{\underline{R}(1_{U-x})}(y) = \tilde{i}_{\underline{R}(1_{U-y})}(x)$,$\tilde{f}_{\underline{R}(1_{U-x})}(y) = \tilde{f}_{\underline{R}(1_{U-y})}(x)$.

$\Longleftrightarrow (SVNHFU2)\tilde{t}_{\overline{R}(1_x)}(y) = \tilde{t}_{\overline{R}(1_y)}(x)$,$\tilde{i}_{\overline{R}(1_x)}(y) = \tilde{i}_{\overline{R}(1_y)}(x)$,$\tilde{f}_{\overline{R}(1_x)}(y) = \tilde{f}_{\overline{R}(1_y)}(x)$.

3) R is transitive $\Longleftrightarrow (SVNHFL3)\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$, $\Longleftrightarrow (SVNHFU3)\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Proof.

1) Due to the duality of SVNHF rough approximation operators, it is only to prove that R is reflexive $\underline{R}(A) \subseteq A$.
If R is reflexive, for all $x \in U$then $\tilde{t}_R(x,x) = 1$ and $\tilde{i}_R(x,x) = \tilde{f}_R(x,x) = 0$, we have
$\tilde{t}_{\underline{R}(A)}(x) = \{\bigwedge_{y\in U}(\tilde{f}_R(x,y) \vee \tilde{t}_A(y))\}$
$= \{\bigwedge_{y\in U}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y))|s = 1,2,....k\}$,
$= \{(\tilde{f}_R^{\sigma(s)}(x,x) \vee \tilde{t}_A^{\sigma(s)}(x) \wedge \left(\bigwedge_{y\neq x}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y)\right) |s = 1,2,....k\}$,

$= \{\tilde{t}_A^{\sigma(s)}(x) \wedge \left(\bigwedge_{y\neq x}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}_A^{\sigma(s)}(y)\right) |s = 1,2,....k\}$,

$\leq \{\tilde{t}_A^{\sigma(s)}(x)|s = 1,2,...k\} = \tilde{t}_A(x)$,

$\tilde{i}_{\underline{R}(A)}(x) = \{\bigvee_{y\in U}(\tilde{i^c}_R(x,y) \wedge \tilde{i}_A(y))\}$
$= \{\bigvee_{y\in U}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y))|t = 1,2,....m\}$,

$= \{(\tilde{i^c}_R^{\sigma(t)}(x,x) \wedge \tilde{i}_A^{\sigma(t)}(x) \vee \left(\bigvee_{y\neq x}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y)\right) |t = 1,2,....m\}$,

$= \{\tilde{i}_A^{\sigma(t)}(x) \vee \left(\bigvee_{y\neq x}(\tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_A^{\sigma(t)}(y)\right) |t = 1,2,....m\}$,

$\geq \{\tilde{i}_A^{\sigma(t)}(x)|t = 1,2,....m\} = \tilde{i}_A(x)$
and
$\tilde{f}_{\underline{R}(A)}(x) = \{\bigvee_{y\in U}(\tilde{t}_R(x,y) \wedge \tilde{f}_A(y))\}$
$= \{\bigvee_{y\in U}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y))|v = 1,2,....n\}$,

$= \{(\tilde{t}_R^{\sigma(v)}(x,x) \wedge \tilde{f}_A^{\sigma(v)}(x) \vee \left(\bigvee_{y\neq x}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y)\right) |v = 1,2,....n\}$,

$= \{\tilde{f}_A^{\sigma(t)}(x) \vee \left(\bigvee_{y\neq x}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}_A^{\sigma(v)}(y)\right) |v = 1,2,....n\}$,

$\geq \{\tilde{f}_A^{\sigma(v)}(x)|v = 1,2,....m\} = \tilde{f}_A(x)$.
From the above discussions, we conclude that $\underline{R}(A) \subseteq A$. Conversely, if (SVNHFL1) holds, for any $x \in U$ then $\tilde{t}^{\sigma(s)}_{\underline{R}(A)}(x) \leq \tilde{t}_A^{\sigma(s)}(x)$, $\tilde{i}^{\sigma(t)}_{\underline{R}(A)}(x) \geq \tilde{i}_A^{\sigma(t)}(x)$ and $\tilde{f}^{\sigma(v)}_{\underline{R}(A)}(x) \geq \tilde{f}_A^{\sigma(v)}(x)$. So take $A = 1_{U-x}$ we have, $\tilde{t}^{\sigma(s)}_{\underline{R}(1_{U-x})}(x) \leq \tilde{t}^{\sigma(s)}_{1_{U-x}}(x) = 0$, $\tilde{i}^{\sigma(t)}_{\underline{R}(1_{U-x})}(x) \geq \tilde{i}^{\sigma(t)}_{1_{U-x}}(x) = 1$ and $\tilde{f}^{\sigma(v)}_{\underline{R}(1_{U-x})}(x) \geq \tilde{f}^{\sigma(v)}_{1_{U-x}}(x) = 1$. From which we conclude that $\tilde{t}_{\underline{R}(1_{U-x})}(x) = \{0\}$ and $\tilde{i}_{\underline{R}(1_{U-x})}(x) = \tilde{f}_{\underline{R}(1_{U-x})}(x) = \{1\}$. On the other hand, we have

$\tilde{t}_{\underline{R}(1_{U-x})}(x) = \{\bigwedge_{y\in V}(\tilde{f}_R(x,y) \vee \tilde{t}_{1_{U-x}}(y))\}$
$= \{\bigwedge_{y\in V}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}^{\sigma(s)}_{1_{U-x}}(y))|s = 1,2,....k\}$,
$= \{(\tilde{f}_R^{\sigma(s)}(x,x) \vee \tilde{t}^{\sigma(s)}_{1_{U-x}}(x) \wedge \left(\bigwedge_{y\neq x}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{t}^{\sigma(s)}_{1_{U-x}}(y)\right) |s = 1,2,....k\}$,

$= \{\tilde{f}_R^{\sigma(s)}(x,x) \wedge \left(\bigwedge_{y\neq x}(\tilde{f}_R^{\sigma(s)}(x,y) \vee 1)\right) |s = 1,2,....k\}$,

$= \{\tilde{f}_R^{\sigma(s)}(x,x))|s = 1,2,....k\} = \tilde{f}_R(x,x)$

$\tilde{i}_{\underline{R}(1_{U-x})}(x) = \{\bigvee_{y\in V}(\tilde{i^c}_R(x,y) \wedge \tilde{i}_{1_{U-x}}(y))\}$
$= \{\bigvee_{y\in V}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}^{\sigma(t)}_{1_{U-x}}(y))|t = 1,2,....m\}$,

$= \{(\tilde{i^c}_R^{\sigma(t)}(x,x) \wedge \tilde{i}^{\sigma(t)}_{1_{U-x}}(x) \vee \left(\bigvee_{y\neq x}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge \tilde{i}^{\sigma(t)}_{1_{U-x}}(y)\right) |t = 1,2,....m\}$,

$= \{(\tilde{i^c}_R^{\sigma(t)}(x,x) \vee \left(\bigvee_{y\neq x}(\tilde{i^c}_R^{\sigma(t)}(x,y) \wedge 0)\right) |t = 1,2,....m\}$,

$= \{\tilde{i^c}_R^{\sigma(t)}(x,x)|t = 1,2,....m\} = \tilde{i^c}_R(x,x)$

$\tilde{f}_{\underline{R}(1_{U-x})}(x) = \{\bigvee_{y\in V}(\tilde{t}_R(x,y) \wedge \tilde{f}_{1_{U-x}}(y))\}$
$= \{\bigvee_{y\in V}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}^{\sigma(v)}_{1_{U-x}}(y))|v = 1,2,....n\}$,

$= \{(\tilde{t}_R^{\sigma(v)}(x,x) \wedge \tilde{f}^{\sigma(v)}_{1_{U-x}}(x) \vee \left(\bigvee_{y\neq x}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{f}^{\sigma(v)}_{1_{U-x}}(y)\right) |v = 1,2,....n\}$,

$= \{(\tilde{t}_R^{\sigma(v)}(x,x) \vee \left(\bigvee_{y\neq x}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge 0)\right) |v = 1,2,....n\}$,

$= \{\tilde{t}_R^{\sigma(v)}(x,x)|v = 1,2,....m\} = \tilde{t}_R(x,x)$.

Thus it follows that $\tilde{t}_{\underline{R}}(x,y) = \{1\}$ and $\tilde{i}_{\underline{R}}(x,y) = \tilde{f}_{\underline{R}}(x,y) = \{0\}$.
Hence, R is reflexive.

2) It follows immediately from Theorem 5.

3) Because of the the duality of SVNHF rough approximation operators, it is only to prove that R is transitive then (SVNHFL3) holds.
If R is transitive, then $\tilde{t}_R^{\sigma(s)}(x,y) \wedge \tilde{t}_R^{\sigma(s)}(y,z) \leq \tilde{t}_R^{\sigma(s)}(x,z)$, $\tilde{i}_R^{\sigma(t)}(x,y) \vee \tilde{i}_R^{\sigma(t)}(y,z) \leq \tilde{i}_R^{\sigma(t)}(x,z)$ and $\tilde{f}_R^{\sigma(v)}(x,y) \vee \tilde{f}_R^{\sigma(v)}(y,z) \leq \tilde{f}_R^{\sigma(v)}(x,z)$, we have

$\tilde{t}_{\underline{R}(\underline{R}(A))}(x) = \bigwedge_{y\in U}(\tilde{f}_R(x,y) \vee \tilde{t}_{\underline{R}(A)}(y))$
$= \bigwedge_{y\in U}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \left(\bigwedge_{z\in U}\tilde{f}_R^{\sigma(s)}(y,z) \vee \tilde{t}^{\sigma(s)}_{(A)}(z)\right) |s = 1,2,....k$
$= \bigwedge_{y\in U}\bigwedge_{z\in U}(\tilde{f}_R^{\sigma(s)}(x,y) \vee \tilde{f}_R^{\sigma(s)}(y,z) \vee \tilde{t}^{\sigma(s)}_{(A)}(z)|s = 1,2,....k$
$\geq \left\{\bigwedge_{z\in U}(\tilde{f}_R^{\sigma(s)}(x,z) \vee \tilde{t}^{\sigma(s)}_{(A)}(z)|s = 1,2,....k\right\}$
$= \tilde{t}_{\underline{R}(A)}(x)$,
$\tilde{i}_{\underline{R}(\underline{R}(A))}(x) = \bigvee_{y\in U}(\tilde{i^c}_R(x,y) \wedge \tilde{i}_{\underline{R}(A)}(y))$

$= \bigvee_{y\in U}((1-\tilde{i}_R(x,y)) \wedge \tilde{i}_{\underline{R}(A)}(y))$
$= \bigvee_{y\in U}((1 - \tilde{i}_R^{\sigma(t)}(x,y)) \wedge \left(\bigvee_{z\in U}(1-\tilde{i}_R^{\sigma(t)}(y,z)) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)\right) |t=1,2,....m$
$= \bigvee_{y\in U}\bigvee_{z\in U}((1-\tilde{i}_R^{\sigma(t)}(x,y)) \wedge (1-\tilde{i}_R^{\sigma(s)}(y,z)) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)|t=1,2,....m$
$= \bigvee_{z\in U}\left(\left[1-\bigwedge_{y\in U}(\tilde{i}_R^{\sigma(t)}(x,y) \wedge \tilde{i}_R^{\sigma(t)}(y,z))\right]\right) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)|t=1,2,....m$
$\le \bigvee_{z\in U}((1-\tilde{i}_R^{\sigma(t)}(x,z)) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)|t=1,2,....m$
$= \bigvee_{z\in U}(\tilde{i^c}_R^{\sigma(t)}(x,z) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)|t=1,2,....m$
$= \tilde{i}_{\underline{R}(A)}(x)$
and

$\tilde{f}_{\underline{R}(\underline{R}(A))}(x) = \bigvee_{y\in U}(\tilde{t}_R(x,y) \wedge \tilde{f}_{\underline{R}(A)}(y))$
$= \left\{\bigvee_{y\in U}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \left(\bigvee_{z\in U}\tilde{t}_R^{\sigma(v)}(y,z) \wedge \tilde{f}_{(A)}^{\sigma(v)}(z)\right) |v=1,2,....n\right\}$
$= \bigvee_{y\in U}\bigvee_{z\in U}(\tilde{t}_R^{\sigma(v)}(x,y) \wedge \tilde{t}_R^{\sigma(v)}(y,z) \wedge \tilde{i}_{(A)}^{\sigma(t)}(z)|v=1,2,....n$
$\le \left\{\bigvee_{z\in U}(\tilde{t}_R^{\sigma(v)}(x,z) \wedge \tilde{f}_{(A)}^{\sigma(v)}(z)|v=1,2,....n\right\}$
$= \tilde{f}_{\underline{R}(A)}(x)$

Hence, we conclude that (SVNHFL3) holds. Conversely, for all $A \in SVNHF(U)\tilde{t}_{\underline{R}(\underline{R}(A))}(x) \ge \tilde{t}_{\underline{R}(A)}(x)$, $\tilde{i}_{\underline{R}(\underline{R}(A))}(x) \le \tilde{i}_{\underline{R}(A)}(x)\tilde{f}_{\underline{R}(\underline{R}(A))}(x) \le \tilde{f}_{\underline{R}(A)}(x)$
,then $\tilde{t}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) \ge \tilde{t}_{\underline{R}(1_{U-\{y\}})}(x)$, $\tilde{i}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) \le \tilde{i}_{\underline{R}(1_{U-\{y\}})}(x)\tilde{f}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) \le \tilde{f}_{\underline{R}(1_{U-\{y\}})}(x)$
On the other hand, we see that
$\tilde{t}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) = \bigwedge_{z\in U}\{(\tilde{f}_R(x,z) \vee \tilde{t}_{\underline{R}(1_{U-\{y\}})}(z))\}$
$= \bigwedge_{z\in U}\{(\tilde{f}_R(x,z) \vee \tilde{t}_R(z,y))\}$
,
$\tilde{i}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) = \bigvee_{z\in U}\{(\tilde{i^c}_R(x,z) \wedge \tilde{i}_{\underline{R}(1_{U-\{y\}})}(z))\}$
$= \bigvee_{z\in U}\{(\tilde{i^c}_R(x,z) \wedge \tilde{i^c}_R(z,y))\}$
and
$\tilde{f}_{\underline{R}(\underline{R}(1_{U-\{y\}}))}(x) = \bigvee_{z\in U}\{(\tilde{t}_R(x,z) \wedge \tilde{f}_{\underline{R}(1_{U-\{y\}})}(z))\}$
$= \bigvee_{z\in U}\{(\tilde{t}_R(x,z) \wedge \tilde{f}_R(z,y))\}$

Note that $\tilde{i}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{f}_R(x,y)$, $\tilde{i}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{i^c}_R(x,y)$ and $\tilde{f}_{\underline{R}(1_{U-\{y\}})}(x) = \tilde{t}_R(x,y)$.
Hence, we conclude that $\bigvee_{z\in U}\{\tilde{t}_R^{\sigma(s)}(x,z) \wedge \tilde{t}_R^{\sigma(s)}(z,y)\} \le \tilde{t}_R^{\sigma(s)}(x,y)$
$\bigwedge_{z\in U}\{\tilde{i}_R^{\sigma(t)}(x,z) \vee \tilde{i}_R^{\sigma(t)}(z,y)\} \ge \tilde{i}_R^{\sigma(t)}(x,y)$,
$\bigwedge_{z\in U}\{\tilde{f}_R^{\sigma(v)}(x,z) \vee \tilde{f}_R^{\sigma(v)}(z,y)\} \ge \tilde{f}_R^{\sigma(v)}(x,y)$.
By the definition of transitivity, we conclude that R is transitive.

IV. The Application of Single Valued Neutrosophic Hesitant Fuzzy Rough Set Model in Medical Diagnoseis

Rough set theory was developed by Pawlak [5] as a mathematical approach to handle imprecision, vagueness, and uncertainty. It has a wide application in many practical problems, especially the use of rough sets in decision making. The concept of single-valued neutrosophic hesitant fuzzy which is a generalization of the fuzzy set first introduced by Jun Ye [12] into the decision-making problems.
In [3] the definition of score function of SVNHF elements was introduced as follows:
Definition 10 [3]. Let $n =< \tilde{t}, \tilde{i}, \tilde{f} >$ be a SVNHFE, then the score function can be fined as:
$S(n) = \frac{1}{3}\{\frac{1}{l_t}\sum\gamma + \frac{1}{l_i}\sum(1-\delta) + \frac{1}{l_f}\sum(1-\eta)\}$, where l_t, l_i and l_f are the numbers of vales of $\tilde{t}, \tilde{i}$ and $\tilde{f}$, respectively in n.

By Definition 3, we can define the sum of $\underline{R}$ and $\overline{R}$ as follows:
Definition 11. Let $\underline{R}$ and $\overline{R}$ be two SVNHFS in U, we define the sum of $\underline{R}$ and $\overline{R}$ as $\underline{R} \oplus \overline{R} = \{< y_j, \tilde{t}_{\underline{R}}(y_j) \oplus \tilde{t}_{\overline{R}}(y_j), \tilde{i}_{\underline{R}}(y_j) \oplus \tilde{i}_{\overline{R}}(y_j), \tilde{f}_{\underline{R}}(y_j) \oplus \tilde{f}_{\overline{R}}(y_j) : y_i \in V\} = \{[\tilde{t}_{\underline{R}}(y_j) + \tilde{t}_{\overline{R}}(y_j) - \tilde{t}_{\underline{R}}(y_j)\tilde{t}_{\overline{R}}(y_j)), \tilde{i}_{\underline{R}}(y_j)\tilde{i}_{\overline{R}}(y_j), \tilde{f}_{\underline{R}}(y_j)\tilde{f}_{\overline{R}}(y_j)]\}$.

In this section, we will apply SVNHF rough set model on two universes to medical diagnosis problems. Suppose that the universe $U = \{x_1, x_2, ..., x_m\}$ denotes a disease set. Let $R \in SVNHFR(U \times V)$ be an SVNHF relation from U to V. For any $(x_i, y_i) \in U \times V$, $\tilde{t}_R(x_j, y_i)$ represents the true membership degree of the relationships between the symptom $x_i(x_i \in U)$ and the disease $y_i(y_i \in V)$, $\tilde{i}_R(x_j, y_i)$ represents the true membership degree of the relationships between the symptom $x_i(x_i \in U)$ and the disease $y_i(y_i \in V)$ and $\tilde{f}_R(x_j, y_i)$ represents the true membership degree of the relationships between the symptom $x_i(x_i \in U)$ and the disease $y_i(y_i \in V)$, which are evaluated by several doctors in advance. In clinical practice, a patient can see different doctors and my get different diagnoses. To decrease the risk of misdiagnosis, we should carefully consider all the doctors' comments. In that case, for any a patient set A who has some symptoms in universe U, patient set A is an SVNHF set on symptom set U. That is, $A = \{< x_i, \tilde{t}_A(x_i), \tilde{i}_A(x_i), \tilde{f}_A(x_i) > |x_i \in U\}$, now the problem is that a decision maker needs to make a reasonable decision about how to judge what kind of the disease y_i patient A is suffering from.

In the following, we present an approach to the decision making for this kind of problem by using the SVNHF rough set theory over two universes.

Algorithm

Step 1. by Definition 7, we calculate the lower and upper approximations $\underline{R}(A)$ and $\overline{R}(A) of A$.
Step 2. from Definition 11, we can obtain $\underline{R} \oplus \overline{R}$
Step 3. on the basis of Definition 10, the score function of SVNHF elements are obtained by us.

Denote
$\lambda_j = s(\underline{R} \oplus \overline{R}) = s(\tilde{t}_{\underline{R}}(y_j) \oplus \tilde{t}_{\overline{R}}(y_j), \tilde{i}_{\underline{R}}(y_j) \oplus \tilde{i}_{\overline{R}}(y_j), \tilde{f}_{\underline{R}}(y_j) \oplus \tilde{f}_{\overline{R}})$

Step 4. the optimal decision is to select y_ℓ if $\lambda_\ell = max\lambda_j, j = 1, 2, ...V$
we conclude that patient A is suffering from the disease y_ℓ.

TABLE I
SYMPTOMS CHARACTERISTIC FOR THE CONSIDERD DIAGNOSES

R	y_1	y_2
x_1	{(0.6,0.2,0.1),(0.2,0.1,0.3),(0.2,0.6,0.1)}	{ (0.2,0.3),(0.1,0.3),(0.4)}
x_2	{ (0.5,0.2),(0.2,0.1),(0.4,0.3) }	{(0.6),(0.2,0.4),(0.4) }
x_3	{ (0.4,0.2,0.5),(0.1),(0.3,0.4) }	{(0.5,0.1),(0.2,0.1,0.4),(0.2,0.1,0.6) }
x_4	{ (0.3,0.4,0.5),(0.1,0.5),(0.3,0.2) }	{(0.5,0.7),(0.2,0.3),(0.8,0.4) }
x_5	{ (0.1,0.1,0.3),(0.4,0.6,0.2),(0.8,0.3,0.1) }	{ (0.4,0.1,0.3),(0.6,0.5),(0.5,0.1,0.2) }

TABLE II
SYMPTOMS CHARACTERISTIC FOR THE CONSIDERED DIAGNOSES

R	y_3	y_4
x_1	{(0.1,0.7),(0.5,0.2), (0.3,0.2)}	{(0.2,0.7,0.1),(0.1,0.6),(0.2)}
x_2	{ (0.2,0.6),(0.1,0.6),(0.3,0.1,0.5) }	{(0.3,0.1,0.4),(0.4,0.2,0.4),(0.8,.2,0.1)}
x_3	{ (0.2,0.8),(0.1,0.9),(0.5,0.6) }	{ (0.8,0.3,0.1),(0.4,0.1,0.3),(0.6,0.4)}
x_4	{ (0.2,0.3,0.6),(0.1,0.2),(0.4,0.2) }	{ (0.5,0.4,0.3),(0.4),(0.8,0.1) }
x_5	{(0.9,0.1,0.4),(0.1),(0.4,0.3) }	{(0.1,0.2),(0.1,0.5,03),(0.1,0.4)}

V. A NUMERICAL EXAMPLE

In this section, we will apply the decision approach proposed in Section IV to a medical diagnose problem.

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ be five symptoms in clinic, where x_i stands for " Headache," "Nausea", "stomach pain", "Vomiting", "temperature", and the universe $V = \{y_1, y_2, y_3, y_4\}$ be four diseases, where y_j stand for " Hepatitis," "peptic ulcer", "malaria", "typhoid", respectively. Let R be SVNHF relation from U to V. Where R is a medical knowledge statistic data of the relationship of the symptom $x_i(x_i \in U)$ and the disease $y_j(y_j \in V)$. The statistic data are given in Tables I and II. In this example, we assume that A represent a patient, and the symptoms of patient A are described by SVNHF set on the universe U. Let

$A = \{(x_1, (0.2, 0.3), (0.2, 0.1, 0.4), (0.1)),$
$(x_2, (0.1, 0.4), (0.5), (0.1, 0.6)),$
$(x_3, (0.1, 0.3, 0.8), (0.2, 0.5), (0.6, 0.2)),$
$((x_4, (0.2, 0.1, 0.4), (0.1, 0.5, 0.1), (0.4, 0.3))\}$

For example $A(x_2)$, doctors cannot present the precise the memberships degree of how pain the stomach of patient A is, but they have certain hesitancy in providing the memberships degrees of how pain the stomach of patient A is. In what follows, we give the decision making process by using the four steps given in Section VI in details.

First, by definition, we calculate the lower and upper approximations $\underline{R}(A)$ and $\overline{R}(A)$ of A. as follows

$\underline{R}(A) = \{(y_1, \{0.2, 0.2, 0.1\}, \{0.5, 0.5, 0.6\}, \{0.4, 0.3, 0.3\}),$
$(y_2, \{0.2, 0.2, 0.2\}, \{0.5, 0.5, 0.5\}, \{0.5, 0.6, 0.6\}),$
$(y_3, \{0.3, 0.2, 0.3\}, \{0.5, 0.6, 0.6\}, \{0.2, 0.6, 0.6\}),$
$(y_4, \{0.2, 0.1, 0.3\}, \{0.5, 0.5, 0.6\}, \{0.6, 0.3, 0.4\})\}$

$\overline{R}(A) = \{(y_1, \{0.2, 0.2, 0.5\}, \{0.1, 0.1, 0.4\}, \{0.2, 0.3, 0.1\}),$
$(y_2, \{0.3, 0.4, 0.4\}, \{0.2, 0.3, 0.3\}, \{0.4, 0.2, 0.4\}),$
$(y_3, \{0.3, 0.4, 0.8\}, \{0.1, 0.2, 0.2\}, \{0.3, 0.2, 0.2\}),$
$(y_4, \{0.2, 0.3, 0.4\}, \{0.2, 0.2, 0.4\}, \{0.2, 0.2, 0.2\})\}$

Then, we have

$\underline{R}(A) \oplus \overline{R}(A) = \{(y_1, \{0.3, 0.3, 0.5\}, \{0.05, 0.05, 0.2\},$
$\{0.08, 0.09, 0.03\}), (y_2, \{0.4, 0.5, 0.5\}, \{0.1, 0.1, 0.1\},$
$\{0.2, 0.1, 0.2\}), (y_3, \{0.5, 0.5, 0.9\}, \{0.05, 0.1, 0.1\},$
$\{0.06, 0.1, 0.1\}), (y_4, \{0.4, 0.4, 0.6\}, \{0.1, 0.1, 0.2\},$
$\{0.1, 0.06, 0.08\})\}$

By definition, we obtain the score functions of SVNHF $\underline{R}(A) + \overline{R}(A)$ as follows:

$S((\underline{R}(A) \oplus \overline{R}(A))(y_1)) = 0.73$
$S((\underline{R}(A) \oplus \overline{R}(A))(y_2)) = 0.72$
$S((\underline{R}(A) \oplus \overline{R}(A))(y_3)) = 0.81$
$S((\underline{R}(A) \oplus \overline{R}(A))(y_4)) = 0.74$

It is clear that the maximum score function is $\lambda_3 = .81$. Hence, the optimal decision is to select y_3. That is, we can conclude that patient A is suffering from the disease malaria y_3.

VI. CONCLUSION

In this paper, we have presented the concept single valued neutrosophic hesitant fuzzy rough sets which is a combination of three powerful topics: neutrosophic, hesitant and rough sets. We defined SVNHF rough approximation operators in term of SVNHF relations. Properties of upper and lower SVNHF rough approximation operators are also investigated. Finally, we develop a general framework for dealing with uncertainty decision-making by using the SVNHF rough sets over two universes. A medical diagnosis problem is also shown to indicate the principle steps of the decision methodology. In the future, we will mainly focus on investigating uncertain measures and knowledge reductions of the SVNHF rough sets.

VII. CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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