

# On Chvátal's Conjecture for the Hamiltonicity of 1-Tough Graphs and Their Complements

Shin-Shin Kao, Yuan-Kang Shih, Hsun Su

**Abstract**—In this paper, we show that the conjecture of Chvátal, which states that any 1-tough graph is either a Hamiltonian graph or its complement contains a specific graph denoted by  $F$ , does not hold in general. More precisely, it is true only for graphs with six or seven vertices, and is false for graphs with eight or more vertices. A theorem is derived as a correction for the conjecture.

**Keywords**—Complement, degree sum, Hamiltonian, tough.

## I. INTRODUCTION

EVER since Chvátal introduced the concept of toughness of graphs, numerous studies have been done, see [1] for a survey. In [2], which was originally published in 1973, Chvátal posted seven conjectures. Five of the conjectures regard the existence of a minimum toughness that guarantees a certain cycle structure in any graph, one of them is about the Hamiltonicity of 2-tough neighborhood-connected graphs, and the other one relates the existence of a Hamiltonian cycle of any 1-tough graph with its complement graph. These conjectures are inspiring and have led to a bountiful harvest of results. So far, the minimum toughness  $t_0$  which makes the conjecture “there exists  $t_0$  such that every  $t_0$ -tough graph is hamiltonian” hold has not been found. The best result by now is published by Bauer et al. [3], who showed that if such a  $t_0$  exists, it must be  $t_0 \geq \frac{9}{4}$ . For Chvátal's conjecture regarding the Hamiltonicity of any 1-tough graph and its complement, which is presented below, much fewer researches are done.

**Conjecture 1.** (see [2]) If  $G$  is 1-tough, then either  $G$  is Hamiltonian or its complement  $\bar{G}$  contains the graph  $F$  in Fig. 1 (a).

In this paper, we are devoted to the study of the above conjecture. Since  $F$  has six vertices, it is obvious that Conjecture 1 deals with graphs with at least six vertices. We shall give graphic examples showing that Conjecture 1 is not true when  $|G| = n \geq 8$ , and a proof that the conjecture holds for  $|G| = n \in \{6, 7\}$ . Our corrections of Chvátal's conjecture will be presented as Theorem 5 and 6. This paper is organized as follows. Notations, terminologies, and some known theorems are given in Section II, and our main results are shown in

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## Section III.

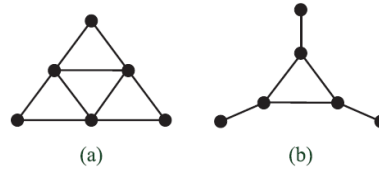


Fig. 1 (a) The graph  $F$ . (b) The complement graph of  $F$ , denoted by  $\bar{F}$

## II. TERMINOLOGY AND KNOWN RESULTS

Let  $G = (V, E)$  be a finite and simple graph with its vertex set  $V$  and edge set  $E$ . Two vertices  $u$  and  $v$  are *adjacent* in  $G$  if  $(u, v) \in E$ . For any  $u \in V$ , the *neighborhood* of  $u$  in  $G$  is defined by  $N_G(u) = \{v | (u, v) \in E\} \subset V$ . The *degree* of  $u$  in  $G$ , denoted by  $\deg_G(u)$ , is the number  $|N_G(u)|$ . The *minimum degree*  $\delta(G)$  of  $G$  is defined as  $\delta(G) = \min\{\deg_G(u) | u \in V\}$ .  $\sigma_k(G)$  denotes the minimum degree sum taken over all independent sets of  $k$  vertices of  $G$ . The *complement graph*  $\bar{G} = (V', E')$  of a graph  $G = (V, E)$  is defined as  $V = V'$  and  $E' = \{(u, v) | (u, v) \text{ does not belong to } E \vee u, v \in V\}$ . For undefined notations and terminologies, we follow [4].

A path  $P$  between two vertices  $v_0$  and  $v_k$  is represented by  $P = \langle v_0, v_1, \dots, v_k \rangle$ , where all vertices are different and every two consecutive vertices are adjacent. We also write the path  $P = \langle v_0, v_1, \dots, v_k \rangle$  as  $\langle v_0, v_1, \dots, v_i, P', v_j, v_{j+1}, \dots, v_k \rangle$ , where  $P'$  denotes the path  $\langle v_i, v_{i+1}, \dots, v_j \rangle$ . A path of  $G$  is called a *Hamiltonian path* if it traverses all vertices of  $V$  exactly once. A cycle of  $G$  is called a *Hamiltonian cycle* if the cycle traverses all vertices of  $V$  exactly once except the beginning vertex and the end vertex. We say that a graph  $G$  is *Hamiltonian* if there exists a Hamiltonian cycle in  $G$ . The *circumference*  $c(G)$  of a graph  $G$  is defined as the length of the longest cycle in  $G$ . We define  $k$  as the *vertex connectivity* of  $G$ , and  $k(G)$  the number of components of  $G$ . Suppose  $G$  is not a complete graph. We say  $G$  is *t-tough* if  $t$  is a nonnegative real number and  $t \leq |S|/k(G - S)$ , where  $S$  is a vertexcut of  $G$ . The maximum real number  $t$  for which  $G$  is  $t$ -tough is called the *toughness* of  $G$ , and the toughness of any complete graph is  $\infty$ . It is known that every Hamiltonian graph is 1-tough, and every 1-tough graph is 2-connected.

Let  $G_1$  and  $G_2$  be two graphs.  $G_1$  and  $G_2$  are called *disjoint* if  $G_1$  and  $G_2$  have no vertex in common. The *union* of two disjoint graphs,  $G_1$  and  $G_2$ , denoted by  $G_1 + G_2$ , is a graph with  $V(G_1 + G_2) = V(G_1) \cup V(G_2)$  and  $E(G_1 + G_2) = E(G_1) \cup E(G_2)$ . The union of  $n$  copies of a graph  $G$  is written as  $nG$ . Obviously,  $\bar{K}_n = nK_1$ . The *join* of two disjoint subgraphs  $G_1$  and  $G_2$ ,

denoted by  $G_1 \vee G_2$ , is the graph obtained from  $G_1 + G_2$  by joining each vertex of  $G_1$  to each vertex of  $G_2$ .

Here, we list some known theorems, which will be used in the following sections.

**Theorem 1.** (see [5], [6].) If  $G$  is a 1-tough graph with  $|G| = n \geq 11$  such that  $\sigma_2(G) \geq n - 4$ , then  $G$  is hamiltonian.

**Theorem 2.** (see [6], [7].) If  $G$  is a 1-tough graph with  $|G| = n \geq 3$ , then  $c(G) \geq \min\{n, \sigma_2(G) + 2\}$ .

We have an immediate result from the above theorem.

**Corollary 1.** If  $G$  is a 1-tough graph with  $|G| = n \geq 3$  such that  $\sigma_2(G) \geq n - 2$ , then  $G$  is Hamiltonian.

**Theorem 3.** (see [8].) Let  $G$  be a 1-tough graph on  $|G| = n \geq 3$  vertices with  $\delta(G) \geq n/3$ . Then  $c(G) \geq 5n/6 + 1$ .

**Theorem 4.** (see [1].) If  $G$  is a 1-tough graph with  $|G| = n \geq 3$  and  $\sigma_3(G) \geq n + k - 2$ , then  $G$  is Hamiltonian.

III. MAIN RESULTS

It is easy to see that the complete bipartite graph  $K_{m,m}$ , where  $m \geq 6$ , is 1-tough, Hamiltonian, and its complement  $\overline{K_{m,m}} = K_m \cup K_m$  contains  $F$  in Fig. 1 (a). Thus,  $K_{m,m}$  provides a family of bipartite graphs which are counterexamples to Conjecture 1. For nonbipartite cases, let  $n \geq 8$ , and  $D_n = \{\overline{K_3} \vee K_2 \vee K_{n-5}\} \cup \{(a,x), (b,y), (c,z)\}$ , where  $\{a, b, c\}$  are the three isolated vertices of  $\overline{K_3}$  and  $\{x, y, z\} \in V(K_{n-5})$ . See Fig. 2 (a) for an illustration.

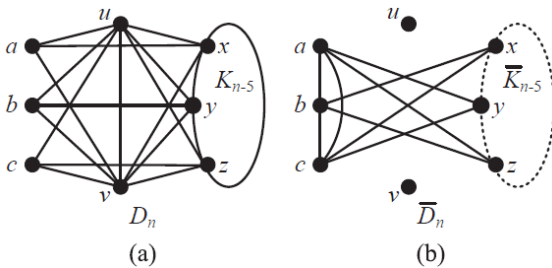


Fig. 2 (a) The graph  $D_n$ . (b) The complement graph of  $D_n$ , denoted by  $\overline{D_n}$

We have the following lemma.

**Lemma 1.** For  $n \geq 8$ ,  $D_n$  is 1-tough, hamiltonian, and its complement graph  $\overline{D_n}$  contains the graph  $F$ .

**Proof.** By brute force,  $D_n$  is 1-tough. (In fact,  $D_8$  is  $\frac{4}{3}$ -tough and  $D_n$  is  $\frac{5}{4}$ -tough for  $n \geq 9$ .) Next, we will show that  $D_n$  is Hamiltonian. Because  $K_{n-5}$  is a complete graph, there exists a Hamiltonian path  $P$  in  $K_{n-5}$  between  $x$  and  $y$ . Thus  $D_n$  has a Hamiltonian cycle  $(x, a, u, c, v, b, y, P, x)$ . On the other hand,  $\overline{D_n}$  contains the edges,  $\{(a, b), (b, c), (c, a), (a, y), (y, c), (b, x), (x, c), (a, z), (z, b)\}$ , which implies that  $\overline{D_n}$  contains the graph  $F$ . Therefore,  $D_n$  serves to illustrate that Conjecture 1 is false.

Theorem 5 affirms that Conjecture 1 is true for graphs with six or seven vertices.

**Theorem 5.** Let  $|G| = n \in \{6,7\}$ . If  $G$  is 1-tough, then either  $G$  is Hamiltonian or its complement  $\overline{G}$  contains the graph  $F$ .

**Proof.** We consider  $G$  with  $|G| = 6$  first. In this case, we want

to show that  $G$  is Hamiltonian and its complement  $\overline{G}$  does not contain  $F$ . By Theorem 3,  $c(G) = 6$ , so  $G$  is a Hamiltonian graph. Assume that  $\overline{G}$  contains  $F$ , then  $G$  must contain fewer edges than  $\overline{F}$ , the complement of  $F$ . See Fig. 1 (b) for an illustration of  $\overline{F}$ . Since the graph  $\overline{F}$  is  $\frac{1}{2}$ -tough,  $G$  cannot be better than  $\frac{1}{2}$ -tough, which violates the known condition that  $G$  is 1-tough. Therefore,  $\overline{G}$  does not contain  $F$ . Next, we consider  $G$  with  $|G| = 7$ . Note that  $G$  being 1-tough implies that  $k \geq 2$ . There are two cases.

**Case 1.**  $\sigma_3(G) \geq 7$ . With Theorem 4,  $G$  contains a Hamiltonian cycle, denoted by  $C_G = \langle 1,2,3,4,5,6,7,1 \rangle$ . Obviously,  $E(G)$  consists of all edges in  $C_G$  and possibly more. Let  $C_7$  be a cycle with length 7 and  $\overline{C_7}$  the complement of  $C_7$ . It is easy to see that  $\overline{C_7}$  does not contain  $F$ , so  $\overline{G}$  cannot contain  $F$ . As a result, Conjecture 1 holds in this case.

**Case 2.**  $\sigma_3(G) \leq 6$ . Since  $k \geq 2$ , this case occurs only when there exists an independent set of three vertices  $\{x, y, z\}$  such that  $deg_G(x) = deg_G(y) = deg_G(z) = 2$ , and  $\sigma_3(G) = 6$ . We shall let  $V(G) = \{x, y, z, a, b, c, d\}$ . Under Case 2, there are three major subcases and totally five possibilities for which we must provide rigorous proofs. Table I gives an illustration for these possible situations. For simplicity, we shall label these subcases by (a), (b), (c) and so on.

TABLE I  
CASE ANALYSIS IN THE PROOF CASE 2 IN THEOREM 5

(a) $ N_G(x) \cup N_G(y) \cup N_G(z)  = 2$
(b) $ N_G(x) \cup N_G(y) \cup N_G(z)  = 3$
(c) $ N_G(x) \cup N_G(y) \cup N_G(z)  = 4$
(d) Two of $N_G(x)$ , $N_G(y)$ and $N_G(z)$ are identical.
(e) All of $N_G(x)$ , $N_G(y)$ and $N_G(z)$ are different.
(f) None of $N_G(a)$ , $N_G(b)$ , $N_G(c)$ and $N_G(d)$ covers $\{x, y, z\}$ .
(g) One of $N_G(a)$ , $N_G(b)$ , $N_G(c)$ and $N_G(d)$ covers $\{x, y, z\}$ .

- a)  $|N_G(x) \cup N_G(y) \cup N_G(z)| = 2$ . Let  $N_G(x) \cup N_G(y) \cup N_G(z) = \{a, b\}$ . Thus, the subgraph induced by  $\{x, y, z, a, b\}$  is  $K_{3,2}$ , and  $G$  is not Hamiltonian. Removing  $\{a, b\}$  results in a graph with at least four components, so  $G$  is  $\frac{1}{2}$ -tough or weaker. It violates the assumption that  $G$  is 1-tough, so this case cannot happen.
- b)  $|N_G(x) \cup N_G(y) \cup N_G(z)| = 3$ . Let  $N_G(x) \cup N_G(y) \cup N_G(z) = \{a, b, c\}$ . Removing  $\{a, b, c\}$  results in a graph with at least four components, so  $G$  is  $\frac{3}{4}$ -tough or weaker. Again, it contradicts the known fact that  $G$  is 1-tough, so this case should not occur.
- c)  $|N_G(x) \cup N_G(y) \cup N_G(z)| = 4$ . There are two possibilities: (d) and (e).
- d) Two of  $N_G(x)$ ,  $N_G(y)$  and  $N_G(z)$  are identical. W.L.O.G., let  $N_G(x) = N_G(y) = \{a, b\}$  and  $N_G(z) = \{c, d\}$ . In this case, removing  $\{a, b\}$  results in a graph with at least three components, so  $G$  is  $\frac{2}{3}$ -tough or weaker, which violates the condition that  $G$  is 1-tough, so this case will not happen.
- e) All of  $N_G(x)$ ,  $N_G(y)$  and  $N_G(z)$  are different. There are two subcases. There are two subcases: (f) and (g).
- f) None of  $N_G(a)$ ,  $N_G(b)$ ,  $N_G(c)$  and  $N_G(d)$  covers  $\{x, y, z\}$ . W.L.O.G., let  $N_G(x) = \{a, b\}$ ,  $N_G(y) = \{b, c\}$ , and  $N_G(z) = \{c, d\}$ . See Fig. 3 for an illustration. If  $(a, d) \in E(G)$ , then

$(a, x, b, y, c, z, d, a)$  is a cycle of length 7, which is a Hamiltonian cycle of  $G$ . The argument in Case 1 shows that  $\bar{G}$  does not contain  $F$ , so Conjecture 1 holds in this case. Now, we discuss the situation when  $(a, d)$  does not belong to  $E(G)$ . The set of edges among  $\{a, b, c, d\}$  contains at most  $\{(a, b), (b, c), (c, d), (a, c), (b, d)\}$ . Removing  $\{b, c\}$  results in a graph with at least three components, so  $G$  is  $\frac{2}{3}$ -tough or weaker. It is not possible.

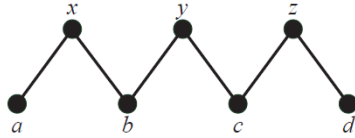


Fig. 3 An illustration for Case 2, (f) in the proof of Theorem 5

g) One of  $N_G(a), N_G(b), N_G(c)$  and  $N_G(d)$  covers  $\{x, y, z\}$ . W.L.O.G., let  $N_G(a)$  be the one covering  $\{x, y, z\}$ , and let  $N_G(x) = \{a, b\}, N_G(y) = \{a, c\}$ , and  $N_G(z) = \{a, d\}$ . It is easy to see that  $G$  must be non-Hamiltonian. Moreover,  $\bar{G}$  contains the triangle with vertices  $\{x, y, z\}$  and the edges  $\{(x, c), (x, d), (y, b), (y, d), (z, b), (z, c)\}$ . That is,  $\bar{G}$  contains  $F$ . Since  $G$  is 1-tough, it can be observed that  $E$  contains  $\{(b, c), (c, d), (b, d)\}$  while the edges  $(a, b), (a, c), (a, d)$  are optional. See Fig. 4 for an illustration. We note that the graph with  $deg_G(a) = 6$  is isomorphic to the graph  $H$  in [2]. Thus, Conjecture 1 is true in this case.

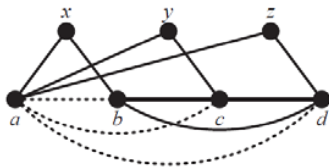


Fig. 4 An illustration for Case 2, (g) in the proof of Theorem 5

From the above derivation, we conclude that for any 1-tough graph  $G$  with  $|G| = 7$ , either  $G$  contains a Hamiltonian cycle or  $G$  is of the form in Fig. 4, of which the complement contains  $F$ .

The following two lemmas are derived in order to obtain the correction for Conjecture 1 for graphs with eight or more vertices. We denote the complement of  $G$  by  $\bar{G}$ . The graph  $F^*$  is shown in Fig. 5.

**Lemma 2.** Let  $G$  be a 1-tough graph with  $|G| = n \geq 11$ . The following three statements are equivalent.

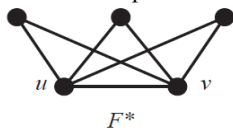


Fig. 5 The graph  $F^*$

- i) There exists some nonadjacent pair  $\{x, y\}$  in  $G$  such that  $deg_G(x) + deg_G(y) \leq n - 5$ .
- ii) There exists an edge  $(x, y)$  of  $\bar{G}$  such that  $deg_{\bar{G}}(x) + deg_{\bar{G}}(y) \geq n + 3$ .
- iii)  $\bar{G}$  contains the graph  $F^*$ .

**Proof.** First of all, we want to show (i) implies (ii). Take the edge  $(u, v)$  of  $\bar{G}$  such that the nonadjacent vertex pair  $\{u, v\}$  in  $G$  satisfies  $deg_G(u) + deg_G(v) \leq n - 5$ . Therefore,  $deg_{\bar{G}}(u) + deg_{\bar{G}}(v)$

$$\begin{aligned}
 &= (n - 1 - deg_G(u)) + (n - 1 - deg_G(v)) \\
 &\geq (2n - 2) - (n - 5) \\
 &= n + 3.
 \end{aligned}$$

Secondly, we need to show (ii) implies (iii). Following (ii), there are  $n - 2$  vertices in  $V(\bar{G}) - \{u, v\}$ . If  $N_{\bar{G}}(u) \cap N_{\bar{G}}(v) = \emptyset$ , then  $deg_{\bar{G}}(u) + deg_{\bar{G}}(v) \leq 1 + 1 + (n - 2) = n$ . It violates (ii). Thus,  $N_{\bar{G}}(u)$  and  $N_{\bar{G}}(v)$  must have at least three common vertices in  $V(\bar{G}) - \{u, v\}$ . This implies that  $\bar{G}$  contains the graph  $F^*$ .

Finally, we will show (iii) implies (i). This part will be shown by deducing a contradiction from the opposite assumption. Assume that  $\bar{G}$  contains the graph  $F^*$ , and  $deg_G(x) + deg_G(y) \geq n - 4$  holds for any nonadjacent pair of vertices  $\{x, y\}$  of  $G$ . With a simple calculation, one can see that  $deg_G(x) + deg_G(y) \leq n + 2$  holds for any edge  $(x, y)$  of  $\bar{G}$ . As in the previous argument, it means that the endvertices  $x$  and  $y$  of any edge  $(x, y)$  in  $\bar{G}$  have at most two common neighbors. Then,  $\bar{G}$  cannot contain  $F^*$ , which violates (iii). Consequently, there must be some nonadjacent pair of vertices  $\{x, y\}$  in  $G$  with  $deg_G(x) + deg_G(y) \leq n - 5$ .

Lemma 3 can be obtained using the similar derivation as in Lemma 2.

**Lemma 3.** Let  $G$  be a 1-tough graph with  $|G| = n \in \{8, 9, 10\}$ . The following three statements are equivalent.

- i) There exists some nonadjacent pair  $\{x, y\}$  in  $G$  with  $deg_G(x) + deg_G(y) \leq n - 3$ .
- ii) There exists an edge  $(x, y)$  of  $\bar{G}$  such that  $deg_G(x) + deg_G(y) \geq n + 1$
- iii) The complement of  $G$ , denoted by  $\bar{G}$ , contains the graph  $K_3$ .

Our correction of Conjecture 1 for graphs with eight or more vertices is presented below.

**Theorem 6.** Let  $G$  be a 1-tough graph with  $|G| = n \geq 8$ . Then

- a) For  $n \geq 11$ , either  $\sigma_2(G) \geq n - 4$  or  $\bar{G}$  contains  $F^*$ .
- b) For  $n \in \{8, 9, 10\}$ , either  $\sigma_2(G) \geq n - 2$  or  $\bar{G}$  contains  $K_3$ .

**Proof.** We will explain (a), where  $n \geq 11$ , in detail and skip the similar discussion for (b). There are two cases.

**Case 1.** Suppose that  $deg_G(x) + deg_G(y) \leq n - 4$  holds for any nonadjacent pair of vertices  $\{x, y\}$  of  $G$ . With Theorem 1,  $G$  is Hamiltonian. Note that the degree-sum condition is the sufficient condition for  $G$  to be Hamiltonian, and the converse is not true.

**Case 2.** Suppose that there exists some nonadjacent pair of vertices  $\{x, y\}$  of  $G$  such that  $deg_G(x) + deg_G(y) \leq n - 5$ . With Lemma 2, it is equivalent to saying that  $\bar{G}$  contains  $F^*$ .

Combining Case 1 and 2, (a) is verified. When we apply Corollary 1 for (b) concerning the case where  $\sigma_2(G) \geq n - 2$ , the similar difficulty appears. The fact that the degree-sum condition provides only the sufficient condition for  $G$  to be Hamiltonian, not the necessary condition prevents us from a

stronger conclusion as in Conjecture 1.

As a result, Theorem 6 corrects Conjecture 1 for  $n \geq 8$  and becomes the best that one can have.

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