

On Chvátal's Conjecture for the Hamiltonicity of 1-Tough Graphs and Their Complements

Shin-Shin Kao, Yuan-Kang Shih, Hsun Su

Section III.

Abstract—In this paper, we show that the conjecture of Chvátal, which states that any 1-tough graph is either a Hamiltonian graph or its complement contains a specific graph denoted by F , does not hold in general. More precisely, it is true only for graphs with six or seven vertices, and is false for graphs with eight or more vertices. A theorem is derived as a correction for the conjecture.

Keywords—Complement, degree sum, Hamiltonian, tough.

I. INTRODUCTION

EVER since Chvátal introduced the concept of toughness of graphs, numerous studies have been done, see [1] for a survey. In [2], which was originally published in 1973, Chvátal posted seven conjectures. Five of the conjectures regard the existence of a minimum toughness that guarantees a certain cycle structure in any graph, one of them is about the Hamiltonicity of 2-tough neighborhood-connected graphs, and the other one relates the existence of a Hamiltonian cycle of any 1-tough graph with its complement graph. These conjectures are inspiring and have led to a bountiful harvest of results. So far, the minimum toughness t_0 which makes the conjecture “there exists t_0 such that every t_0 -tough graph is hamiltonian” hold has not been found. The best result by now is published by Bauer et al. [3], who showed that if such a t_0 exists, it must be $t_0 \geq \frac{9}{4}$. For Chvátal's conjecture regarding the Hamiltonicity of any 1-tough graph and its complement, which is presented below, much fewer researches are done.

Conjecture 1. (see [2]) If G is 1-tough, then either G is Hamiltonian or its complement \bar{G} contains the graph F in Fig. 1 (a).

In this paper, we are devoted to the study of the above conjecture. Since F has six vertices, it is obvious that Conjecture 1 deals with graphs with at least six vertices. We shall give graphic examples showing that Conjecture 1 is not true when $|G| = n \geq 8$, and a proof that the conjecture holds for $|G| = n \in \{6, 7\}$. Our corrections of Chvátal's conjecture will be presented as Theorem 5 and 6. This paper is organized as follows. Notations, terminologies, and some known theorems are given in Section II, and our main results are shown in

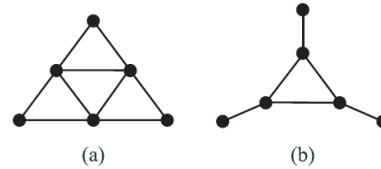


Fig. 1 (a) The graph F . (b) The complement graph of F , denoted by \bar{F}

II. TERMINOLOGY AND KNOWN RESULTS

Let $G = (V, E)$ be a finite and simple graph with its vertex set V and edge set E . Two vertices u and v are *adjacent* in G if $(u, v) \in E$. For any $u \in V$, the *neighborhood* of u in G is defined by $N_G(u) = \{v | (u, v) \in E\} \subset V$. The *degree* of u in G , denoted by $\deg_G(u)$, is the number $|N_G(u)|$. The *minimum degree* $\delta(G)$ of G is defined as $\delta(G) = \min\{\deg_G(u) | u \in V\}$. $\sigma_k(G)$ denotes the minimum degree sum taken over all independent sets of k vertices of G . The *complement* graph $\bar{G} = (V', E')$ of a graph $G = (V, E)$ is defined as $V = V'$ and $E' = \{(u, v) | (u, v) \text{ does not belong to } E \forall u, v \in V\}$. For undefined notations and terminologies, we follow [4].

A path P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \dots, v_k \rangle$, where all vertices are different and every two consecutive vertices are adjacent. We also write the path $P = \langle v_0, v_1, \dots, v_k \rangle$ as $\langle v_0, v_1, \dots, v_i, P', v_j, v_{j+1}, \dots, v_k \rangle$, where P' denotes the path $\langle v_i, v_{i+1}, \dots, v_j \rangle$. A path of G is called a *Hamiltonian path* if it traverses all vertices of V exactly once. A cycle of G is called a *Hamiltonian cycle* if the cycle traverses all vertices of V exactly once except the beginning vertex and the end vertex. We say that a graph G is *Hamiltonian* if there exists a Hamiltonian cycle in G . The *circumference* $c(G)$ of a graph G is defined as the length of the longest cycle in G . We define k as the *vertex connectivity* of G , and $k(G)$ the number of components of G . Suppose G is not a complete graph. We say G is *t-tough* if t is a nonnegative real number and $t \leq |S|/k(G - S)$, where S is a vertexcut of G . The maximum real number t for which G is t -tough is called the *toughness* of G , and the toughness of any complete graph is ∞ . It is known that every Hamiltonian graph is 1-tough, and every 1-tough graph is 2-connected.

Let G_1 and G_2 be two graphs. G_1 and G_2 are called *disjoint* if G_1 and G_2 have no vertex in common. The *union* of two disjoint graphs, G_1 and G_2 , denoted by $G_1 + G_2$, is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. The union of n copies of a graph G is written as nG . Obviously, $\bar{K}_n = nK_1$. The *join* of two disjoint subgraphs G_1 and G_2 ,

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denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2 .

Here, we list some known theorems, which will be used in the following sections.

Theorem 1. (see [5], [6].) If G is a 1-tough graph with $|G| = n \geq 11$ such that $\sigma_2(G) \geq n - 4$, then G is hamiltonian.

Theorem 2. (see [6], [7].) If G is a 1-tough graph with $|G| = n \geq 3$, then $c(G) \geq \min\{n, \sigma_2(G) + 2\}$.

We have an immediate result from the above theorem.

Corollary 1. If G is a 1-tough graph with $|G| = n \geq 3$ such that $\sigma_2(G) \geq n - 2$, then G is Hamiltonian.

Theorem 3. (see [8].) Let G be a 1-tough graph on $|G| = n \geq 3$ vertices with $\delta(G) \geq n/3$. Then $c(G) \geq 5n/6 + 1$.

Theorem 4. (see [1].) If G is a 1-tough graph with $|G| = n \geq 3$ and $\sigma_3(G) \geq n + k - 2$, then G is Hamiltonian.

III. MAIN RESULTS

It is easy to see that the complete bipartite graph $K_{m,m}$, where $m \geq 6$, is 1-tough, Hamiltonian, and its complement $\overline{K_{m,m}} = K_m \cup K_m$ contains F in Fig. 1 (a). Thus, $K_{m,m}$ provides a family of bipartite graphs which are counterexamples to Conjecture 1. For nonbipartite cases, let $n \geq 8$, and $D_n = \{K_3 \vee K_2 \vee K_{n-5}\} \cup \{(a,x), (b,y), (c,z)\}$, where $\{a,b,c\}$ are the three isolated vertices of $\overline{K_3}$ and $\{x,y,z\} \in V(K_{n-5})$. See Fig. 2 (a) for an illustration.

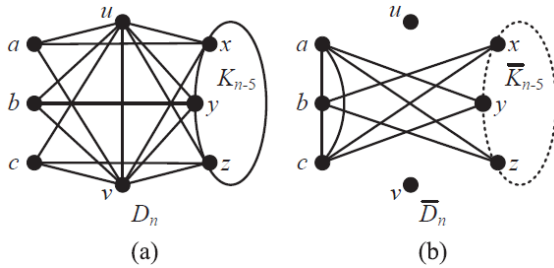


Fig. 2 (a) The graph D_n . (b) The complement graph of D_n , denoted by $\overline{D_n}$

We have the following lemma.

Lemma 1. For $n \geq 8$, D_n is 1-tough, hamiltonian, and its complement graph $\overline{D_n}$ contains the graph F .

Proof. By brute force, D_n is 1-tough. (In fact, D_8 is $\frac{4}{3}$ -tough and D_n is $\frac{5}{4}$ -tough for $n \geq 9$.) Next, we will show that D_n is Hamiltonian. Because K_{n-5} is a complete graph, there exists a Hamiltonian path P in K_{n-5} between x and y . Thus D_n has a Hamiltonian cycle $(x, a, u, c, v, b, y, P, x)$. On the other hand, $\overline{D_n}$ contains the edges, $\{(a,b), (b,c), (c,a), (a,y), (y,c), (b,x), (x,c), (a,z), (z,b)\}$, which implies that $\overline{D_n}$ contains the graph F . Therefore, D_n serves to illustrate that Conjecture 1 is false.

Theorem 5 affirms that Conjecture 1 is true for graphs with six or seven vertices.

Theorem 5. Let $|G| = n \in \{6,7\}$. If G is 1-tough, then either G is Hamiltonian or its complement \overline{G} contains the graph F .

Proof. We consider G with $|G| = 6$ first. In this case, we want

to show that G is Hamiltonian and its complement \overline{G} does not contain F . By Theorem 3, $c(G) = 6$, so G is a Hamiltonian graph. Assume that \overline{G} contains F , then G must contain fewer edges than \overline{F} , the complement of F . See Fig. 1 (b) for an illustration of \overline{F} . Since the graph \overline{F} is $\frac{1}{2}$ -tough, G cannot be better than $\frac{1}{2}$ -tough, which violates the known condition that G is 1-tough. Therefore, \overline{G} does not contain F . Next, we consider G with $|G| = 7$. Note that G being 1-tough implies that $k \geq 2$. There are two cases.

Case 1. $\sigma_3(G) \geq 7$. With Theorem 4, G contains a Hamiltonian cycle, denoted by $C_G = \langle 1,2,3,4,5,6,7,1 \rangle$. Obviously, $E(G)$ consists of all edges in C_G and possibly more. Let C_7 be a cycle with length 7 and $\overline{C_7}$ the complement of C_7 . It is easy to see that $\overline{C_7}$ does not contain F , so \overline{G} cannot contain F . As a result, Conjecture 1 holds in this case.

Case 2. $\sigma_3(G) \leq 6$. Since $k \geq 2$, this case occurs only when there exists an independent set of three vertices $\{x,y,z\}$ such that $\deg_G(x) = \deg_G(y) = \deg_G(z) = 2$, and $\sigma_3(G) = 6$. We shall let $V(G) = \{x,y,z,a,b,c,d\}$. Under Case 2, there are three major subcases and totally five possibilities for which we must provide rigorous proofs. Table I gives an illustration for these possible situations. For simplicity, we shall label these subcases by (a), (b), (c) and so on.

TABLE I
CASE ANALYSIS IN THE PROOF CASE 2 IN THEOREM 5

(a)	$ N_G(x) \cup N_G(y) \cup N_G(z) = 2$
(b)	$ N_G(x) \cup N_G(y) \cup N_G(z) = 3$
(c)	$ N_G(x) \cup N_G(y) \cup N_G(z) = 4$
(d)	Two of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are identical.
(e)	All of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are different.
(f)	None of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$.
(g)	One of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$.

- $|N_G(x) \cup N_G(y) \cup N_G(z)| = 2$. Let $N_G(x) \cup N_G(y) \cup N_G(z) = \{a,b\}$. Thus, the subgraph induced by $\{x,y,z,a,b\}$ is $K_{3,2}$, and G is not Hamiltonian. Removing $\{a,b\}$ results in a graph with at least four components, so G is $\frac{1}{2}$ -tough or weaker. It violates the assumption that G is 1-tough, so this case cannot happen.
- $|N_G(x) \cup N_G(y) \cup N_G(z)| = 3$. Let $N_G(x) \cup N_G(y) \cup N_G(z) = \{a,b,c\}$. Removing $\{a,b,c\}$ results in a graph with at least four components, so G is $\frac{3}{4}$ -tough or weaker. Again, it contradicts the known fact that G is 1-tough, so this case should not occur.
- $|N_G(x) \cup N_G(y) \cup N_G(z)| = 4$. There are two possibilities: (d) and (e).
- Two of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are identical. W.L.O.G., let $N_G(x) = N_G(y) = \{a,b\}$ and $N_G(z) = \{c,d\}$. In this case, removing $\{a,b\}$ results in a graph with at least three components, so G is $\frac{2}{3}$ -tough or weaker, which violates the condition that G is 1-tough, so this case will not happen.
- All of $N_G(x)$, $N_G(y)$ and $N_G(z)$ are different. There are two subcases. There are two subcases: (f) and (g).
- None of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x,y,z\}$. W.L.O.G., let $N_G(x) = \{a,b\}$, $N_G(y) = \{b,c\}$, and $N_G(z) = \{c,d\}$. See Fig. 3 for an illustration. If $(a,d) \in E(G)$, then

$\langle a, x, b, y, c, z, d, a \rangle$ is a cycle of length 7, which is a Hamiltonian cycle of G . The argument in Case 1 shows that \bar{G} does not contain F , so Conjecture 1 holds in this case. Now, we discuss the situation when (a, d) does not belong to $E(G)$. The set of edges among $\{a, b, c, d\}$ contains at most $\{(a, b), (b, c), (c, d), (a, c), (b, d)\}$. Removing $\{b, c\}$ results in a graph with at least three components, so G is $\frac{2}{3}$ -tough or weaker. It is not possible.

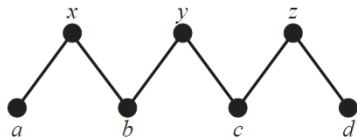


Fig. 3 An illustration for Case 2, (f) in the proof of Theorem 5

- g) One of $N_G(a)$, $N_G(b)$, $N_G(c)$ and $N_G(d)$ covers $\{x, y, z\}$. W.L.O.G., let $N_G(a)$ be the one covering $\{x, y, z\}$, and let $N_G(x) = \{a, b\}$, $N_G(y) = \{a, c\}$, and $N_G(z) = \{a, d\}$. It is easy to see that G must be non-Hamiltonian. Moreover, \bar{G} contains the triangle with vertices $\{x, y, z\}$ and the edges $\{(x, c), (x, d), (y, b), (y, d), (z, b), (z, c)\}$. That is, \bar{G} contains F . Since G is 1-tough, it can be observed that E contains $\{(b, c), (c, d), (b, d)\}$ while the edges $(a, b), (a, c), (a, d)$ are optional. See Fig. 4 for an illustration. We note that the graph with $\deg_G(a) = 6$ is isomorphic to the graph H in [2]. Thus, Conjecture 1 is true in this case.

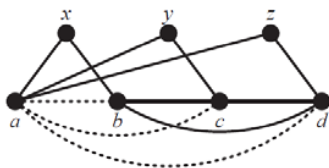


Fig. 4 An illustration for Case 2, (g) in the proof of Theorem 5

From the above derivation, we conclude that for any 1-tough graph G with $|G| = 7$, either G contains a Hamiltonian cycle or G is of the form in Fig. 4, of which the complement contains F .

The following two lemmas are derived in order to obtain the correction for Conjecture 1 for graphs with eight or more vertices. We denote the complement of G by \bar{G} . The graph F^* is shown in Fig. 5.

Lemma 2. Let G be a 1-tough graph with $|G| = n \geq 11$. The following three statements are equivalent.

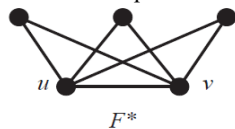


Fig. 5 The graph F^*

- There exists some nonadjacent pair $\{x, y\}$ in G such that $\deg_G(x) + \deg_G(y) \leq n - 5$.
- There exists an edge (x, y) of \bar{G} such that $\deg_{\bar{G}}(x) + \deg_{\bar{G}}(y) \geq n + 3$.
- \bar{G} contains the graph F^* .

Proof. First of all, we want to show (i) implies (ii). Take the edge (u, v) of \bar{G} such that the nonadjacent vertex pair $\{u, v\}$ in G satisfies $\deg_G(u) + \deg_G(v) \leq n - 5$. Therefore, $\deg_{\bar{G}}(u) + \deg_{\bar{G}}(v)$

$$\begin{aligned} &= (n - 1 - \deg_G(u)) + (n - 1 - \deg_G(v)) \\ &\geq (2n - 2) - (n - 5) \\ &= n + 3. \end{aligned}$$

Secondly, we need to show (ii) implies (iii). Following (ii), there are $n - 2$ vertices in $V(\bar{G}) - \{u, v\}$. If $N_{\bar{G}}(u) \cap N_{\bar{G}}(v) = \emptyset$, then $\deg_{\bar{G}}(u) + \deg_{\bar{G}}(v) \leq 1 + 1 + (n - 2) = n$. It violates (ii). Thus, $N_{\bar{G}}(u)$ and $N_{\bar{G}}(v)$ must have at least three common vertices in $V(\bar{G}) - \{u, v\}$. This implies that \bar{G} contains the graph F^* .

Finally, we will show (iii) implies (i). This part will be shown by deducing a contradiction from the opposite assumption. Assume that \bar{G} contains the graph F^* , and $\deg_G(x) + \deg_G(y) \geq n - 4$ holds for any nonadjacent pair of vertices $\{x, y\}$ of G . With a simple calculation, one can see that $\deg_G(x) + \deg_G(y) \leq n + 2$ holds for any edge (x, y) of \bar{G} . As in the previous argument, it means that the endvertices x and y of any edge (x, y) in \bar{G} have at most two common neighbors. Then, \bar{G} cannot contain F^* , which violates (iii). Consequently, there must be some nonadjacent pair of vertices $\{x, y\}$ in G with $\deg_G(x) + \deg_G(y) \leq n - 5$.

Lemma 3 can be obtained using the similar derivation as in Lemma 2.

Lemma 3. Let G be a 1-tough graph with $|G| = n \in \{8, 9, 10\}$. The following three statements are equivalent.

- There exists some nonadjacent pair $\{x, y\}$ in G with $\deg_G(x) + \deg_G(y) \leq n - 3$.
- There exists an edge (x, y) of \bar{G} such that $\deg_{\bar{G}}(x) + \deg_{\bar{G}}(y) \geq n + 1$.
- The complement of G , denoted by \bar{G} , contains the graph K_3 .

Our correction of Conjecture 1 for graphs with eight or more vertices is presented below.

Theorem 6. Let G be a 1-tough graph with $|G| = n \geq 8$. Then

- For $n \geq 11$, either $\sigma_2(G) \geq n - 4$ or \bar{G} contains F^* .
- For $n \in \{8, 9, 10\}$, either $\sigma_2(G) \geq n - 2$ or \bar{G} contains K_3 .

Proof. We will explain (a), where $n \geq 11$, in detail and skip the similar discussion for (b). There are two cases.

Case 1. Suppose that $\deg_G(x) + \deg_G(y) \leq n - 4$ holds for any nonadjacent pair of vertices $\{x, y\}$ of G . With Theorem 1, G is Hamiltonian. Note that the degree-sum condition is the sufficient condition for G to be Hamiltonian, and the converse is not true.

Case 2. Suppose that there exists some nonadjacent pair of vertices $\{x, y\}$ of G such that $\deg_G(x) + \deg_G(y) \leq n - 5$. With Lemma 2, it is equivalent to saying that \bar{G} contains F^* .

Combining Case 1 and 2, (a) is verified. When we apply Corollary 1 for (b) concerning the case where $\sigma_2(G) \geq n - 2$, the similar difficulty appears. The fact that the degree-sum condition provides only the sufficient condition for G to be Hamiltonian, not the necessary condition prevents us from a

stronger conclusion as in Conjecture 1.

As a result, Theorem 6 corrects Conjecture 1 for $n \geq 8$ and becomes the best that one can have.

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