

Stability of Property (gm) under Perturbation and Spectral Properties Type Weyl Theorems

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Abstract—A Banach space operator T obeys property (gm) if the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues are exactly those points λ of the spectrum for which $T - \lambda I$ is a left Drazin invertible. In this article, we study the stability of property (gm) , for a bounded operator acting on a Banach space, under perturbation by finite rank operators, by nilpotent operators, by quasi-nilpotent operators, or more generally by algebraic operators commuting with T .

Keywords—Weyl's theorem, Weyl spectrum, polaroid operators, property (gm) , property (m) .

I. INTRODUCTION

THROUGHOUT this paper let $\mathcal{B}(\mathcal{X})$ denote the algebra of bounded operators acting on an infinite complex Banach space \mathcal{X} . We use I to denote the identity operator on \mathcal{X} , and $\mathcal{K}(\mathcal{X})$ to denote the ideal of all compact operators on \mathcal{X} and $\mathcal{F}(\mathcal{X})$ to denote the ideal of all finite rank operators on \mathcal{X} . We shall denote the spectrum, the point spectrum and the approximate point spectrum of $T \in \mathcal{B}(\mathcal{X})$ by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$, respectively. Throughout this paper, the set of all complex numbers and the complex conjugate of a complex number λ will be denoted by \mathbb{C} and $\bar{\lambda}$, respectively. The closure of a set S will be denoted by \bar{S} and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. If K is a subset of \mathbb{C} , then $\text{iso } K$ denotes the set of all isolated points of K and $\text{acc } K$ denotes the set of all points of accumulation of K . We use T^* to denote the adjoint of $T \in \mathcal{B}(\mathcal{X})$. For an arbitrary operator $T \in \mathcal{B}(\mathcal{X})$, $\ker(T)$ denotes its kernel and $\mathfrak{R}(T)$ denotes its range. We set $\alpha(T) = \dim \ker(T)$ and $\beta(T) = \dim \mathcal{X}/\mathfrak{R}(T)$. Let $a := a(T)$ be the ascent of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be the descent of an operator T ; i.e., the smallest nonnegative integer q such that $\mathfrak{R}(T^q) = \mathfrak{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [21, Proposition 38.3]. Moreover, $0 < a(T - \lambda I) = d(T - \lambda I) < \infty$ precisely when λ is a pole of the resolvent of T , see Heuser [21, Proposition 50.2].

Following [20] we say that $T \in \mathcal{B}(\mathcal{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the only analytic function $f : U_\lambda \rightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathcal{B}(\mathcal{X})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from

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the identity Theorem for analytic function it easily follows that $T \in \mathcal{B}(\mathcal{X})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [22, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Denote by

$$SF_+(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) : \alpha(T) < \infty \text{ and } \mathfrak{R}(T) \text{ is closed}\}$$

the class of all upper semi-Fredholm operators, and by

$$SF_-(\mathcal{X}) := \{T \in \mathcal{B}(\mathcal{X}) : \beta(T) < \infty\}$$

the class of all lower semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $SF_\pm(\mathcal{X}) := SF_+(\mathcal{X}) \cup SF_-(\mathcal{X})$, while the class of all Fredholm operator is defined by $\mathfrak{F}(\mathcal{X}) := SF_+(\mathcal{X}) \cap SF_-(\mathcal{X})$. For a semi-Fredholm operator T we define the index, $\text{ind}(T)$, by $\text{ind}(T) = \alpha(T) - \beta(T)$. The upper semi-Weyl operators are defined as the class of Fredholm operators with index less than or equal to 0, while the class of Weyl operators are defined as the class of Fredholm operators of index 0. These classes of operators generate the following spectra: the Weyl spectrum defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Weyl operator}\},$$

the upper semi-Weyl spectrum defined by

$$\sigma_{SF_+}^-(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not an upper semi-Weyl operator}\}.$$

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be Browder (resp. upper semi-Browder, lower semi-Browder) if T is Fredholm and $a(T) = d(T) < \infty$ (resp. T is upper semi-Fredholm and $a(T) < \infty$, T is lower semi-Fredholm and $d(T) < \infty$). The Browder spectrum of $T \in \mathcal{B}(\mathcal{X})$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a Browder operator}\},$$

the upper semi-Browder spectrum is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not an upper semi-Browder operator}\}.$$

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is a Drazin invertible if and only if it has a finite ascent and descent. The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible}\}.$$

Let $\pi(T) := \{\lambda \in \mathbb{C} : a(T - \lambda) = d(T - \lambda) < \infty\}$ be the set of poles. Then $\pi^0(T) := \{\lambda \in \pi(T) : \alpha(T - \lambda) < \infty\}$ is the set of poles of finite rank. We observe that $\pi(T) = \sigma(T) \setminus \sigma_D(T)$. An operator $T \in \mathcal{B}(\mathcal{X})$ is called left Drazin

invertible, $T \in LD(\mathcal{X})$, if $a(T) < \infty$ and $\Re(T^{a(T)+1})$ is closed. The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left Drazin invertible}\}.$$

Let $\pi_a(T) := \{\lambda \in \sigma_a(T) : T - \lambda \text{ is not a left Drazin invertible}\}$ be the set of left poles of T . Then $\pi_a^0(T) := \{\lambda \in \pi_a(T) : \alpha(T - \lambda) < \infty\}$ is the set of left poles of T of finite rank. We observe that $\pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T)$. According also to [21], the space $\Re((T - \lambda)^{\alpha(T-\lambda)+1})$ is closed for each $\lambda \in \pi(T)$. Hence we have always $\pi(T) \subset \pi_a(T)$ and $\pi^0(T) \subset \pi_a^0(T)$. We say that *a-Browder's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in \mathfrak{a}\mathfrak{B}$, if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_a^0(T)$.

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be a *Riesz operator* if $T - \lambda \in \mathfrak{F}(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators.

Suppose that $T \in \mathcal{B}(\mathcal{X})$ and R is a Riesz operator commuting with T . Then it follows from [26, Theorem 1] and [29, Proposition 5] that

$$\sigma_b(T) = \sigma_b(T + R); \quad (1)$$

$$\sigma_w(T) = \sigma_w(T + R); \quad (2)$$

$$\sigma_{ub}(T) = \sigma_{ub}(T + R); \quad (3)$$

$$\sigma_{SF_+^-}(T) = \sigma_{SF_+^-}(T + R). \quad (4)$$

Let $E(T) := \{\lambda \in \text{iso } \sigma(T) : \alpha(T - \lambda) > 0\}$ be the set of all isolated eigenvalues of T and $E_a(T) := \{\lambda \in \text{iso } \sigma_a(T) : \alpha(T - \lambda) > 0\}$ be the set of all eigenvalues of T that are isolated in $\sigma_a(T)$. Then $E^0(T) := \{\lambda \in E(T) : \alpha(T - \lambda) < \infty\}$ is the set of all isolated eigenvalues of T of finite multiplicity and $E_a^0(T) := \{\lambda \in E_a(T) : \alpha(T - \lambda) < \infty\}$ is the set of all eigenvalues of T that are isolated in $\sigma_a(T)$ of finite multiplicity. According to Coburn [19], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\Delta(T) = \pi^0(T)$. According to Rakočević [25], an operator $T \in \mathcal{B}(X)$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$. It is known [25] that an operator satisfying *a-Weyl's theorem* satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in \mathcal{B}(\mathcal{X})$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $\Re(T^n)$ viewed as a map from $\Re(T^n)$ to $\Re(T^n)$ (in particular $T_{[0]} = T$). If for some integer n the range space $\Re(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm operator*. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [12]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm operator* [11]. An operator T is called a *B-Weyl operator* if it is a *B-Fredholm operator* of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl operator}\}$ [13].

Following [14], we say that *generalized Weyl's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in g\mathcal{W}$ if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$ and that *generalized Browder's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in g\mathfrak{B}$, if $\Delta^g(T) = \pi(T)$. It is proved

in [9, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [15, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \pi(T)$, it is proved in [16, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(\mathcal{X})$ be the class of all *upper semi-B-Fredholm operators*,

$$SBF_+(\mathcal{X}) := \{T \in SBF_+(\mathcal{X}) : \text{ind}(T) \leq 0\}.$$

The *upper B-Weyl spectrum* of T is defined by

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+(\mathcal{X})\}.$$

We say that *generalized a-Weyl's theorem* holds for $T \in \mathcal{B}(\mathcal{X})$, $T \in ga\mathcal{W}$, if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$ and that $T \in \mathcal{B}(\mathcal{X})$ obeys *generalized a-Browder's theorem*, $T \in ga\mathfrak{B}$, if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [9, Theorem 2.2] that generalized *a-Browder's theorem* is equivalent to *a-Browder's theorem*, and it is known from [15, Theorem 3.11] that an operator satisfying generalized *a-Weyl's theorem* satisfies *a-Weyl's theorem*, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [16, Theorem 2.10] that generalized *a-Weyl's theorem* is equivalent to *a-Weyl's theorem*.

II. PROPERTY (gm) FOR BOUNDED LINEAR OPERATORS

Definition 1. Let $T \in \mathcal{B}(\mathcal{X})$. We say that T obeys

- (i) property (gw) if $\Delta_a^g(T) = E(T)$ [10].
- (ii) property (gR) if $\sigma_a(T) \setminus \sigma_{LD}(T) = E(T)$ [6].
- (iii) property (gt) if $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$ [27].

In [27, Theorem 2.6] the author proved that T obeys property (gt) if and only if T obeys property (gw) and $\sigma(T) = \sigma_a(T)$.

Definition 2. ([28]) Let $T \in \mathcal{B}(\mathcal{X})$. Then we say that T obeys property (gm) if

$$\sigma(T) \setminus \sigma_{LD}(T) = E(T).$$

Generalized Weyl's theorem corresponds to the half of property (gm), in the following sense:

Theorem 1. ([28]) If $T \in \mathcal{B}(\mathcal{X})$ then the following assertions are equivalent:

- 1) Property (gm) holds for T ;
- 2) T satisfies generalized Weyl's theorem and $\sigma_{LD}(T) = \sigma_{BW}(T)$.

Theorem 2. Let $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:

- (i) Property (gt) holds for T ;
- (ii) T satisfies property (gm) and $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$.

Proof: (i) \implies (ii) As T has property (gt), we have T satisfies generalized Browder's theorem and so $\sigma_{LD}(T) =$

$\sigma_{SBF_+^-}(T)$. Therefore,

$$E(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T satisfies property (gm) .

(ii) \implies (i) Suppose that T obeys property (m) and $\sigma_{LD}(T) = \sigma_{SBF_+^-}(T)$. Then

$$E(T) = \sigma(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T).$$

That is, T obeys property (gt) . ■

Remark 1. Let $T \in \mathcal{B}(\mathcal{X})$. If T^* has the SVEP, then it is known from [23, Page 35] that $\sigma(T) = \sigma_a(T)$ and from [3, Theorem 2.9] we have $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_D(T)$. Hence $E_a(T) = E(T)$, $\Delta^g(T) = \Delta_a^g(T)$, $\Delta_+^g(T) = \Delta^g(T)$ and $\sigma(T) \setminus \sigma_{LD}(T) = \Delta^g(T)$.

Theorem 3. Let $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions are equivalent:

- (i) Property (gm) holds for T ;
- (ii) T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$.

Proof: (i) \implies (ii) Assume that T obeys property (gm) . It then follows from Theorem 1 that T satisfies generalized Weyl's theorem and $\sigma_{LD}(T) = \sigma_{BW}(T)$ and hence T satisfies generalized Browder's theorem and $\pi(T) = E(T)$. Therefore

$$\pi_a(T) = \sigma_a(T) \setminus \sigma_{LD}(T) \subseteq \sigma(T) \setminus \sigma_{LD}(T) = E(T) = \pi(T) \subseteq \pi_a(T).$$

So, $E(T) = \pi_a(T)$, i.e., T obeys property (gR) and $\sigma(T) = \sigma_a(T)$.

(ii) \implies (i) Suppose that T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$. Then

$$\pi_a(T) = E(T) = \sigma_a(T) \setminus \sigma_{LD}(T) = \sigma(T) \setminus \sigma_{LD}(T).$$

That is, T obeys property (gm) . ■

A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely isoloid (respectively, finitely a -isoloid) if every isolated point of $\sigma(T)$ (respectively, every isolated point of $\sigma_a(T)$) is an eigenvalue of T having finite multiplicity.

Theorem 4. Suppose that $T \in \mathcal{B}(\mathcal{X})$ is a finitely a -isoloid operator and suppose that there is an injective quasi-nilpotent operator Q which commutes with T . Then T obeys property (gm) .

Proof: First we note that $E(T) = \emptyset$. Indeed, suppose that $\lambda \in E(T)$. Then λ is an isolated point of $\sigma(T)$ and hence belong to $\sigma_a(T)$. Thus $\lambda \in \text{iso } \sigma_a(T)$, so that $0 < \alpha(T - \lambda) < \infty$, since T is finitely a -isoloid. But from [6, Lemma 3.6] we also have $\alpha(T - \lambda) = 0$, and this is impossible. Therefore, $E(T) = \emptyset$.

In order to show property (gm) holds for T , it suffices to prove that $\sigma(T) \setminus \sigma_{LD}(T) = \emptyset$. Let $\lambda \in \sigma(T) \setminus \sigma_{LD}(T)$. Then $\lambda \in \sigma(T)$ and $T - \lambda$ is left Drazin invertible. We distinguish two cases. Firstly, if $\lambda \in \sigma_a(T)$. By [2, Theorem 2.7] λ is an isolated point of $\sigma_a(T)$, and since T is finitely a -isoloid we then have $\alpha(T - \lambda) < \infty$. Again by [6, Lemma 3.6] we then conclude that $T - \lambda$ is injective. On the other hand, by [4, Lemma 2.4], we have $T - \lambda \in SF_+(\mathcal{X})$, so $T - \lambda$ has closed range and hence $T - \lambda$ is bounded below, i.e., $\lambda \notin \sigma_a(T)$,

a contradiction. If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \pi_a(T)$. Hence $\lambda \in \sigma_{LD}(T)$, a contradiction. Therefore, $\sigma(T) \setminus \sigma_{LD}(T) = \emptyset$, and consequently T satisfies property (gm) . ■

III. PROPERTY (gm) UNDER PERTURBATIONS BY FINITE RANK OPERATORS

We begin with the following lemmas in order to give the proof of the main result in this section.

Lemma 1. ([24, Lemma 2.1]) Let $T \in \mathcal{B}(\mathcal{X})$. If F is an arbitrary finite rank operator on \mathcal{X} , such that $FT = TF$, then for all $\mu \in \mathbb{C}$:

$$\mu \in \text{acc } \sigma(T) \iff \mu \in \text{acc } \sigma(T + F).$$

Remark 2. If $T \in \mathcal{B}(\mathcal{X})$ is an isoloid and F is an arbitrary finite rank operator on \mathcal{X} , such that $FT = TF$, then it follows from Lemma 1 that

$$E(T + F) \cap \sigma(T) \subseteq E(T).$$

Remark 3. We conclude from [17, Theorem 2.1] that if $T \in \mathcal{B}(\mathcal{X})$ and $F \in \mathcal{F}(\mathcal{X})$ such that $TF = FT$, then

$$\sigma_{LD}(T) = \sigma_{LD}(T + F). \quad (5)$$

Recall that $T \in \mathcal{B}(\mathcal{X})$ is isolated, provided that all isolated points of $\sigma(T)$ are eigenvalues of T . $T \in \mathcal{B}(\mathcal{X})$ is a -isolated provided that all isolated points of $\sigma_a(T)$ are eigenvalues of T . It is well-known that $\partial\sigma(T) \subseteq \sigma_a(T)$, so all isolated points of $\sigma(T)$ are also isolated points of $\sigma_a(T)$. Now it is obvious that if T is a -isolated, then it is also isolated.

Theorem 5. Let $T \in \mathcal{B}(\mathcal{X})$. Suppose that F is an arbitrary finite rank operator and $TF = FT$. If T is isoloid and property (gm) holds for T , then property (gm) holds for $T + F$.

Proof: It is enough to prove that $0 \in \sigma(T + F) \setminus \sigma_{LD}(T + F)$ if and only if $0 \in E(T + F)$.

Firstly we prove that if $0 \in \sigma(T + F) \setminus \sigma_{LD}(T + F)$, then $T + F$ is left Drazin invertible and $0 < \alpha(T + F)$. We need to prove that $0 \in \text{iso } \sigma(T + F)$. It follows that $T \in LD(\mathcal{X})$, so $0 \notin \sigma_{LD}(T)$. It is possible that $0 \notin \sigma(T)$. In this case we get from Lemma 1 that $0 \notin \text{acc } \sigma(T)$ and hence $0 \notin \text{acc } \sigma(T + F)$, so $0 \in E(T + F)$. The second possibility is that $0 \in \sigma(T)$. Since property (gm) holds for T , we get that $0 \notin \text{acc } \sigma(T)$ and again $0 \in E(T + F)$.

To prove the opposite implication, suppose that $0 \in E(T + F)$. Then $0 \in \text{iso } \sigma(T + F)$ and $0 < \alpha(T + F)$. Hence $0 \notin \text{acc } \sigma(T)$ and so it follows that $0 \leq \alpha(T)$. Again we distinguish two cases. Firstly, if $0 \notin \sigma(T)$, then $T \in LD(\mathcal{X})(\mathcal{X})$ and by Remark 3 $T + F \in LD(\mathcal{X})$, $0 \in \sigma(T + F) \setminus \sigma_{LD}(T + F)$. On the other hand, if $0 \in \sigma(T)$ then $0 \in \text{iso } \sigma(T)$. Since T is isoloid, we get that $0 < \alpha(T)$ and $0 \notin \sigma_{LD}(T)$. Now, we have $T \in LD(\mathcal{X})$, $T + F \in LD(\mathcal{X})$ and $0 \in \sigma(T + F) \setminus \sigma_{LD}(T + F)$. ■

Example 1. Let $S : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ be an injective quasinilpotent operator which is not nilpotent. We define T on the Banach space $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$. Then $\sigma(T) = \sigma_a(T) = \{0, 1\}$ and $E(T) = \{1\}$. It follows

that $\sigma_{BW}(T) = \{0\}$ and hence $\sigma_{SBF_+^-}(T) = \sigma_{LD}(T) = \{0\}$. Hence $\sigma(T) \setminus \sigma_{LD}(T) = E(T)$ and T obeys property (gm) .

We define the operator U on $\ell^2(\mathbb{N})$ by $U(x_1, x_2, \dots) := (-x_1, 0, 0, \dots)$ and $F = U \oplus 0$ on the Banach space $\mathcal{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then F is a finite rank operator commuting with T . On the other hand, $\sigma(T + F) = \sigma_a(T + F) = \{0, 1\}$ and $E(T + F) = \{0, 1\}$. As $\sigma_{LD}(T + F) = \sigma_{LD}(T) = \{0\}$, then $\sigma(T + F) \setminus \sigma_{LD}(T + F) = \{1\} \neq E(T + F)$ and $T + F$ does not satisfy property (gm) .

Theorem 6. Let $T \in \mathcal{B}(\mathcal{X})$ and let F be a finite rank operator commuting with T . If T satisfies property (gm) , then the following properties are equivalent.

- (i) $T + F$ satisfies property (gm) ;
- (ii) $E(T) = E(T + F)$.

Proof: Assume that $T + F$ satisfies property (gm) , then

$$\sigma(T + F) \setminus \sigma_{LD}(T + F) = E(T + F).$$

As $\sigma(T + F) = \sigma(T)$ and $\sigma_{LD}(T) = \sigma_{LD}(T + F)$ then $\sigma(T) \setminus \sigma_{LD}(T) = E(T + F)$. Since T obeys property (gm) , then $E(T) = \sigma(T) \setminus \sigma_{LD}(T)$. So, $E(T) = E(T + F)$. Conversely, assume that $E(T + F) = E(T)$, then as T obeys property (gm) we have

$$E(T + F) = E(T) = \sigma(T) \setminus \sigma_{LD}(T) = \sigma(T + F) \setminus \sigma_{LD}(T + F)$$

and hence $T + F$ obeys property (gm) . ■

Lemma 2. ([30]) Let $T \in \mathcal{B}(\mathcal{X})$ and let $F \in \mathcal{B}(\mathcal{X})$ with $F^n \in \mathcal{F}(\mathcal{X})$ for some $n \in \mathbb{N}$. If T commutes with F , then

$$\sigma_{BW}(T) = \sigma_{BW}(T + F). \quad (6)$$

$$\sigma_D(T) = \sigma_D(T + F). \quad (7)$$

$$\sigma_{LD}(T) = \sigma_{LD}(T + F). \quad (8)$$

Theorem 7. Let $T \in \mathcal{B}(\mathcal{X})$ be an isoloid and let $F \in \mathcal{B}(\mathcal{X})$ with $F^n \in \mathcal{F}(\mathcal{X})$ for some $n \in \mathbb{N}$. If T commutes with F , then

$$E(T) = E(T + F).$$

Proof: Let $\lambda \in E(T + F)$. Then λ is an isolated point of $\sigma(T + F)$, and since $\alpha(T + F - \lambda) > 0$ we then have $\lambda \in \sigma(T + F) = \sigma(T)$. Therefore, it follows from Remark 2 that $\lambda \in E(T)$. By symmetry, we have the other inclusion. ■

Theorem 8. Let $T \in \mathcal{B}(\mathcal{X})$ be an isoloid obeys property (gm) and let $F \in \mathcal{B}(\mathcal{X})$ with $F^n \in \mathcal{F}(\mathcal{X})$ for some $n \in \mathbb{N}$. If T commutes with F , then $T + F$ obeys property (gm) .

Proof: As T obeys property (gm) . Then

$$\begin{aligned} E(T) &= \sigma(T) \setminus \sigma_{LD}(T) \\ &= \sigma(T + F) \setminus \sigma_{LD}(T + F) \quad (\text{by Lemma 2}) \\ &= E(T + F) \quad (\text{by Theorem 7}). \end{aligned}$$

Hence, $T + F$ obeys property (gm) . ■

IV. PROPERTY (gm) UNDER PERTURBATION BY QUASI-NILPOTENT OPERATORS

First, observe that if Q is quasi-nilpotent and commutes with $T \in \mathcal{B}(\mathcal{X})$ then

$$\sigma(T) = \sigma(T + Q) \quad \text{and} \quad \sigma_a(T) = \sigma_a(T + Q). \quad (9)$$

In particular both equalities holds for commuting nilpotent operators.

Suppose that $T \in \mathcal{B}(\mathcal{X})$ and that $N \in \mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T . Then from the proof of [18, Theorem 3.5], we have

$$\alpha(T + N) > 0 \iff \alpha(T) > 0. \quad (10)$$

Hence by Equation (9), we have the following equation:

$$E(T + N) = E(T). \quad (11)$$

Lemma 3. Suppose that $T \in \mathcal{B}(\mathcal{X})$ and that $N \in \mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T . Then

$$\sigma_{LD}(T + N) = \sigma_{LD}(T). \quad (12)$$

Proof: It follows from [30, Corollary 3.8] that $\pi_a(T + N) = \pi_a(T)$. Then

$$\begin{aligned} \sigma_{LD}(T + N) &= \sigma_a(T + N) \setminus \pi_a(T + N) \\ &= \sigma_a(T) \setminus \pi_a(T + N) \quad (\text{by Equation 9}) \\ &= \sigma_a(T) \setminus \pi_a(T) \\ &= \sigma_{LD}(T). \end{aligned}$$

So, the proof of the lemma is achieved. ■

Theorem 9. Suppose that $T \in \mathcal{B}(\mathcal{X})$ has property (gm) and that $N \in \mathcal{B}(\mathcal{X})$ is a nilpotent operator commuting with T . Then $T + N$ has property (gm) .

Proof: As T obeys property (gm) , we have

$$\begin{aligned} E(T + N) &= E(T) \quad (\text{by Equation 11}) \\ &= \sigma(T) \setminus \sigma_{LD}(T) \\ &= \sigma(T + N) \setminus \sigma_{LD}(T + N) \quad (\text{by Equation (9) and Lemma 3}) \end{aligned}$$

That is, $T + N$ obeys property (gm) . ■

The following example shows that property (gm) is not stable under commuting quasi-nilpotent perturbations.

Example 2. Let $Q : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ be a quasi-nilpotent operator defined by

$$Q(x_1, x_2, \dots) := \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots \right) \quad \text{for all } (x_n) \in \frac{x_2}{2}.$$

Then Q is quasi-nilpotent, $\sigma(Q) = \sigma_{LD}(Q) = \{0\}$ and $E(T) = \{0\}$. Take $T = 0$. Clearly, T satisfies property (gm) , but $T + Q = Q$ fails to satisfy property (gm) .

A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to be polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T and that $T \in \mathcal{B}(\mathcal{X})$ is said to be a -polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T . It is known that T is polaroid if and only if T^* is polaroid and evidently,

$$T \text{ } a\text{-polaroid} \implies T \text{ polaroid}, \quad (13)$$

while, in general, the converse does not hold.

Theorem 10. Let $T \in \mathcal{B}(\mathcal{X})$ obeys property (gm) . If T is a -polaroid and finitely isoloid, Q is a quasi-nilpotent operator which commutes with T , then $T + Q$ obeys property (gm) .

Proof: It follows from [6, Theorem 4.8] that $T + Q$ is a -polaroid and hence by [6, Theorem 3.2], we have $T + Q$ obeys property (gR) . As T obeys property (gm) , we have by Theorem 3 that T satisfies property (gR) and $\sigma(T) = \sigma_a(T)$. Therefore,

$$\begin{aligned} E(T + Q) &= \sigma_a(T + Q) \setminus \sigma_{LD}(T + Q) \\ &= \sigma_a(T) \setminus \sigma_{LD}(T + Q) \\ &= \sigma(T) \setminus \sigma_{LD}(T + Q) \\ &= \sigma(T + Q) \setminus \sigma_{LD}(T + Q). \end{aligned}$$

That is, $T + Q$ obeys property (gm) . ■

V. PROPERTY (gm) UNDER PERTURBATIONS BY ALGEBRAIC OPERATORS

We shall consider algebraic perturbations of operators satisfying property (gm) .

A bounded linear operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

Let $H_{nc}(T)$ denotes the set of all complex-valued functions f , defined and regular in some neighborhood of $\sigma(T)$, such that f is not constant on the connected components of its domain of definition.

Theorem 11. Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with T .

- (i) If T^* is hereditarily polaroid and has SVEP, then $T + K$ obeys property (gm) .
- (ii) If T is hereditarily polaroid and has SVEP, then $T^* + K^*$ obeys property (gm) .

Proof: (i) Obviously, K^* is algebraic and commutes with T^* . Moreover, by [7, Theorem 2.15], we have $T^* + K^*$ is polaroid, or equivalently, $T + K$ is polaroid. Since T^* has SVEP then by [5, Theorem 2.14], we have $T^* + K^*$ has SVEP. Therefore, $T + K$ obeys property (gm) by [28, Theorem 3.4 (i)].

(ii) It follows from the proof of Theorem 2.15 of [7] that $T + K$ is polaroid and hence by duality $T^* + K^*$ is polaroid. Since T has SVEP then it follows from [5, Theorem 2.14] that $T + K$ has SVEP. Therefore, $T^* + K^*$ obeys property (m) by [28, Theorem 3.3 (ii)]. ■

Theorem 12. Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with T .

- (i) If T^* is hereditarily polaroid and has SVEP, then $f(T + K)$ obeys property (gm) for all $f \in H_{nc}(\sigma(T))$.
- (ii) If T is hereditarily polaroid and has SVEP, then $f(T^* + K^*)$ obeys property (gm) for all $f \in H_{nc}(\sigma(T))$.

Proof: (i) We conclude from [7, Theorem 2.15] that $T + K$ is polaroid and hence by [8, Lemma 3.11], we have $f(T + K)$ is polaroid and from [5, Theorem 2.14] that $T^* + K^*$ has SVEP. The SVEP of $T^* + K^*$ entails the SVEP for $f(T^* + K^*)$ by [1, Theorem 2.40]. So, $f(T + K)$ obeys property (m) by [28, Theorem 3.4 (i)].

(ii) The proof of part (ii) is analogous. ■

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